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Research Article

On Fixed Point Results in Controlled Metric Spaces

Jamshaid Ahmad,[1] **Abdullah Eqal Al-Mazrooei**,[1] **Hassen Aydi**,[2,3] **and Manuel De la Sen**[4]

[1]*Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia*
[2]*University of Sousse, Higher Institute of Information Technology and Communication Techniques, H. Sousse 4000, Tunisia*
[3]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[4]*Institute of Research and Development of Processes IIDP, University of the Basque Country, Campus of Leioa. Leioa (Bizkaia), P.O. Box 48940, Spain*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

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In this article, we introduce Reich type contractions and (α, F)-contractions in the class of controlled metric spaces and establish some new related fixed point theorems. Our results are generalizations of some known results of literature. Some examples and certain consequences are given to illustrate significance of presented results.

1. Introduction and Preliminaries

In 1993, Czerwik [1] reintroduced a new class of generalized metric spaces, called as b-metric spaces, as generalizations of metric spaces.

Definition 1 ([1]). Let Y be a nonempty set and $s \geq 1$. A function $d_b : Y \times Y \longrightarrow [0,\infty)$ is said to be a b *-metric* if for all $\varsigma, \omega, \tau \in Y$,

($b1$) $d_b(\varsigma, \omega) = 0$ iff $\varsigma = \omega$
($b2$) $d_b(\varsigma, \omega) = d_b(\omega, \varsigma)$ for all $\varsigma, \omega \in Y$
($b3$) $d_b(\varsigma, \tau) \leq s[d_b(\varsigma, \omega) + d_b(\omega, \tau)]$

The pair (Y, d_b) is then called a b-metric space. Subsequently, many fixed point results on such spaces were given (see [2–7]).

Kamran et al. [8] initiated the concept of extended b -metric spaces.

Definition 2. Let Y be a nonempty set and $p : Y \times Y \longrightarrow [1,\infty)$ be a function. A function $d_e : Y \times Y \longrightarrow [0,\infty)$ is called an extended b *-metric* if for all $\varsigma, \omega, \tau \in Y$,

(i) $d_e(\varsigma, \omega) = 0$ iff $\varsigma = \omega$
(ii) $d_e(\varsigma, \omega) = d_e(\omega, \varsigma)$
(iii) $d_e(\varsigma, \omega) \leq p(\varsigma, \omega)[d_e(\varsigma, \tau) + d_e(\tau, \omega)]$

The pair (Y, d_e) is called an extended b-metric space.

Very recently, a new kind of a generalized b-metric space was introduced by Mlaiki et al. [9].

Definition 3. Let Y be a nonempty set and $p : Y \times Y \longrightarrow [1,\infty)$ be a function. A function $\mathscr{C} : Y \times Y \longrightarrow [0,\infty)$ is called a controlled metric if for all $\varsigma, \omega, \tau \in Y$,

(i) $\mathscr{C}(\varsigma, \omega) = 0$ iff $\varsigma = \omega$
(ii) $\mathscr{C}(\varsigma, \omega) = C(\omega, \varsigma)$
(iii) $\mathscr{C}(\varsigma, \omega) \leq p(\varsigma, \tau)\mathscr{C}(\varsigma, \tau) + p(\omega, \tau)\mathscr{C}(\omega, \tau)$

The pair $(Y, \mathscr{C})$ is called a controlled metric space (see also [10]).

The Cauchy and convergent sequences in controlled metric type spaces are defined in this way.

Definition 4 ([9]). Let $(Y, \mathscr{C})$ be a controlled metric space and $\{\varsigma_n\}_{n\geq 0}$ be a sequence in Y. Then,

(i) The sequence $\{\varsigma_n\}$ converges to some ς in Y; if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathscr{C}(\varsigma_n, \varsigma) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty}\varsigma_n = \varsigma$

(ii) The sequence $\{\varsigma_n\}$ is Cauchy; if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathscr{C}(\varsigma_m, \varsigma_n) < \varepsilon$ for all $m, n \geq N$

(iii) The controlled metric space $(Y, \mathscr{C})$ is called complete if every Cauchy sequence is convergent

Definition 5 ([9]). Let $(Y, \mathscr{C})$ be a controlled metric space. Let $\varsigma \in Y$ and $\varepsilon > 0$.

(i) The open ball $B(\varsigma, \varepsilon)$ is

$$B(\varsigma, \varepsilon) = \{\omega \in Y : \mathscr{C}(\varsigma, \omega) \lhd \varepsilon\} \tag{1}$$

(ii) The mapping $\Omega : Y \longrightarrow Y$ is said to be continuous at $\varsigma \in Y$; if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\Omega(B(\varsigma, \delta)) \subseteq B(\Omega\varsigma, \varepsilon)$

Very recently, Wardowski [11] introduced a new type of contractions, called F-contractions and established some new related fixed point theorems in the context of complete metric spaces.

Definition 6. Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a function satisfying

(F_1) F is strictly increasing, that is, for all $t_1, t_2 \in \mathbb{R}^+$ such that $t_1 < t_2$ implies that $F(t_1) < F(t_2)$.

(F_2) For every sequence $\{t_n\}$ of positive real numbers, $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} F(t_n) = -\infty$ are equivalent.

(F_3) There is $h \in (0, 1)$ so that $\lim_{t\to 0^+} t^h F(t) = 0$.

Let $\sqsubseteq$ be the set of above functions F satisfying (F_1)-(F_3) (to be consistent with Wardowski [11]). A self-mapping Ω on the metric space (Y, η) is said to be an F-contraction if there are a function F satisfying (F_1)-(F_3) and a constant $\tau > 0$ so that

$$\eta(\Omega\varsigma, \Omega\omega) > 0 \Rightarrow \tau + F(\eta(\Omega\varsigma, \Omega\omega)) \leq F(\eta(\varsigma, \omega)) \tag{2}$$

for all $\varsigma, \omega \in Y$.

Theorem 7 [11]. *Let (Y, η) be a complete metric space and $\Omega : Y \longrightarrow Y$ be an F -contraction, then Ω admits a unique fixed point.*

The authors in [11] manifested that a Banach contraction is a specific case of F-contractions, while there are many F -contractions which need not be a Banach contraction. For more details, we refer the readers to ([12–22]).

In this paper, we first define Reich [23, 24] and (α, F)-contractions in the setting of controlled metric spaces and prove some new fixed point results. We also provide some examples to illustrate significance of the established results.

2. Results on Reich Type Contractions

Theorem 8. *Let $(Y, \mathscr{C})$ be a complete controlled metric space. Let $\Omega : Y \longrightarrow Y$ be so that there are $\alpha, \beta, \gamma \in (0, 1)$ with $k = \alpha + \beta/1 - \gamma < 1$,*

$$\mathscr{C}(\Omega\varsigma, \Omega\omega) \leq \alpha\mathscr{C}(\varsigma, \omega) + \beta\mathscr{C}(\varsigma, \Omega\varsigma) + \gamma\mathscr{C}(\omega, \Omega\omega) \tag{3}$$

for all $\varsigma, \omega \in Y$. For $\varsigma_0 \in Y$, take $\varsigma_n = \Omega^n\varsigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \frac{p(\varsigma_{i+1}, \varsigma_{i+2})p(\varsigma_{i+1}, \varsigma_m)}{p(\varsigma_i, \varsigma_{i+1})} < \frac{1}{k}. \tag{4}$$

Suppose that $\lim_{n\to\infty}p(\varsigma_n, \varsigma)$ and $\lim_{n\to\infty}p(\varsigma, \varsigma_n)$ exist, are finite, and $\gamma\lim_{n\to\infty}p(\varsigma_n, \varsigma) < 1$ for every $\varsigma \in Y$, then Ω possesses a unique fixed point.

Proof. The considered sequence $\{\varsigma_n\}$ verifies $\varsigma_{n+1} = \Omega\varsigma_n$ for all $n \in \mathbb{N}$. Obviously, if there exists $n_0 \in \mathbb{N}$ for which $\varsigma_{n_0+1} = \varsigma_{n_0}$, then $\Omega\varsigma_{n_0} = \varsigma_{n_0}$, and the proof is finished. Thus, we suppose that $\varsigma_{n+1}{=}\varsigma_n$ for every $n \in \mathbb{N}$. Thus, by (1), we have

$$\begin{aligned}\mathscr{C}(\varsigma_n, \varsigma_{n+1}) &= \mathscr{C}(\Omega\varsigma_{n-1}, \Omega\varsigma_n) \leq \alpha\mathscr{C}(\varsigma_{n-1}, \varsigma_n) \\ &\quad + \beta\mathscr{C}(\varsigma_{n-1}, \Omega\varsigma_{n-1}) + \gamma\mathscr{C}(\varsigma_n, \Omega\varsigma_n) \\ &= \alpha\mathscr{C}(\varsigma_{n-1}, \varsigma_n) + \beta\mathscr{C}(\varsigma_{n-1}, \varsigma_n) + \gamma\mathscr{C}(\varsigma_n, \varsigma_{n+1})\end{aligned} \tag{5}$$

which implies that

$$\mathscr{C}(\varsigma_n, \varsigma_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma}\mathscr{C}(\varsigma_{n-1}, \varsigma_n) = k\mathscr{C}(\varsigma_{n-1}, \varsigma_n). \tag{6}$$

Thus, we have

$$\mathscr{C}(\varsigma_n, \varsigma_{n+1}) \leq k\mathscr{C}(\varsigma_{n-1}, \varsigma_n) \leq k^2\mathscr{C}(\varsigma_{n-2}, \varsigma_{n-1}) \leq \cdots \leq k^n\mathscr{C}(\varsigma_0, \varsigma_1). \tag{7}$$

For all $n, m \in \mathbb{N}(n < m)$, we have

$$\begin{aligned}\mathscr{C}(\varsigma_n, \varsigma_m) &\leq p(\varsigma_n, \varsigma_{n+1})\mathscr{C}(\varsigma_n, \varsigma_{n+1}) + p(\varsigma_{n+1}, \varsigma_m)\mathscr{C}(\varsigma_{n+1}, \varsigma_m) \\ &\leq p(\varsigma_n, \varsigma_{n+1})\mathscr{C}(\varsigma_n, \varsigma_{n+1}) + p(\varsigma_{n+1}, \varsigma_m)p(\varsigma_{n+1}, \varsigma_{n+2}) \\ &\quad \cdot \mathscr{C}(\varsigma_{n+1}, \varsigma_{n+2}) + p(\varsigma_{n+1}, \varsigma_m)p(\varsigma_{n+2}, \varsigma_m)\mathscr{C}(\varsigma_{n+2}, \varsigma_m) \\ &\leq p(\varsigma_n, \varsigma_{n+1})\mathscr{C}(\varsigma_n, \varsigma_{n+1}) + p(\varsigma_{n+1}, \varsigma_m)p(\varsigma_{n+1}, \varsigma_{n+2}) \\ &\quad \cdot \mathscr{C}(\varsigma_{n+1}, \varsigma_{n+2}) + p(\varsigma_{n+1}, \varsigma_m)p(\varsigma_{n+2}, \varsigma_m)p(\varsigma_{n+2}, \varsigma_{n+3}) \\ &\quad \cdot \mathscr{C}(\varsigma_{n+2}, \varsigma_{n+3}) + p(\varsigma_{n+1}, \varsigma_m)p(\varsigma_{n+2}, \varsigma_m)p(\varsigma_{n+3}, \varsigma_m) \\ &\quad \cdot \mathscr{C}(\varsigma_{n+3}, \varsigma_m) \leq \cdots \leq p(\varsigma_n, \varsigma_{n+1})\mathscr{C}(\varsigma_n, \varsigma_{n+1}) \\ &\quad + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} p(\varsigma_j, \varsigma_m)\right)p(\varsigma_i, \varsigma_{i+1})\mathscr{C}(\varsigma_i, \varsigma_{i+1}) \\ &\quad + \prod_{i=n+1}^{m-1} p(\varsigma_i, \varsigma_m)\mathscr{C}(\varsigma_{m-1}, \varsigma_m).\end{aligned} \tag{8}$$

This implies that

$$\begin{aligned}\mathscr{C}(\varsigma_n,\varsigma_m) &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1})\\&\quad+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}p(\varsigma_j,\varsigma_m)\right)p(\varsigma_i,\varsigma_{i+1})\mathscr{C}(\varsigma_i,\varsigma_{i+1})\\&\quad+\left(\prod_{i=n+1}^{m-1}p(\varsigma_i,\varsigma_m)\right)p(\varsigma_{m-1},\varsigma_m)\mathscr{C}(\varsigma_{m-1},\varsigma_m)\\&\le p(\varsigma_n,\varsigma_{n+1})k^n\mathscr{C}(\varsigma_0,\varsigma_1)\\&\quad+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}p(\varsigma_j,\varsigma_m)\right)p(\varsigma_i,\varsigma_{i+1})k^i\mathscr{C}(\varsigma_0,\varsigma_1)\\&\quad+\left(\prod_{i=n+1}^{m-1}p(\varsigma_i,\varsigma_m)\right)p(\varsigma_{m-1},\varsigma_m)k^{m-1}\mathscr{C}(\varsigma_0,\varsigma_1)\\&= p(\varsigma_n,\varsigma_{n+1})k^n\mathscr{C}(\varsigma_0,\varsigma_1)\\&\quad+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}p(\varsigma_j,\varsigma_m)\right)p(\varsigma_i,\varsigma_{i+1})k^i\mathscr{C}(\varsigma_0,\varsigma_1).\end{aligned} \tag{9}$$

Let

$$S_l=\sum_{i=0}^{l}\left(\prod_{j=0}^{i}p(\varsigma_j,\varsigma_m)\right)p(\varsigma_i,\varsigma_{i+1})k^i\mathscr{C}(\varsigma_0,\varsigma_1). \tag{10}$$

Consider

$$\upsilon_i=\left(\prod_{j=0}^{i}p(\varsigma_j,\varsigma_m)\right)p(\varsigma_i,\varsigma_{i+1})k^i\mathscr{C}(\varsigma_0,\varsigma_1). \tag{11}$$

We have

$$\frac{\upsilon_{i+1}}{\upsilon_i}=p(\varsigma_{i+1},\varsigma_m)\frac{p(\varsigma_{i+1},\varsigma_{i+2})}{p(\varsigma_i,\varsigma_{i+1})}k. \tag{12}$$

In view of condition (4) and the ratio test, we ensure that the series $\sum_i\upsilon_i$ converges. Thus, $\lim_{n\to\infty}S_n$ exists. Hence, the real sequence $\{S_n\}$ is Cauchy.

Now, using (9), we get

$$\mathscr{C}(\varsigma_n,\varsigma_m)\le\mathscr{C}(\varsigma_0,\varsigma_1)[k^n p(\varsigma_n,\varsigma_{n+1})+(S_{m-1}-S_n)]. \tag{13}$$

Above, we used $p(\varsigma,\omega)\ge 1$. Letting $n,m\longrightarrow\infty$ in (13), we obtain

$$\lim_{n,m\to\infty}\mathscr{C}(\varsigma_n,\varsigma_m)=0. \tag{14}$$

Thus, the sequence $\{\varsigma_n\}$ is Cauchy in the complete controlled metric space $(Y,\mathscr{C})$. So, there is some $\varsigma^*\in Y$ so that

$$\lim_{n\to\infty}\mathscr{C}(\varsigma_n,\varsigma^*)=0\,; \tag{15}$$

that is, $\varsigma_n\longrightarrow\varsigma^*$ as $n\longrightarrow\infty$. Now, we will prove that ς^* is a fixed point of Ω. By (3) and condition (iii), we get

$$\begin{aligned}\mathscr{C}(\varsigma^*,\Omega\varsigma^*) &\le p(\varsigma^*,\varsigma_{n+1})\mathscr{C}(\varsigma^*,\varsigma_{n+1})+p(\varsigma_{n+1},\Omega\varsigma^*)\mathscr{C}(\varsigma_{n+1},\Omega\varsigma^*)\\&= p(\varsigma^*,\varsigma_{n+1})\mathscr{C}(\varsigma^*,\varsigma_{n+1})+p(\varsigma_{n+1},\Omega\varsigma^*)\mathscr{C}(\Omega\varsigma_n,\Omega\varsigma^*)\\&\le p(\varsigma^*,\varsigma_{n+1})\mathscr{C}(\varsigma^*,\varsigma_{n+1})+p(\varsigma_{n+1},\Omega\varsigma^*)[\alpha\mathscr{C}(\varsigma_n,\varsigma^*)\\&\quad+\beta\mathscr{C}(\varsigma_n,\Omega\varsigma_n)+\gamma\mathscr{C}(\varsigma^*,\Omega\varsigma^*)]\\&= p(\varsigma^*,\varsigma_{n+1})\mathscr{C}(\varsigma^*,\varsigma_{n+1})+p(\varsigma_{n+1},\Omega\varsigma^*)[\alpha\mathscr{C}(\varsigma_n,\varsigma^*)\\&\quad+\beta\mathscr{C}(\varsigma_n,\varsigma_{n+1})+\gamma\mathscr{C}(\varsigma^*,\Omega\varsigma^*)].\end{aligned} \tag{16}$$

Taking the limit as $n\longrightarrow\infty$ and using (5, 6) and the fact that $\lim_{n\to\infty}p(\varsigma_n,\varsigma)$ and $\lim_{n\to\infty}p(\varsigma,\varsigma_n)$ exist, are finite, we obtain that

$$\mathscr{C}(\varsigma^*,\Omega\varsigma^*)\le\left[\gamma\lim_{n\to\infty}p(\varsigma_{n+1},\Omega\varsigma^*)\right]\mathscr{C}(\varsigma^*,\Omega\varsigma^*). \tag{17}$$

Suppose that $\varsigma^*\ne\Omega\varsigma^*$, having in mind that $[\gamma\lim_{n\to\infty}p(\varsigma_{n+1},\Omega\varsigma^*)]<1$, so

$$0<\mathscr{C}(\varsigma^*,\Omega\varsigma^*)\le\left[\gamma\lim_{n\to\infty}p(\varsigma_{n+1},\Omega\varsigma^*)\right]\mathscr{C}(\varsigma^*,\Omega\varsigma^*)<\mathscr{C}(\varsigma^*,\Omega\varsigma^*). \tag{18}$$

It is a contradiction. This yields that $\varsigma^*=\Omega\varsigma^*$. The uniqueness of the fixed point follows easily. It completes the proof.

Example 9. Consider $Y=\{0,1,2\}$. Take the controlled metric $\mathscr{C}$ defined as

$$\mathscr{C}(0,1)=\frac{1}{2},\mathscr{C}(0,2)=\frac{11}{20},\mathscr{C}(1,2)=\frac{3}{20}, \tag{19}$$

where $p:Y\times Y\longrightarrow[1,\infty)$ is symmetric such that

$$p(0,0)=p(1,1)=p(2,2)=p(1,2)=1,p(0,2)=2,p(0,1)=\frac{3}{2}. \tag{20}$$

Given $\Omega:Y\longrightarrow Y$ as

$$\Omega 0=2\text{ and }\Omega 1=\Omega 2=1. \tag{21}$$

Consider $\alpha=1/11$ and $\beta=\gamma=3/11$. Take $\varsigma_0=0$, then $\varsigma_1=2$ and $\varsigma_n=1$ for all $n\ge 2$. Clearly, (4) is satisfied. On the other hand, note that (3) holds for all $\varsigma,\omega\in Y$. All other hypotheses of Theorem 8 are verified, and so Ω has a unique fixed point, which is $u=1$.

Example 10. Let $Y=[0,1]$. Consider the controlled metric type $\mathscr{C}$ defined as

$$\mathscr{C}(\varsigma,\omega)=|\varsigma-\omega|^2, \tag{22}$$

where $p(\varsigma,\omega)=\varsigma+\omega+1$ for $\varsigma,\omega\in Y$. Take $\Omega\varsigma=\varsigma^2/4$. Consider $\alpha=1/4$ and $\beta=\gamma=1/3$. Take $\varsigma_0=0$; so, (4) is satisfied.

Also, (3) holds. All conditions in Theorem 8 are fulfilled, and so, there is a unique fixed point, which is $u = 0$.

Corollary 11 (see. [9]). *Let $(Y, \mathscr{C})$ be a complete controlled metric space. Let $\Omega : Y \longrightarrow Y$ be s that there are $\alpha \in (0, 1)$ and*

$$\mathscr{C}(\Omega\varsigma, \Omega\omega) \leq \alpha\mathscr{C}(\varsigma, \omega) \quad (23)$$

for all $\varsigma, \omega \in Y$. For $\varsigma_0 \in Y$, take $\varsigma_n = \Omega^n\varsigma_0$. Assume that $\sup_{m\geq 1}\lim_{i\to\infty} p(\varsigma_{i+1}, \varsigma_{i+2})p(\varsigma_{i+1}, \varsigma_m)/p(\varsigma_i, \varsigma_{i+1}) < 1/\alpha$.(36)

Suppose that $\lim_{n\to\infty}p(\varsigma_n, \varsigma)$ and $\lim_{n\to\infty}p(\varsigma, \varsigma_n)$ exist, are finite, and $\gamma\lim_{n\to\infty}p(\varsigma_n, \varsigma) < 1$ for every $\varsigma \in Y$, then Ω possesses a unique fixed point.

Proof. Taking $\beta = \gamma = 0$ in Theorem 8.

3. Results on (α, F)-Contractions

In 2012, Samet et al. [25] initiated the notion of α-admissible mappings and proved some related fixed point results in the context of complete metric spaces.

Definition 12 ([25]). Let Y be a nonempty set, and $\alpha : Y \times Y \longrightarrow [0,\infty)$ be a given function. A self-mapping Ω on Y is called α -admissible if

$$\varsigma, \omega \in Y, \alpha(\varsigma, \omega) \geq 1 \Rightarrow \alpha(\Omega\varsigma, \Omega\omega) \geq 1. \quad (24)$$

Definition 13. Let $(Y, \mathscr{C})$ be a controlled metric space. A mapping $\Omega : Y \longrightarrow Y$ is said to be an *(α, F)*-contraction if there are some $\alpha : Y \times Y \longrightarrow \mathbb{R}^+$,$F \in \sqsubseteq$, and $\tau > 0$ so that

$$\tau + \alpha(\varsigma, \omega)F((\mathscr{C}(\Omega\varsigma, \Omega\omega)) \leq F(\mathscr{C}(\varsigma, \omega)) \quad (25)$$

for all $\varsigma, \omega \in Y$ with $\mathscr{C}(\Omega\varsigma, \Omega\omega) > 0$.

Theorem 14. *Let $(Y, \mathscr{C})$ be a complete controlled metric space. Let $\Omega : Y \longrightarrow Y$ be an (α, F)-contraction so that*

(i) Ω is α-admissible

(ii) There is $\varsigma_0 \in Y$ so that $\alpha(\varsigma_0, \Omega\varsigma_0) \geq 1$

(iii) Ω is continuous

(iv) For $\varsigma_0 \in Y$, define the Picard sequence $\{\varsigma_n = \Omega^n\varsigma_0\}$ such that

$$\sup_{m\geq 1}\lim_{i\to\infty} \frac{p(\varsigma_{i+1}, \varsigma_{i+2})p(\varsigma_{i+1}, \varsigma_m)}{p(\varsigma_i, \varsigma_{i+1})} < 1. \quad (26)$$

Assume that $\lim_{n\to\infty}p(\varsigma_n, \varsigma)$ and $\lim_{n\to\infty}p(\varsigma, \varsigma_n)$ exist and are finite, for every $\varsigma \in Y$, then Ω possesses a unique fixed point.

Proof. Let $\varsigma_0 \in Y$ be such that $\alpha(\varsigma_0, \Omega\varsigma_0) \geq 1$. We define a sequence $\{\varsigma_n\}$ in Y by $\varsigma_{n+1} = \Omega\varsigma_n$ for all $n \in \mathbb{N}$. Clearly, if there is n_0 so that $\varsigma_{n_0+1} = \varsigma_{n_0}$, then the proof is finished. So, assume that ς_{n+1}=ς_n for each $n \in \mathbb{N}$, by using (i) and (ii), it is obvious that

$$\alpha(\varsigma_n, \varsigma_{n+1}) \geq 1 \quad (27)$$

for all $n \in \mathbb{N}$. By (25), we have

$$\tau + F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) = \tau + F(\mathscr{C}(\Omega\varsigma_{n-1}, \Omega\varsigma_n)) \leq \tau + \alpha(\varsigma_n, \varsigma_{n+1})F(\mathscr{C}(\Omega\varsigma_{n-1}, \Omega\varsigma_n)). \quad (28)$$

Since Ω is an (α, F)-contraction, we can write

$$\tau + F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) \leq \tau + \alpha(\varsigma_n, \varsigma_{n+1})F(\mathscr{C}(\Omega\varsigma_{n-1}, \Omega\varsigma_n)) \leq F(\mathscr{C}(\varsigma_{n-1}, \varsigma_n)). \quad (29)$$

Thus, from (29), we get

$$F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) \leq F(\mathscr{C}(\varsigma_{n-1}, \varsigma_n)) - \tau \leq F(\mathscr{C}(\varsigma_{n-2}, \varsigma_{n-1})) - 2\tau \leq F(\mathscr{C}(\varsigma_{n-3}, \varsigma_{n-2})) - 3\tau \leq \cdots \leq F(\mathscr{C}(\varsigma_0, \varsigma_1)).-n\tau \quad (30)$$

Thus, by (29), we have

$$F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) \leq \leq F(\mathscr{C}(\varsigma_0, \varsigma_1)) - n\tau. \quad (31)$$

Letting $n \longrightarrow \infty$ in (31), we get

$$\lim_{n\to\infty} F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) = -\infty. \quad (32)$$

By (F_2), we get

$$\lim_{n\to\infty} \mathscr{C}(\varsigma_n, \varsigma_{n+1}) = 0. \quad (33)$$

Now, by (F_3), there is $h \in (0, 1)$ so that

$$\lim_{n\to\infty} [\mathscr{C}(\varsigma_n, \varsigma_{n+1})]^h F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) = 0. \quad (34)$$

From (25), we have

$$[\mathscr{C}(\varsigma_n, \varsigma_{n+1})]^h F(\mathscr{C}(\varsigma_n, \varsigma_{n+1})) - [\mathscr{C}(\varsigma_n, \varsigma_{n+1})]^h F(\mathscr{C}(\varsigma_0, \varsigma_{n+1})) \leq -n\tau[\mathscr{C}(\varsigma_n, \varsigma_{n+1})]^h \leq 0. \quad (35)$$

On taking limit as $n \longrightarrow \infty$, we obtain

$$\lim_{n\to\infty} n[\mathscr{C}(\varsigma_n, \varsigma_{n+1})]^h = 0. \quad (36)$$

Hence, $\lim_{n\to\infty} n^{1/h}\mathscr{C}(\varsigma_n, \varsigma_{n+1}) = 0$, and there exists $n_1 \in \mathbb{N}$ such that $n^{1/h}\mathscr{C}(\varsigma_n, \varsigma_{n+1}) \leq 1$ for all $n \geq n_1$. So, we have

$$\mathscr{C}(\varsigma_n, \varsigma_{n+1}) \leq \frac{1}{n^{1/h}} \quad (37)$$

for all $n \geq n_1$. Consider the triangle inequality for $q \geq 1$ to have

$$\begin{aligned}\mathscr{C}(\varsigma_n,\varsigma_{n+q}) &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + p(\varsigma_{n+1},\varsigma_{n+q})\mathscr{C}(\varsigma_{n+1},\varsigma_{n+q})\\ &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + p(\varsigma_{n+1},\varsigma_{n+q})p(\varsigma_{n+1},\varsigma_{n+2})\\ &\quad\cdot \mathscr{C}(\varsigma_{n+1},\varsigma_{n+2}) + p(\varsigma_{n+1},\varsigma_{n+q})p(\varsigma_{n+2},\varsigma_{n+q})\mathscr{C}(\varsigma_{n+2},\varsigma_{n+q})\\ &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + p(\varsigma_{n+1},\varsigma_{n+q})p(\varsigma_{n+1},\varsigma_{n+2})\\ &\quad\cdot \mathscr{C}(\varsigma_{n+1},\varsigma_{n+2}) + p(\varsigma_{n+1},\varsigma_{n+q})p(\varsigma_{n+2},\varsigma_{n+q})\\ &\quad\cdot p(\varsigma_{n+2},\varsigma_{n+3})\mathscr{C}(\varsigma_{n+2},\varsigma_{n+3}) + p(\varsigma_{n+1},\varsigma_{n+q})\\ &\quad\cdot p(\varsigma_{n+2},\varsigma_{n+q})p(\varsigma_{n+3},\varsigma_{n+q})\mathscr{C}(\varsigma_{n+3},\varsigma_{n+q}) \le \cdots\\ &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + \sum_{i=n+1}^{n+q-2}\left(\prod_{j=n+1}^{i} p(\varsigma_j,\varsigma_{n+q})\right)\\ &\quad\cdot p(\varsigma_i,\varsigma_{i+1})\mathscr{C}(\varsigma_i,\varsigma_{i+1}) + \prod_{i=n+1}^{n+q-1} p(\varsigma_i,\varsigma_{n+q})\mathscr{C}(\varsigma_{n+q-1},\varsigma_{n+q}).\end{aligned} \tag{38}$$

It implies that

$$\begin{aligned}\mathscr{C}(\varsigma_n,\varsigma_{n+q}) &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + \sum_{i=n+1}^{n+q-2}\left(\prod_{j=n+1}^{i} p(\varsigma_j,\varsigma_{n+q})\right)\\ &\quad\cdot p(\varsigma_i,\varsigma_{i+1})\mathscr{C}(\varsigma_i,\varsigma_{i+1}) + \left(\sum_{i=n+1}^{n+q-1} p(\varsigma_i,\varsigma_{n+q})\right)\\ &\quad\cdot p(\varsigma_{n+q-1},\varsigma_{n+q})\mathscr{C}(\varsigma_{n+q-1},\varsigma_{n+q}) = p(\varsigma_n,\varsigma_{n+1})\\ &\quad\cdot \mathscr{C}(\varsigma_n,\varsigma_{n+1}) + \sum_{i=n+1}^{n+q-1}\left(\sum_{j=n+1}^{i} p(\varsigma_j,\varsigma_{n+q})\right)\\ &\quad\cdot p(\varsigma_i,\varsigma_{i+1})\mathscr{C}(\varsigma_i,\varsigma_{i+1}) \le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1})\\ &\quad+ \sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)p(\varsigma_i,\varsigma_{i+1})\mathscr{C}(\varsigma_i,\varsigma_{i+1})\\ &\le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + \sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)\\ &\quad\cdot p(\varsigma_i,\varsigma_{i+1})\frac{1}{i^{1/k}}.\end{aligned} \tag{39}$$

Now, consider

$$\begin{aligned}&\sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)p(\varsigma_i,\varsigma_{i+1})\frac{1}{n^{1/k}}\\ &\quad= \sum_{i=n+1}^{n+q-1}\frac{1}{i^{1/k}}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)p(\varsigma_i,\varsigma_{i+1})\\ &\quad\le \sum_{i=n+1}^{\infty}\frac{1}{i^{1/k}}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)p(\varsigma_i,\varsigma_{i+1}) = \sum_{i=n+1}^{\infty} U_i V_i,\end{aligned} \tag{40}$$

where

$$\begin{aligned}U_i &= \frac{1}{i^{1/k}},\\ V_i &= p(\varsigma_i,\varsigma_{i+1})\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q}).\end{aligned} \tag{41}$$

Since $1/k > 0$, $\sum_{i=n+1}^{\infty}(1/i^{1/k})$ converges and also $\{V_i\}_i$ is increasing and bounded above, thus, $\lim_{i\to\infty}\{V_i\}$, which is nonzero, exists. Hence, $\{\sum_{i=n+1}^{\infty} U_i V_i\}_n$ converges. Let us consider the partial sum

$$S_q = \sum_{i=0}^{q}\left(\prod_{j=0}^{i} p(\varsigma_j,\varsigma_{n+q})\right)p(\varsigma_i,\varsigma_{i+1})\frac{1}{i^{1/k}}. \tag{42}$$

Now, from (39), we have

$$\mathscr{C}(\varsigma_n,\varsigma_{n+q}) \le p(\varsigma_n,\varsigma_{n+1})\mathscr{C}(\varsigma_n,\varsigma_{n+1}) + (S_{n+q-1} - S_n). \tag{43}$$

By the ratio test and using the condition (26), we guarantee the existence of $\lim_{n\to\infty} S_n$. Hence, the real sequence $\{S_n\}$ is Cauchy. Now, taking the limit $n \longrightarrow +\infty$ in (43), we get

$$\lim_{n\to\infty}\mathscr{C}(\varsigma_n,\varsigma_{n+q}) = 0\,; \tag{44}$$

that is, $\{\varsigma_n\}$ is a Cauchy sequence in $(Y,\mathscr{C})$, which is complete, so $\{\varsigma_n\}$ converges to some $u \in Y$. We claim that $\Omega u = u$. Since $\varsigma_n \longrightarrow u$ as $n \longrightarrow \infty$ and Ω is continuous, we have $\Omega\varsigma_n \longrightarrow \Omega u$ as $n \longrightarrow \infty$. Thus, we have

$$\mathscr{C}(u,\Omega u) = \lim_{n\to\infty}\mathscr{C}(\varsigma_{n+1},\Omega u) = \lim_{n\to\infty}\mathscr{C}(\Omega\varsigma_n,\Omega u) = 0, \tag{45}$$

and hence, $u = \Omega u$. Thus, u is a fixed point of Ω. Its uniqueness is obvious.

Corollary 15. *Let $(Y,\mathscr{C})$ be a complete controlled metric space, and let $\Omega : Y \longrightarrow Y$ be continuous so that*

$$\tau + F((\mathscr{C}(\Omega\varsigma,\Omega\omega)) \le F(\mathscr{C}(\varsigma,\omega)) \tag{46}$$

for all $\varsigma,\omega \in Y$. For $\varsigma_0 \in Y$, take $\varsigma_n = \Omega^n\varsigma_0$. Suppose that

$$\sup_{m\ge 1}\lim_{i\to\infty}\frac{p(\varsigma_{i+1},\varsigma_{i+2})p(\varsigma_{i+1},\varsigma_m)}{p(\varsigma_i,\varsigma_{i+1})} < 1. \tag{47}$$

Assume that $\lim_{n\to\infty}p(\varsigma_n,\varsigma)$ and $\lim_{n\to\infty}p(\varsigma,\varsigma_n)$ exist and are finite, for every $\varsigma \in Y$, then Ω possesses a unique fixed point.

Proof. Taking $\alpha : Y \times Y \longrightarrow [0,\infty)$ by $\alpha(\varsigma,\omega) = 1$, for all $\varsigma,\omega \in Y$ in Theorem 14.

Corollary 16. *Let $(Y,\mathscr{C})$ be a complete extended b -metric space and $\Omega : Y \longrightarrow Y$ be a continuous, α -admissible and (α, F)-contraction so that there is $\varsigma_0 \in Y$ in order that $\alpha(\varsigma_0,\Omega\varsigma_0) \ge 1$. Suppose that*

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{p(\varsigma_{i+1},\varsigma_{i+2})p(\varsigma_{i+1},\varsigma_m)}{p(\varsigma_i,\varsigma_{i+1})}<1. \tag{48}$$

If in addition, $\lim_{n\to\infty}p(\varsigma_n,\varsigma)$ *and* $\lim_{n\to\infty}p(\varsigma,\varsigma_n)$ *exist and are finite, for every* $\varsigma\in Y$*, then* Ω *has a unique fixed point.*

Corollary 17. *Let* $(Y,\mathscr{C})$ *be a complete b -metric space and* $\Omega:Y\longrightarrow Y$ *be a continuous,* α *-admissible and (* α*,* F *)-contraction so that there is* $\varsigma_0\in Y$ *with* $\alpha(\varsigma_0,\Omega\varsigma_0)\geq 1$*. Then,* Ω *has a unique fixed point.*

Proof. Taking $p:Y\times Y\longrightarrow[1,\infty)$ by $p(\varsigma,\omega)=p(\omega,\tau)$, for all $\varsigma,\omega,\tau\in Y$ in Theorem 14.

Corollary 18. *Let* $(Y,\mathscr{C})$ *be a complete metric space and* $\Omega:Y\longrightarrow Y$ *be a continuous,* α *-admissible and (* α*,* F *)-contraction so that there is* $\varsigma_0\in Y$ *in order that* $\alpha(\varsigma_0,\Omega\varsigma_0)\geq 1$*. Then,* Ω *has a unique fixed point.*

Proof. Taking $p:Y\times Y\longrightarrow[1,\infty)$ by $p(\varsigma,\omega)=1$, for all $\varsigma,\omega\in Y$ in Theorem 14.

Corollary 19 (see. [11]). *Let* $(Y,\mathscr{C})$ *be a complete metric space and* $\Omega:Y\longrightarrow Y$ *be an* F *-contraction. Then,* Ω *has a unique fixed point.*

Proof. Taking $\alpha:Y\times Y\longrightarrow[0,\infty)$ by $\alpha(\varsigma,\omega)=1$, for all $\varsigma,\omega\in Y$ in Corollary 18.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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