Finiteness in the Card Game of War

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1 The Game of War

The Game of War is an internationally popular children's card game with a very basic set of rules. At the start, the pack is shuffled and divided into two equal parts; then each player reveals their top card. The player with the highest value card then collects both cards and returns them to the bottom of their hand. This process continues until one player loses all cards.

Many who play this game in childhood lose patience before the end as the back and forth nature of card transfer often allows a player to almost reach the point of victory only to see their hand reduced dramatically in a couple of minutes. In effect children playing are conducting basic mathematical experiments and observing the development of chaotic dynamics. It is often wrongly assumed that this game is deterministic and the result is set once the cards have been dealt. However this is not so: the rules of the game do not stipulate in which order the winner of each play returns the cards to the bottom of their hand - own card first and then rival's or vice versa.

We shall at first consider a model with an arbitrary (even) number of cards and only one suit. Hence a situation where both players present the same value card cannot occur; in the classic game, if this does occur, it would lead to a process called 'war' and players would continue to lay cards until one plays a winner thus claiming all cards played.

So consider a game where the players strictly control the order in which the cards are returned to the bottom of their hand; in this case there is a chance that the game will never finish. Such a cyclic game is shown in Fig. 1 where $n = 6$. The card of the 'left' player is always returned to the bottom of the winner's hand before the card of the 'right' player. Such a never ending game

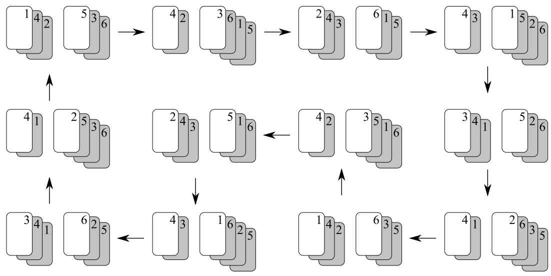

Figure 1: A never ending game, $n = 6$.

can also occur with a standard pack if the above rules are followed. An example of an initial shuffling for such a game is given on Fig. 2 (throughout the paper we assume that the ace is the highest value card). Observe that after the first two moves are made, the last two cards in the hand of player L are $(A\heartsuit, K\clubsuit)$, and in the hand of player R – $(K\heartsuit, A\clubsuit)$, then after the next two moves the last 4 cards are $(A\heartsuit, K\clubsuit, A\diamondsuit, K\spadesuit)$ for the player L and $(K\heartsuit, A\clubsuit, K\diamondsuit, A\spadesuit)$ for

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player R, and so on. It can be easily seen that the order of the card values is preserved, and after 26 moves the players will end up with precisely the same distribution of card values in their hands (although suits would get shuffled).

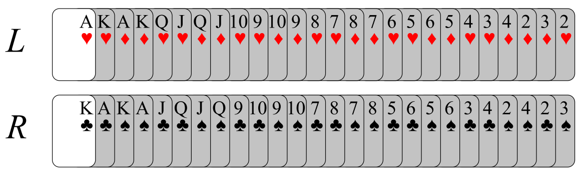

Figure 2: A never ending game, standard pack of 52 cards.

Hence we have established that when rigid rules are used, it might be possible to never finish the game. But what if players use both possible ways of returning cards to their hands, and on each particular move choose such rules at random? In this case, it is not immediately clear whether the players have a nonzero chance to reach the end of the game, however, it is possible to give an answer to this question using a probabilistic model of the game. We show that if the players use both rules to return the cards to their hand, the mathematical expectation of the number of moves in the game is finite, i.e., there is a zero chance of never finishing the game.

2 Mathematical model

We now focus on a model case in which there are cards from 1 to n. We assume that each player possesses certain peculiarities which means when collecting cards their card will be placed on the bottom with probability p_1^i and second from bottom with p_2^i, where the index i identifies the player (i is either L or R). This is illustrated on Fig. 3 where player L wins, and chooses to place his card on the bottom with probability p_1^L, and uses the other rule with probability p_2^L. Let C be the pack of cards, ei-

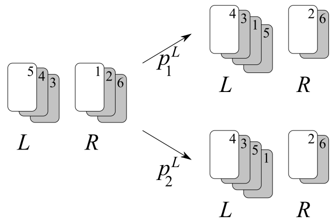

Figure 3: Probabilities with which player L chooses the order to place the cards in his hand (model pack of 6 cards)

ther numbered (from 1 to n, where n is any even number) or standard (with 52 cards, valued from 2 to ace and having 4 suits). We use L to denote the cards in the hand of the 'left' player, and R to denote the 'right' player's hand. In the model case L and R are ordered sets of numbers, one of which can be empty; this corresponds to the end of the game.

Each division of the pack into two ordered sets is called a game *state*. That is, the game starts with a state with both sets L and R having an equal number of members, and ends with the state in which one of the sets is empty (the *final* state). Each game play (when the players show their top cards, compare them and then put the ordered pair to the bottom of the winner's hand) is a *transition* from one state to another, because it starts with two ordered sets of cards, and ends with two (different) ordered sets, which correspond to the players' hands.

Such dynamics in which a transition to the next state happens with fixed probabilities independent on the preceding choices is called Markov chain. In the theory of Markov chains one of the fundamental facts is the following: assume that any initial state is possible. The mathematical expectation of the number of moves before the *absorption* (reaching the final state) is finite if and only if the final state can be reached from any state [3, Chap.3]. Usually a Markov chain is represented as a directed graph in which the vertices correspond to states, and edges correspond to the transitions. An edge leaves one vertex and reaches another if and only if there exists a transition from the former to the latter with a nonzero probability. It is not difficult to see that in our case each non-final vertex (or state) has only got two outgoing edges: once both players have revealed their cards, the winner, by putting the top cards to the bottom of his hand, defines such two transitions with probabilities p_1^i and p_2^i, where i is either L or R, depending on who wins this particular game. We call a vertex *attaining* if it has got a final state as one of its descendants and *wandering* otherwise. It is obvious that a descendant of a wandering vertex is again wandering, and a predecessor of an attaining one is again attaining. A graph is called *absorbing* if all the vertices are attaining. That is, the graph of our game is absorbing if and only if for every state (each division on the pack into two hands) it is possible to finish playing the game in a finite number of moves. The difference between absorbing and non-absorbing graphs can be seen on Fig. 4. Both graphs *i)* and *ii)* have

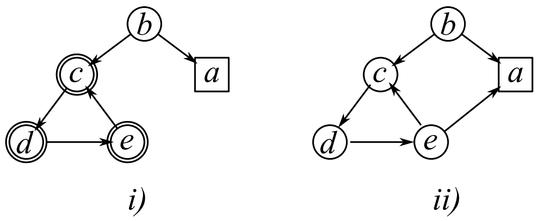

Figure 4: Difference between absorbing and non-absorbing graphs: the graph *i)* is not absorbing, but the graph *ii)* obtained by adding one extra edge between e and a to the graph *i)* is absorbing.

got the same final state a. Observe that the graph *i)* is not absorbing: it can be easily seen that it is impossible to reach the state a from any of the vertices c, d and e, while the graph *ii)*, which differs from *i)* only by one additional edge that leads from e to a, is absorbing, as one can get into a from any other vertex.

3 Proof of the main result

We have already established that each state (except for final) has got exactly two outgoing edges. Now assume that each of the players has got at least two cards in their hands. There are exactly two incoming edges corresponding to the possible preceding plays in which either of the players won (see Fig. 5 for an illustration for a game with 6 cards).

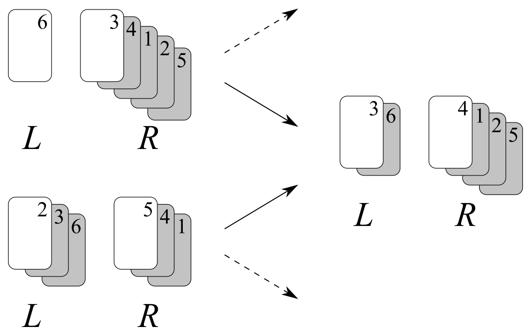

Figure 5: If the players have got at least two cards each, then this vertex has got exactly two predecessors.

If one of the players has got only one card left, he could not have won the preceding play, and hence there is only one possibility for the winner. This is illustrated on Fig. 6 for a game with 6 cards.

Using this crucial observation, we are going to show that *the game graph has got no wandering vertices* (i.e. it is possible to finish the game starting from any state). We show this by assuming that there is at least one wandering vertex, and then establishing that this leads to a contradiction.

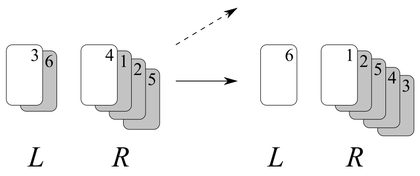

Figure 6: If one of the players has got only one card left, this vertex has got a unique direct predecessor.

By $\mathcal{W}$ denote the set of all wandering vertices in our graph. Observe that each edge going out from a wandering vertex goes into a wandering vertex again, otherwise we would get a contradiction with the definition of a wandering vertex. Therefore, the total number of different edges that leave the vertices in $\mathcal{W}$ is not less than the total number of different edges that lead into $\mathcal{W}$. Taking into account that each non-terminal vertex has got exactly two outgoing edges and either one or two incoming, we immediately get two results: first of all, **if wandering vertices exist, then they comprise an *isolated subgraph*, i.e. it is impossible to get into there from any vertex outside of this subgraph**; second, each wandering vertex has got exactly two predecessors, and hence **each wandering vertex corresponds to the state in which each player has got at least two cards**.

Now pick up any wandering vertex and conduct the following 'back-tracking' procedure: if the vertex has got two direct predecessors, consider the one in which the left player has got less cards than in the current state (that is, the left player won the preceding play). Observe that this is always possible (see Fig. 5), as we can always backtrack through the play in which the left player was the winner. If we continue going back in this manner, we will finally reach the state in which the left player has got only one card left, and hence has got only one direct predecessor; this can not correspond to a wandering vertex. This means that a wandering vertex can be reached from a non-wandering, which contradicts our earlier finding (that wandering vertices constitute an isolated subgraph). This can only mean that the graph does not have any wandering vertices at all, and hence for each state there is a path to a final one.

Now we can use a well-established fact (see [3, Chap.3]) that if a graph corresponding to a finite Markov chain is absorbing, the mathematical expectation of the number of moves needed to reach the final state is finite, which is exactly what we expected to prove in the first place.

We will not discuss the proof for the classical game of war in detail because the basic mathematical ideas are already contained in the proof of the model game. We only note that the main idea in studying the real card game is the following obvious statement: *if a subgraph of an oriented graph, which consists of all the vertices of the original graph, and might not include some of the edges, does not have any wandering vertices, then the original graph does not have them either.*

We would like to point out that this card game was studied by other authors, however, they focused on other aspects of the game. In particular, Jacob Haqq-Misra [2] uses numerical simulation (employing Monte-Carlo method) to find out how the advantage in the initial distribution of cards influences the outcome of the standard game; Ben-Naim and Krapivsky [1] discuss a stochastic model of the game.

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