HIROSHIMA MATH. J.
9 (1979), 623–626

On Fitting's Lemma

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(Received March 15, 1979)

We assume throughout the paper that R is a ring with identity and that all the R-modules are unital. Given a right ideal K of R, $I_R(K)$ will denote the idealizer of K in R: $I_R(K)=\{x\in R \mid xK\subseteq K\}$.

Let M be a right R-module. The intersection of maximal submodules of M_R is denoted by $J(M_R)$, and M_R is said to be *semisimple* if $J(M_R)=0$. If for each $f\in \mathrm{End}(M_R)$ there exists a positive integer n such that $M=\mathrm{Ker}\, f^n \oplus \mathrm{Im}\, f^n$, then M_R is said to satisfy *Fitting's lemma*. In this paper, we consider the following properties:

(I) Every injective endomorphism of any finitely generated right R-module is an isomorphism.

(S) Every surjective endomorphism of any finitely generated right R-module is an isomorphism.

(F) Every finitely generated right R-module satisfies Fitting's lemma.

In [1, Proposition 2.3], E. P. Armendariz, J. W. Fisher and R. L. Snider proved that M_R satisfies Fitting's lemma if and only if $\mathrm{End}(M_R)$ is strongly π-regular. They also proved that R possesses the property (I) if and only if $(R)_n$ is strongly π-regular for each positive integer n (see [1, Theorem 1.1]).

In what follows, we shall prove first that R possesses the property (F) if and only if R does (I) and (S) (Theorem 1). Next, we shall prove that any right V-ring with primitive factor rings Noetherian possesses the property (S) (Corollary 3). As a combination of Theorem 1 and Corollary 3, we shall give an alternative proof of [1, Theorem 2.5]. Finally, we shall show that the property (F) is Morita invariant.

Now, we begin with the following theorem.

THEOREM 1. *A ring R possesses the property* (F) *if and only if R possesses the properties* (I) *and* (S).

PROOF. It suffices to prove that (I) and (S) imply (F). Suppose R possesses the properties (I) and (S). Let M be a finitely generated right R-module. Then we have an exact sequence $R_R^{(n)} \xrightarrow{h} M_R \longrightarrow 0$ for some positive integer n. Let f be an arbitrary endomorphism of M_R. Since $R_R^{(n)}$ is projective there exists some $\bar{f}\in \mathrm{End}(R_R^{(n)})$ such that $h\bar{f}=fh$. Since $\mathrm{End}(R_R^{(n)})$ is strongly π-regular by [1, Theorem 1.1], there exists $g \in \operatorname{End}(R_R^{(n)})$ such that $\bar{f}=\bar{f}^{m+1}g$ with some positive integer m. Then, $f^m(M)=f^m h(R^{(n)})=h\bar{f}^{m+1}g(R^{(n)})\subseteq h\bar{f}^{m+1}(R^{(n)})=f^{m+1}h(R^{(n)})=f^{m+1}(M)$, which proves $f^m(M)=f^{m+1}(M)$. We consider the finitely generated right R-module $M'=f^m(M)$. Since $(f^m \mid M'): M'_R \to M'_R$ is surjective, $\operatorname{Ker}(f^m \mid M')=0$ by hypothesis. This implies $f^m(M)\cap \operatorname{Ker} f^m=0$. Now, it is easy to see that $M=f^m(M)\oplus \operatorname{Ker} f^m$.

The next was announced in [1].

COROLLARY 1. *If R is a strongly π-regular PI-ring which is an integral extension of its center, then R possesses the property* (F). *In particular, every commutative π-regular ring possesses the property* (F).

PROOF. R possesses (I) and (S) by [1, Theorem 1.1 and Theorem 2.2].

COROLLARY 2 ([1, Proposition 2.7]). *Let $\{R_d \mid d\in D\}$ be a directed set of rings with the property* (F). *If $R=\lim_{\to} R_d$, then R possesses* (F).

PROOF. By [2, Theorem 2], R possesses (S). On the other hand, R does (I) by [1, Theorem 1.1].

THEOREM 2. *If R is a ring with primitive factor rings Noetherian and M is a finitely generated semisimple right R-module, then every surjective endomorphism of M_R is an isomorphism.*

PROOF. Let $\{a_1,\dots, a_n\}$ be a generating system of M_R. Writing the elements of $M^{(n)}$ as $(x_1, x_2,\dots, x_n)$, we can regard $M^{(n)}$ as a right $(R)_n$-module. Then $M^{(n)}$ is a cyclic $(R)_n$-module generated by $(a_1, a_2,\dots, a_n)$. Since $J(M^{(n)}_{(R)_n})=J(M_R)^{(n)}=0$, $M^{(n)}$ is a semisimple $(R)_n$-module. As usual, $\operatorname{End}(M_R)$ and $\operatorname{End}(M^{(n)}_{(R)_n})$ may be identified. So, to our end, it suffices to prove our theorem for cyclic M_R. Then we can assume that $M=R/A$ with some right ideal A of R. First, we shall show that $\cap_P MP=0$ where P runs over all the primitive ideals of R. Let K be an arbitrary maximal right ideal of R containing A. If we set $Q=\{r\in R \mid Rr\subseteq K\}$, then Q is a primitive ideal of R. Since $MQ=Q/A\subseteq K/A$, it follows that $\cap_P MP \subseteq J(M_R)=0$. Now, we can prove that any surjective endomorphism f of M_R is injective. If not, there exists a non-zero x in M such that $f(x)=0$. Then, by the last formula, there exists a primitive ideal P of R such that $x\notin MP$. If we set $\bar{M}=M/MP$, then f induces an epimorphism $\bar{f}: \bar{M}_{R/P}\to \bar{M}_{R/P}$ whose kernel contains a non-zero $x+MP$. Since R/P is Noetherian, the endomorphism $\bar{f}$ is an isomorphism, a contradiction.

COROLLARY 3. *If R is a right V-ring with primitive factor rings Noetherian, then R possesses the property* (S).

PROOF. Note that every right module over a right V-ring R is semisimple by [5, Theorem 2.1] or [6, Theorem 4].

COROLLARY 4 ([1, Theorem 2.5]). *If R is a regular ring with primitive factor rings Artinian, then R possesses the property* (F).

PROOF. Since R is a right V-ring by [6, Theorem 5], R possesses (S) by Corollary 3. Now, let $\bar{R}$ be an arbitrary prime factor ring of R. If $\bar{R}$ is not simple, then by [4, Theorem X.11.3] $\bar{R}$ contains a nontrivial central idempotent, a contradiction. Hence, by [3, Theorem 2.1] $(R)_n$ is strongly π-regular ($n=1, 2,\dots$), and then R possesses the property (I) by [1, Theorem 1.1]. Consequently, R does (F) by Theorem 1.

PROPOSITION 1. *The following are equivalent*:

1) *R possesses the property* (S).

2) *Every right ideal K of $(R)_n$ ($n=1, 2,\dots$) possesses the following property: If $xy-1\in K$ with $x\in I_{(R)_n}(K)$ and $y\in(R)_n$, then y is in $I_{(R)_n}(K)$ and $yx-1\in K$.*

PROOF. 2)$\Rightarrow$1) Let M be a right R-module generated by $a_1,\dots, a_n$, and $K=\{z\in(R)_n \mid (a_1,\dots, a_n)z=(0,\dots, 0)\}$. Given $g\in \mathrm{End}(M_R)$, we can write $g(a_1,\dots, a_n)=(a_1,\dots, a_n)(r_{ij})$ with some $(r_{ij})\in I_{(R)_n}(K)$. Then, the map ϕ: $\mathrm{End}(M_R)\rightarrow I_{(R)_n}(K)/K$ defined by $\phi(g)=(r_{ij})+K$ is a ring isomorphism. If f is a surjective endomorphism of M_R and $\phi(f)=x+K$ then $(a_1,\dots, a_n)xy=(a_1,\dots, a_n)$ with some $y\in(R)_n$. Since $xy-1\in K$, we then have $y\in I_{(R)_n}(K)$ and $yx-1\in K$ by hypothesis. Obviously, $fg=gf=1$ for $g=\phi^{-1}(x+K)$.

1)$\Rightarrow$2) Let K be a right ideal of $(R)_n$. Then, $\mathrm{End}((R^{(n)}/e_{11}K)_R)$ is ring isomorphic to $I_{(R)_n}(K)/K$ as above, and the reverse process in 2)$\Rightarrow$1) enables us to see that 1)$\Rightarrow$2).

COROLLARY 5. *If $(R)_n$ is integral over the center of R for each positive integer n, then R possesses the property* (S).

PROOF. Let K be an arbitrary right ideal of $(R)_n$. Assume that $xy-1\in K$ with $x\in I_{(R)_n}(K)$ and $y\in(R)_n$. Since $(R)_n$ is integral over its center C, we can write $y^{m+1}=\sum_{i=0}^{m} c_i y^i$ with some $c_i\in C$ and some non-negative integer m. Then, it is easy to see that $y\equiv x^m y^{m+1}\equiv\sum_{i=0}^{m} c_i x^{m-i}$ (mod K). Hence, y is in $I_{(R)_n}(K)$ and $yx-1\in K$. According to Proposition 1, this means that R possesses the property (S).

COROLLARY 6. *Every algebraic algebra R over a non-denumerable field K possesses the property* (F).

PROOF. It is known that $(R)_n$ is algebraic over K for any positive integer

n (see [4, Theorem X.14.2]). Then, by Corollary 5, R possesses the property (S). Furthermore, $(R)_n$ is strongly π-regular. Hence, by [1, Theorem 1.1] and Theorem 1, R possesses the property (F).

THEOREM 3. *The property* (F) *is Morita invariant.*

PROOF. According to Theorem 1, it suffices to prove that the properties (I) and (S) are Morita invariant. First, if R possesses the property (I) (resp. (S)), then so does $(R)_n$ by [1, Theorem 1.1] (resp. Proposition 1). Now, let e be an arbitrary non-zero idempotent of R. If $(R)_n$ is strongly π-regular, then so is $(eRe)_n = e(R)_n e$. Hence, again by [1, Theorem 1.1], eRe possesses the property (I) when R does. Finally, assume that R possesses the property (S). Let K be an arbitrary right ideal of $(eRe)_n$, and $xy-e \in K$ with $x \in I_{(eRe)_n}(K)$ and $y \in (eRe)_n$. Since $x \in I_{(R)_n}(K(R)_n + (1-e)(R)_n)$ and $xy-1 \in K(R)_n + (1-e)(R)_n$, by Proposition 1 it follows that $y \in I_{(R)_n}(K(R)_n + (1-e)(R)_n)$ and $yx-1 \in K(R)_n + (1-e)(R)_n$. Recalling that x and y are in $(eRe)_n$, we can easily see that $y \in I_{(eRe)_n}(K)$ and $yx-1 \in K$. Therefore, again by Proposition 1, eRe possesses the property (S).

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