ILLINOIS JOURNAL OF MATHEMATICS
Volume 22, Number 4, December 1978

# ON FLAT FIBRATIONS BY THE AFFINE LINE

BY

T. KAMBAYASHI[1] AND M. MIYANISHI

A recent joint work [1] of Dolgačëv and Veĭsfeĭler studies in the main the geometric structures of unipotent group schemes over an integral ring. As a natural generalization of their own results the following *conjecture* is set forth (see [1, 3.8.3ff]).

Let $\phi\colon X \to S$ be a flat affine morphism of finite type; assume that $S$ is locally noetherian, normal and integral, and that the fibre $\phi^{-1}(P)$ of $\phi$ above each point $P$ of $S$ is isomorphic to the affine $n$-space $\mathbf{A}^n$ over the residue field $\kappa(P)$ of $P$. Then, $X$ is an $\mathbf{A}^n$-bundle over $S$ relative to the Zariski topology.

In the paper cited above, the authors obtain various results in the direction of this conjecture while working under the assumption of an $S$-group scheme structure on $X$.

In the present paper we propose to settle the conjecture affirmatively in the special case where $n = 1$. (It is understood that V. I. Danilov possesses unpublished results to the same effect; cf. [1, 3.8.5].) What we actually prove are the following two theorems.

THEOREM 1. *Let $\phi\colon X \to S$ be an affine, faithfully flat morphism of finite type. Assume that $S$ is locally noetherian, locally factorial and integral scheme, and that the generic fibre of $\phi$ is $\mathbf{A}^1$ and all other fibres are geometrically integral. Then, $X$ is an $\mathbf{A}^1$-bundle over $S$.*

THEOREM 2. *Let $k$ be an algebraically closed field, let $S$ be a regular, integral $k$-scheme of finite type, and let $\phi\colon X \to S$ be an affine, faithfully flat morphism of finite type. Assume that each fibre of $\phi$ is geometrically integral and the general fibres of $\phi$ are isomorphic to $\mathbf{A}^1$ over $k$. Then, there exist a regular, integral $k$-scheme $S'$ of finite type and a faithfully flat, finite, radical morphism $S' \to S$ such that $X \times_S S'$ is an $\mathbf{A}^1$-bundle over $S'$. If in particular the characteristic of $k$ is zero, $X$ is an $\mathbf{A}^1$-bundle over $S$.*

A variation of the conjecture above, wherein $S$ is a curve and $\mathbf{A}^n$ is replaced throughout by the projective $n$-space $\mathbf{P}^n$, is in fact a proven theorem (see Maruyama [9, Theorem 0.1]). It seems that the exact relationship between this variation and the conjecture above stated remains to be clarified.

Received June 8, 1977.
[1] Supported in part by a National Science Foundation research grant.

## 1. Proof of Theorem 1

1.1. Let $S$ be a locally noetherian, integral scheme, and let $\phi\colon X \to S$ be an affine, flat morphism of finite type. The fibres of $\phi$ above the closed points of $S$ will be referred to as *closed fibres*, while the fibre above the generic point of $S$ will be called *the generic fibre*. By *the general fibres of* $\phi$ we shall mean *all* fibres above the closed points belonging to an *unspecified* nonempty open set of $S$. The morphism $\phi\colon X \to S$, or more conventionally $X$ by itself, is called an *affine ruled variety over* $S$ if for every point $P$ on $S$ (including the generic point) the fibre $\phi^{-1}(P)$ above $P$ is isomorphic to the affine line $\mathbf{A}^1_{\kappa(P)}$ over the residue field $\kappa(P)$ of $P$. The morphism $\phi$, or again simply $X$, is said to be an $\mathbf{A}^1$-*bundle over* $S$ if there exists an open covering $\{U_i \to S\}$ relative to the Zariski topology on $S$ such that $X \times_S U_i$ is isomorphic to the affine line $\mathbf{A}^1_{U_i} := \mathbf{A}^1 \times_{\mathbf{Z}} U_i$ over $U_i$ for all $i$. A scheme $S$ is said to be *locally factorial* if for every point $P$ on $S$ the local ring $\mathcal{O}_{P,S}$ is a factorial ring (= a unique factorization domain). A discrete valuation ring of rank 1 will be called a *principal* valuation ring.

The proof of Theorem 1 will be given below in several reduction steps.

1.2. We shall begin with the following elementary result, which is a special case of a theorem of Nagata [11].

LEMMA. *Let* $\mathfrak{o}$ *be a principal valuation ring and let* $A$ *be a flat* $\mathfrak{o}$*-algebra of finite type. Let* $K$ *be the quotient field of* $\mathfrak{o}$, $t$ *a uniformisant of* $\mathfrak{o}$ *and* $k$ *the residue field of* $\mathfrak{o}$*; and let* $A_K$ *and* $A_k$ *denote respectively* $K \otimes_{\mathfrak{o}} A$ *and* $k \otimes_{\mathfrak{o}} A$. *Assume that* $A_K$ *and* $A_k$ *are integral domains. Then:*

(i) *If* $A_K$ *is a normal ring, so is* $A$.
(ii) *If* $A_K$ *is factorial, so is* $A$.

*Proof.* We shall prove only (ii), as the proof of (i) is a routine exercise. By flatness there is a natural inclusion $\mathfrak{o} \subset A$, and $A$ is in turn contained in $A_K$ and is noetherian. Since $A_k$ is integral, $tA$ is a prime ideal in $A$ and $\bigcap_{\nu \geq 0} t^\nu A = (0)$. Let $\mathfrak{p}$ be an arbitrary prime of height 1 in $A$. If $t \in \mathfrak{p}$ then clearly $tA = \mathfrak{p}$. In case $t \notin \mathfrak{p}$, the ideal $\mathfrak{p}A_K$ is prime of height 1 in the factorial domain $A_K = A[t^{-1}]$, whence $\mathfrak{p}A_K = fA_K$, where we may and shall take $f \in A - tA$. Let $b \in \mathfrak{p}$ be arbitrary, and write $b = ft^m a$ with integer $m$ and $a \in A - tA$. If $m < 0$, then $fa = bt^{-m} \in tA$, an absurdity. Consequently, $m \geq 0$ and $\mathfrak{p} \subseteq fA$. It follows that $\mathfrak{p} = fA$ because $f \in \mathfrak{p}$.

1.3. LEMMA. *Let* $(\mathfrak{o}, t\mathfrak{o})$ *be a principal valuation ring with residue field* $k$ *and quotient field* $K$. *Let* $A$ *be a flat* $\mathfrak{o}$*-algebra of finite type. Assume that* $A_K := K \otimes_{\mathfrak{o}} A$ *is* $K$*-isomorphic to a one-variable polynomial ring* $K[x]$ *and that* $A_k := k \otimes_{\mathfrak{o}} A$ *is a geometrically integral domain over* $k$. *Then,* $A$ *is* $\mathfrak{o}$*-isomorphic to a one-variable polynomial ring.*

*Proof.* Because $A$ is factorial by Lemma 1.2 (or, rather, because of the simple fact that $\bigcap_{\nu \geq 0} t^\nu A = (0)$), we may assume that $x \in A$ and $x$ is prime to the uniformisant $t$ of $\mathfrak{o}$. We may write $A = \mathfrak{o}[x, y_1, \ldots, y_m]$. Since $A \subset A_K = K[x]$, there exist integers $\alpha(i) \geq 0$ such that

$$t^{\alpha(i)} y_i = \phi_i(x) := \lambda_{i0} + \lambda_{i1} x + \cdots + \lambda_{ir(i)} x^{r(i)} \tag{1}$$

with $\lambda_{ij} \in \mathfrak{o}$ for $1 \leq i \leq m$ and $0 \leq j \leq r(i)$, where we may assume with each $i$ that if $\alpha(i) > 0$ then not all of $\lambda_{i0}, \lambda_{i1}, \ldots, \lambda_{ir(i)}$ are divisible by $t$. Let us put $\alpha_x := \text{Max}\ \{\alpha(1), \ldots, \alpha(m)\}$. Consider the following assertion:

*P*($n$). If $x \in A$ is found as above with $\alpha_x = n$, then there is some $x_1 \in A$ such that $A = \mathfrak{o}[x_1]$.

We shall prove the assertion $P(n)$ by induction on $n$. $P(0)$ is obviously true. We prove $P(n)$ assuming $P(r)$ to be true for all $r < n$. By applying the canonical (reduction modulo $t$) homomorphism $\rho \colon A \to A/tA = A_k$ to the both sides of (1) for each $i$ with $\alpha(i) = \alpha_x$, we get

$$\rho(\lambda_{i0}) + \rho(\lambda_{i1})\rho(x) + \cdots + \rho(\lambda_{ir(i)})\rho(x)^{r(i)} = 0 \tag{2}$$

with at least one of the coefficients $\rho(\lambda_{ij}) \neq 0$. Since $A_k$ is an integral domain, the equation (2) is a nontrivial algebraic equation of $\rho(x)$ over $k$. Since $A_k$ is geometrically integral, the field $k$ is algebraically closed in the quotient field of $A_k$, whence $\rho(x) \in k$. Let $\mu \in \mathfrak{o}$ be such that $\rho(\mu) = \rho(x)$, and write $x - \mu = t^\beta x'$ with a positive integer $\beta$ and $x' \in A - tA$. Then, noting $\phi_i(\mu) \in t\mathfrak{o}$ and by substituting $\mu + t^\beta x'$ for $x$ in (1), we obtain, after cancellation of $t$,

$$t^{\alpha'(i)} y_i \in \mathfrak{o}[x'] \quad \text{for } 1 \leq i \leq m \text{ and } K[x] = K[x']$$

where $\alpha_{x'} = \text{Max}\ \{\alpha'(1), \ldots, \alpha'(m)\} < n = \alpha_x$. Since $P(\alpha_{x'})$ is assumed to be true, the conclusion of $P(n)$ holds. Q.E.D.

1.4. It is easy to see, as shown in Paragraph 1.5 below, that Theorem 1 follows from Lemma 1.3 in the special case where $\dim S = 1$. In order to prove the theorem over $S$ with $\dim S \geq 2$ we need the following:

LEMMA. *Let* $(A, \mathfrak{m})$ *be a factorial local ring of dimension* $\geq 2$ *with residue field* $k$. *Let* $R$ *be a flat* $A$*-algebra of finite type. Assume that* $R_\mathfrak{p} := A_\mathfrak{p} \otimes_A R$ *is* $A_\mathfrak{p}$*-isomorphic to a one-variable polynomial ring* $A_\mathfrak{p}[t_\mathfrak{p}]$ *for every nonmaximal prime ideal* $\mathfrak{p}$ *of* $A$ *and that* $\bar{R} := R/\mathfrak{m}R$ *is geometrically regular over* $k$. *Then,* $R$ *is* $A$*-isomorphic to a one-variable polynomial ring* $A[t]$.

*Proof.* The proof consists of four steps.

(I) Let $X := \text{Spec}\ R$, $S := \text{Spec}\ A$ and let $\phi \colon X \to S$ be the flat morphism corresponding to the canonical injection $A \subset R$. $\phi$ is in fact faithfully flat, and each fibre of $\phi$ is geometrically regular. Therefore, $\phi$ is smooth. Since $S$ is normal, this implies that $X$ is normal [3, IV (6.5.4)]. Thus, $R$ is a normal domain.

(II) Let $U := S - \{\mathfrak{m}\}$. Since $R$ is finitely generated over $A$ and $R_{\mathfrak{p}} = A_{\mathfrak{p}}[t_{\mathfrak{p}}]$ for each $\mathfrak{p} \in U$, there is $f_{\mathfrak{p}} \in A - \mathfrak{p}$ such that $R[f_{\mathfrak{p}}^{-1}] = A[f_{\mathfrak{p}}^{-1}][t_{\mathfrak{p}}]$, whence we know the existence of an open covering $\mathscr{V} = \{V_i\}_{i \in I}$ of $U$ such that

$$V_i := \operatorname{Spec}(A[f_i^{-1}])$$

with $f_i \in A$ and $R[f_i^{-1}] = A[f_i^{-1}][t_i]$ for each $i \in I$. This shows that $X_U := \phi^{-1}(U) = X \times_S U$ can be viewed as an $\mathbf{A}^1$-bundle over $U$. Set

$$A_i := A[f_i^{-1}],\ A_{ij} := A[f_i^{-1}, f_j^{-1}] \quad \text{and} \quad A_{ijl} := A[f_i^{-1}, f_j^{-1}, f_l^{-1}]$$

for $i, j, l \in I$. Since $A_{ij}[t_i] = R[f_i^{-1}, f_j^{-1}] = A_{ij}[t_j]$ and $A_{ij}$ is an integral domain, we get $t_j = \alpha_{ji} t_i + \beta_{ji}$ with units $\alpha_{ji}$ in $A_{ij}$ and $\beta_{ji} \in A_{ij}$ for each pair $i, j$ of elements of the index set $I$, where, furthermore, the $\alpha$'s and the $\beta$'s are subject to the relations in $A_{ijl}$ that read as follows:

$$\alpha_{li} = \alpha_{lj}\alpha_{ji} \quad \text{and} \quad \beta_{li} = \alpha_{lj}\beta_{ji} + \beta_{lj}.$$

Consequently, $\{\alpha_{ij}\}_{(i,j) \in I \times I}$ gives rise to an invertible sheaf $\mathscr{L}$ which one views as an element of $H^1(U, \mathcal{O}_U^*)$. However, $H^1(U, \mathcal{O}_U^*) = (0)$ because $(A, \mathfrak{m})$ is a factorial domain [5, Exp. XI, 3.5 and 3.10]. Thus, by replacing $\mathscr{V}$ by a finer open covering of $U$ if necessary, we may assume that

(3) $\quad t_j = t_i + \beta_{ji}$ with $\beta_{ji} \in A_{ji}$ such that $\beta_{li} = \beta_{ji} + \beta_{lj}$ for $i, j, l \in I$.

Hence, $\{\beta_{ij}\}_{(i,j) \in I \times I}$ defines an element $\xi \in H^1(U, \mathcal{O}_U)$.

(III) Consider $X_U = \phi^{-1}(U) = X \times_S U$ and let $Y := X - X_U$. By the local cohomology theory we have the commutative diagram

$$\begin{array}{ccccc} H^1(X_U, \mathcal{O}_X) & \xrightarrow{\sim} & H^2_Y(X, \mathcal{O}_X) & \xrightarrow{\sim} & \varinjlim_n \operatorname{Ext}^2_R(R/\mathfrak{m}^n R, R) \\ \uparrow \theta_U & & \uparrow \theta_{\mathfrak{m}} & & \uparrow \theta_A \\ H^1(U, \mathcal{O}_S) & \xrightarrow{\sim} & H^2_{\{\mathfrak{m}\}}(S, \mathcal{O}_S) & \xrightarrow{\sim} & \varinjlim_n \operatorname{Ext}^2_A(A/\mathfrak{m}^n, A) \end{array}$$

where the terms in the upper and lower rows are respectively $R$-modules and $A$-modules, and $\theta_U$, $\theta_{\mathfrak{m}}$, and $\theta_A$ are homomorphisms induced by the canonical injection $\mathcal{O}_S \hookrightarrow \phi_* \mathcal{O}_X$. (For the definitions and relevant results in local cohomology theory, consult [5] or [6].) Since $R$ is $A$-flat and $\varinjlim_n$ commutes with $R \otimes_A ?$, we have

$$\varinjlim_n \operatorname{Ext}^2_R(R/\mathfrak{m}^n R, R) \cong R \otimes_A \varinjlim_n \operatorname{Ext}^2_A(A/\mathfrak{m}^n, A)$$

and $\theta_A$ is identified with the homomorphism $u \mapsto 1 \otimes u$ for $u$ belonging to $\varinjlim_n \operatorname{Ext}^2_A(A/\mathfrak{m}^n, A)$. Since $R$ is $A$-flat, $\theta_A$ is then injective. The commutative diagram above shows, hence, that $\theta_U$ is injective. On the other hand, $X_U$ has an open covering $\phi^{-1}(\mathscr{V}) = \{\phi^{-1}(V_i); i \in I\}$, and the element $\theta_U(\xi) \in H^1(X_U, \mathcal{O}_X)$ is represented by a Čech 1-cocycle $\{\beta_{ij}\}$ with respect to $\phi^{-1}(\mathscr{V})$. The relation (3) implies that $\{\beta_{ij}\}$ is in fact a 1-coboundary because

$$t_i \in \Gamma(\phi^{-1}(V_i), \mathcal{O}_X) = A_i[t_i].$$

Thus, $\theta_U(\xi) = 0$, and we find $\xi = 0$ because $\theta_U$ is injective. It follows that $X_U$ has a section and is, in fact, a trivial $\mathbf{A}^1$-bundle $\mathbf{A}^1_U$.

(IV) Replacing $\mathscr{V}$ by a finer open covering of $U$ if necessary, we may assume that $\beta_{ji} = \gamma_j - \gamma_i$ with $\gamma_i \in A_i$ for all $i, j \in I$. Then, $t_i - \gamma_i = t_j - \gamma_j$ for all $i$ and all $j$, so if we put $t := t_i - \gamma_i$ then $t \in \Gamma(X_U, \mathcal{O}_X)$. On the other hand, since codim $(Y, X) \geq 2$ and $R$ is normal, $\mathcal{O}_X$ is $Y$-closed [3, IV (5.10.5)]. Hence, $t \in \Gamma(X_U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = R$. Now, look at the $A$-subalgebra $A[t]$ of $R$, and let $Z := \operatorname{Spec}(A[t])$. Then, $\phi$ decomposes as

$$X \xrightarrow{\phi_1} Z \xrightarrow{\phi_2} S,$$

where $\phi_1$ and $\phi_2$ are the morphisms corresponding to the injections $A \hookrightarrow A[t] \hookrightarrow R$. By step (III), $R_\mathfrak{p} = A_\mathfrak{p}[t]$ for each $\mathfrak{p} \in U$. This implies that $\phi_1|_U\colon X_U \to \phi_2^{-1}(U) = Z \times_S U$ is a $U$-isomorphism. Notice that $\mathcal{O}_Z$ is $(Z - \phi_2^{-1}(U))$-closed because codim $(Z - \phi_2^{-1}(U), Z) \geq 2$ and $Z$ is normal. Then we have

$$A[t] = \Gamma(Z, \mathcal{O}_Z) = \Gamma(\phi_2^{-1}(U), \mathcal{O}_Z) \xrightarrow{\sim} \Gamma(X_U, \mathcal{O}_X) = R,$$

an isomorphism given by $(\phi_1|_U)^*$. Therefore, $R = A[t]$. Q.E.D.

1.5. *Proof of Theorem 1.* Since $\phi$ is affine, it suffices clearly to prove the theorem under the hypothesis that $X$ and $S$ are affine schemes. The proof consists of two steps.

(I) Let $A := \Gamma(S, \mathcal{O}_S)$ and $R := \Gamma(X, \mathcal{O}_X)$. The homomorphism $A \to R$ induced by $\phi$ is injective, and makes $R$ a flat $A$-algebra of finite type. For each prime ideal $\mathfrak{p}$ of $A$, let $R_\mathfrak{p} := A_\mathfrak{p} \otimes_A R$. By induction on $n :=$ height $(\mathfrak{p})$ we shall establish the following assertion:

$P(n)$. $R_\mathfrak{p}$ is a one-variable polynomial ring over $A_\mathfrak{p}$ provided $\mathfrak{p}$ is of height $n$.

Indeed, $P(0)$ follows from the assumption of the theorem. As for $P(1)$, $A_\mathfrak{p}$ is a principal valuation ring in that case, so the assertion is supported by Lemma 1.3. We now prove $P(n)$ assuming $P(r)$ to hold for every $r < n$. To simplify notations let us write $R$ and $A$ instead of $R_\mathfrak{p}$ and $A_\mathfrak{p}$, respectively. Now, $A$ is a factorial local ring of dimension $\geq 2$ with maximal ideal $\mathfrak{m}$. By virtue of [3, II (7.1.7)] one can find a principal valuation ring $\mathfrak{o}$ such that the quotient field $K$ of $\mathfrak{o}$ agrees with that of $A$ and that $\mathfrak{o}$ dominates $A$. Then $\mathfrak{o} \otimes_A R$ is a flat $\mathfrak{o}$-algebra of finite type, $K \otimes_\mathfrak{o} (\mathfrak{o} \otimes_A R) = K \otimes_A R$ is a one-variable polynomial ring over $K$, and

$$(\mathfrak{o}/t\mathfrak{o}) \otimes_\mathfrak{o} (\mathfrak{o} \otimes_A R) = (\mathfrak{o}/t\mathfrak{o}) \otimes_{A/\mathfrak{m}} (R/\mathfrak{m}R)$$

is geometrically integral, where $t$ is a uniformisant of $\mathfrak{o}$. By Lemma 1.3, $\mathfrak{o} \otimes_A R$ is then a one-variable polynomial ring over $\mathfrak{o}$. It follows that $(\mathfrak{o}/t\mathfrak{o}) \otimes_{A/\mathfrak{m}} (R/\mathfrak{m}R)$ is geometrically regular and, consequently, $R/\mathfrak{m}R$ is geometrically regular over $A/\mathfrak{m}$. This observation and $P(r)$ for $0 \leq r < n$ together imply that

$A$ and $R$ satisfy all assumptions in Lemma 1.4. Thus, by that lemma, we conclude that $R$ is a one-variable polynomial ring over $A$.

(II) Since $R$ is finitely generated over $A$, step (I) implies that for each prime ideal $\mathfrak{p}$ of $A$ there exists an element $f \in A$ such that $f \notin \mathfrak{p}$ and $R[f^{-1}]$ is a one-variable polynomial ring over $A[f^{-1}]$. Thus, for the Zariski open set $U_f := \operatorname{Spec}(A[f^{-1}]) \subseteq S$, an isomorphism $X \times_S U_f = \mathbf{A}^1 \otimes_{\mathbf{Z}} U_f$ obtains, and $S$ is clearly covered by finitely many such $U_f$'s. This completes the proof of Theorem 1.

## 2. Proof of Theorem 2

2.1. Let $k$ be a field. A $k$-scheme $X$ is called a *form of* $\mathbf{A}^1$ *over* $k$, or simply a *$k$-form of* $\mathbf{A}^1$, if for an algebraic extension field $k'$ of $k$ there exists a $k'$-isomorphism $X \otimes_k k' \xrightarrow{\sim} \mathbf{A}^1_k \otimes_k k' = \mathbf{A}^1_{k'}$. When that is so, there is a purely inseparable extension field $k''$ of $k$ such that $X \otimes_k k''$ is $k''$-isomorphic to $\mathbf{A}^1_{k''}$. It is obvious that, for a $k$-scheme $X$ and an algebraic extension field $k'$ of $k$, $X$ is a $k$-form of $\mathbf{A}^1$ if and only if $X \otimes_k k'$ is a $k'$-form of $\mathbf{A}^1$. A $k$-form of $\mathbf{A}^1$ is evidently an affine smooth $k$-scheme. A $k$-form of $\mathbf{A}^1$ may be characterized as a one-dimensional $k$-smooth scheme of geometric genus zero having exactly one purely inseparable point at infinity. For detailed study on $k$-forms of $\mathbf{A}^1$, see [7, Section 6] and [8].

2.2. A key result to prove Theorem 2 is the following:

LEMMA. *Let $k$ be a field of characteristic $p \geq 0$, let $S$ be a geometrically integral $k$-scheme of finite type, and let $\phi\colon X \to S$ be an affine, flat morphism of finite type. Assume that the general fibres of $\phi$ are forms of $\mathbf{A}^1$ over their respective residue fields at the base scheme $S$. Then, the generic fibre $X_K$ is a $K$-form of $\mathbf{A}^1$, where $K$ denotes the function field of $S$ over $k$. If in particular $p = 0$, $X_K$ is $K$-isomorphic to $\mathbf{A}^1_K$.*

*Proof.* The proof consists of four steps.

(I) Let $\bar{k}$ be an algebraic closure of $k$. Let

$$\bar{S} := S \otimes_k \bar{k}, \quad \bar{X} := X \otimes_k \bar{k} \quad \text{and} \quad \bar{\phi} := \phi \otimes_k \bar{k}.$$

Then $\bar{S}$ is an integral $\bar{k}$-scheme, and the general fibres of $\bar{\phi}$ are $\bar{k}$-isomorphic to $\mathbf{A}^1_{\bar{k}}$. The stated conditions for $\phi$ are clearly present for $\bar{\phi}$, too. Let $\bar{K} := \bar{k} \otimes_k K$. As remarked in 2.1, the generic fibre $X_K$ of $\phi$ is a $K$-form of $\mathbf{A}^1$ if and only if the generic fibre $\bar{X}_{\bar{K}}$ of $\bar{\phi}$ is a $\bar{K}$-form of $\mathbf{A}^1$. These observations show that in proving the lemma at hand we may assume from the outset that $k$ is algebraically closed and that the general fibres are $k$-isomorphic to $\mathbf{A}^1_k$. Furthermore, we may assume with no loss of generality that $S$ is smooth over $k$ because the set of all $k$-smooth points of $S$ is a nonempty open set. We shall assume these additional conditions in the steps that follow.

(II) Let $C$ denote the generic fibre $X_K$ of $\phi$. $C$ is an affine curve over $K$, whose function field $K(C)$ is a regular extension field of $K$ [3, IV (9.7.7), III (9.2.2)]. For each positive integer $n$ we let $K_n := K^{p^{-n}}$. If $p = 0$, $K_n$ is understood to mean $K$ for every $n$. By virtue of [2, Theorem 5, p. 99], there exists a positive integer $N$ such that a complete $K_N$-normal model of $K_N(C) := K_N \otimes_K K(C)$ is smooth over $K_N$. We fix such an $N$ once and for all. Let $S_N$ be the normalization of $S$ in $K_N$. Since $S$ is smooth over $k$ and $k$ is algebraically closed, $S_N$ is smooth over $k$ and the normalization morphism $S_N \to S$ is identified with the $N$th power of the Frobenius morphism of $S_N$.

(III) Let $\tilde{C}_N$ be a complete normal model of $K_N(C)$ over $K_N$. Then, $\tilde{C}_N$ is a smooth projective curve over $K_N$. Thus, $\tilde{C}_N$ is a closed subscheme in the projective space $\mathbf{P}^m_{K_N}$ defined by a finite set of homogeneous equations

$$\{f_\lambda(X_0, \ldots, X_m) = 0;\ \lambda \in \Lambda\}.$$

One can then find a nonempty open set $U$ of $S_N$ such that all the coefficients of all $f_\lambda$'s, as elements of $K_N = k(S_N)$, are defined on $U$. Let $\tilde{X}_N$ be the closed subscheme of $\mathbf{P}^m_k \times_k U$ defined by the same set of homogeneous equations

$$\{f_\lambda(X_0, \ldots, X_m) = 0;\ \lambda \in \Lambda\},$$

and let $\tilde{\phi}_N: \tilde{X}_N \to U$ be the projection onto $U$. The generic fibre of $\tilde{\phi}_N$, which coincides with $\tilde{C}_N$, is geometrically regular. Applying the generic flatness theorem [3, IV (6.9.1)] and the Jacobian criterion of smoothness, we may assume, by shrinking $U$ to a smaller nonempty open set if need be, that $\tilde{\phi}_N$ is smooth over $U$. Now, look at the morphism $\phi_N: X_N := X \times_S U \to U$ obtained from $\phi: X \to S$ by the base change $U \to S$. Since $\tilde{C}_N$ is a completion of the generic fibre $C_N := C \otimes_K K_N$ of $\phi_N$, we have a birational $U$-mapping $\psi_N: X_N \to \tilde{X}_N$ such that $\phi_N = \tilde{\phi}_N \psi_N$. Since $\psi_N$ is everywhere defined on $C_N$, we may assume, by replacing $U$ by a smaller open set if necessary, that $\psi_N: X_N \to \tilde{X}_N$ is an open immersion of $U$-schemes.

(IV) It now suffices to show that $X_K$ is a $K$-form of $\mathbf{A}^1$ under the following additional hypotheses:

(i) There exist a projective smooth morphism $\tilde{\phi}: \tilde{X} \to S$ and an open immersion $\psi: X \to \tilde{X}$ such that $\phi = \tilde{\phi}\psi$.

(ii) Every closed fibre of $\phi$ is $k$-isomorphic to $\mathbf{A}^1_k$.

Then, every closed fibre of $\tilde{\phi}$ is $k$-isomorphic to $\mathbf{P}^1_k$ by virtue of conditions (i) and (ii). Since $\tilde{\phi}$ is faithfully flat and arithmetic genus is invariant under flat deformations [4, Exp. 221, p. 5], [3, III, Section 7], we have the arithmetic genus $p_a(\tilde{X}_K) = 0$ for the generic fibre $\tilde{X}_K$ of $\tilde{\phi}$, which is a smooth projective curve defined over $K$. We shall next show that $\tilde{X}_K - \psi(X_K)$ has only one point and that point is purely inseparable over $K$. Let $\eta$ be a point on $\tilde{X}_K - \psi(X_K)$ and let $T$ be the closure of $\eta$ in $\tilde{X}$. Then, $T \subseteq \tilde{X} - \psi(X)$, the restriction $\tilde{\phi}_T: T \to S$ of $\tilde{\phi}$ onto $T$ is a dominating morphism, and $\deg \tilde{\phi}_T = [K(\eta): K]$. Notice that $\tilde{\phi}_T$ is a generically one-to-one morphism because for each closed point $P$ on $S$,

$$\tilde{\phi}_T^{-1}(P) \subseteq \tilde{\phi}^{-1}(P) - \psi\phi^{-1}(P) = \mathbf{P}_k^1 - \mathbf{A}_k^1 = \{\text{one point}\}.$$

This implies that $\tilde{\phi}_T$ is a birational morphism if $p = 0$ and a radical morphism if $p > 0$. Thus, $K(\eta)$ is purely inseparable over $K$. If $\eta'$ is a point of $\tilde{X}_K - \psi(X_K)$ distinct from $\eta$, let $T'$ be the closure of $\eta'$ in $\tilde{X}$. Then, $T' \subseteq \tilde{X} - \psi(X)$ and $T \neq T'$. Then, for a general closed point $P$ on $S$, $\tilde{\phi}^{-1}(P) - \psi\phi^{-1}(P)$ would have two distinct points, and this is a contradiction. Thus, $\tilde{X}_K - \psi(X_K)$ has only one point, and this point is purely inseparable over $K$. As $\psi$ is an open immersion, this last fact combined with the fact that $p_a(\tilde{X}_K) = 0$ tells us in view of 2.1 that $X_K$ is a $K$-form of $\mathbf{A}^1$, as desired (cf. [7, 6.7.7]). Q.E.D.

2.3. Now we are able to proceed to the following:

*Proof of Theorem 2.* Notice that $k$ is assumed to be algebraically closed. Using the same notations as in 2.2 (especially as in step (III)), we know that for a sufficiently large integer $N$ the generic fibre of $\phi_N: X_N \to U$ is $k(S_N)$-isomorphic to $\mathbf{A}^1_{k(S_N)}$, where $k(S_N)$ is the function field of $S_N$ over $k$. Let $S' := S_N$. Then, $S'$ is a regular, integral $k$-scheme of finite type and the canonical morphism $S' \to S$ is a faithfully flat, finite, radical morphism. Let $X' := X \times_S S'$ and $\phi' := \phi \times_S S'$. Then, $\phi'$ is a faithfully flat, affine morphism of finite type, the generic fibre of $\phi'$ is $k(S')$-isomorphic to $\mathbf{A}^1_{k(S')}$, and every fibre of $\phi'$ is geometrically integral. Thus, all conditions of Theorem 1 are present for $S'$, $X'$, and $\phi'$. Hence $X'$ is an $\mathbf{A}^1$-bundle over $S'$. If $p = 0$, it is clear that $X$ is already an $\mathbf{A}^1$-bundle over $S$. This completes the proof of Theorem 2.

## 3. Comments and discussions

Various remarks to Theorems 1 and 2 will be given in this section.

3.1. While the affine line $\mathbf{A}^1$, and hence the one-dimensional additive group $\mathbf{G}_a$, are stable under flat, geometrically integral specializations as shown in the text above, the one-dimensional torus $\mathbf{G}_m$ may well be specialized into $\mathbf{G}_a$, as shown by the following:

*Example.* Let $k[x, u, t] := (k[t])[X, U]/(U(1 + tX) - 1)$, which contains the polynomial ring $k[t]$ in a natural manner. Let

$$\phi: G := \operatorname{Spec}(k[x, u, t]) \to \mathbf{A}^1 = \operatorname{Spec}(k[t])$$

be the corresponding morphism. The scheme $G$ is made into an $\mathbf{A}^1$-group scheme through the group law defined by

$$(x, u)(x', u') := (x + x' + txx', uu').$$

Here, the fibre above $(t = 0)$ is $\mathbf{G}_a$, and all other closed fibres as well as the generic fibre are isomorphic to $\mathbf{G}_m$.

3.2. If in the example of 3.1 the base ring $k[t]$ is replaced by the one-variable power series ring $k[\![t]\!]$, one can see at once that in Theorem 2 the base scheme $S$ must be assumed to be of finite type over $k$.

3.3. A flat specialization of $\mathbf{A}^n$ $(n \geq 2)$ is not necessarily isomorphic to $\mathbf{A}^n$, as shown by the next.

*Example.* Let $k$ be an algebraically closed field, and let $C$ be a smooth affine plane curve of genus $> 0$ contained as a closed subscheme in $\mathbf{A}_k^2 := \operatorname{Spec}(k[x, y])$. Let $f(x, y) = 0$ be an irreducible equation for $C$. Let $\mathfrak{o} := k[t]_{(t)}$ be the local ring of $\mathbf{A}_k^1 := \operatorname{Spec}(k[t])$ at $t = 0$, let $K := k(t)$, and let

$$A := \mathfrak{o}[x, y, z]/(tz - f(x, y)).$$

Let $X := \operatorname{Spec}(A)$, $S := \operatorname{Spec}(\mathfrak{o})$, and let $\phi: X \to S$ be the morphism induced by the natural inclusion $\mathfrak{o} \hookrightarrow A$. Then, $\phi$ is a faithfully flat, affine morphism of finite type, the generic fibre $X_K$ of $\phi$ is isomorphic to $\mathbf{A}_K^2$, and the closed fibre is $k$-isomorphic to $C \times_k \mathbf{A}_k^1$ which could not be isomorphic to $\mathbf{A}_k^2$. (Flatness of $\phi$ follows from [3, IV (14.3.8)].)

3.4. In the characteristic zero case we have the following, superficially stronger, version of Theorem 2.

*Let $k$ be a field of characteristic zero, let $S$ be a locally factorial, geometrically integral $k$-scheme of finite type, and let $\phi: X \to S$ be a faithfully flat, affine morphism of finite type. Assume that every fibre of $\phi$ is geometrically integral. Then, the following conditions are equivalent to one another:*

(i) *$X$ is an $\mathbf{A}^1$-bundle over $S$.*
(ii) *$X$ is an affine ruled variety over $S$.*
(iii) *The general fibres of $\phi$ are $k$-isomorphic to $\mathbf{A}^1$.*
(iv) *The generic fibre of $\phi$ is $k(S)$-isomorphic to $\mathbf{A}_{k(S)}^1$.*

*Proof.* It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (iv) follows from Lemma 2.2. (iv) $\Rightarrow$ (i) follows from Theorem 1.

3.5. In the positive characteristic case there can be a flat fibration of a curve in which every closed fibre is $\mathbf{A}^1$ and yet the generic fibre is nonisomorphic to $\mathbf{A}^1$.

*Example.* Let $A := k[t] \hookrightarrow R := k[t, X, Y]/(Y^p - X - tX^p)$ be the natural inclusion, and $\phi: X := \operatorname{Spec}(R) \to S := \operatorname{Spec}(A)$ be the corresponding morphism, where $k$ denotes an algebraically closed field of characteristic $p > 0$. In this example, the generic fibre is a purely inseparable $k(t)$-form of $\mathbf{A}^1$ studied in our joint works [7, Section 6], [8], while all closed fibres are $k$-isomorphic to $\mathbf{A}^1$.

3.6. In the notation of Theorem 2, if $S$ is rational over $k$, then $X$ is a unirational variety over $k$. It is an interesting problem to find examples of unirational, irrational varieties by finding fibrations $\phi: X \to S$ as in Theorem 2. This is partially done in [10] by making use of quasielliptic fibrations.

3.7. For a fibration $\phi: X \to S$, the property that a fibre is geometrically integral is not preserved under generalizations, as shown by the following:

*Example.* Let $k$ be a field, and let

$$A := k[\![X, Y]\!] \quad R := A[T, U]/(X^2T - YU^2 - U - Y)$$

be the natural inclusion mapping. For the maximal ideal $\mathfrak{m}$ of $A$, $R/\mathfrak{m}R \cong k[T]$, while for a prime ideal $\mathfrak{p} \subset A$ of height 1 with $X \in \mathfrak{p}$,

$$(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \otimes_A R \cong (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})[T, U]/(YU^2 + U + Y),$$

which is not geometrically integral over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

3.8. A very recent announcement of results [12] by Bass, Connell, and Wright is noteworthy. Their main result asserts that every $\mathbf{A}^n$-bundle over an affine scheme in fact arises from a vector bundle over the same base. As a consequence, the $\mathbf{A}^1$-bundle $X$ in our Theorem 1 above may now be considered a line bundle over $S$, provided $S$ is affine.

REFERENCES

1. B. Ju. Veĭsfeĭler and I. V. Dolgačëv, *Unipotent group schemes over integral rings*, Math. USSR-Izvestija, vol. 8 (1974), pp. 761–800 (= English translation of Izv. Akad. Nauk SSSR, Ser. Mat., Tom 38 (1974), no. 4, pp. 757–799).
2. C. Chevalley, *Introduction to the theory of algebraic functions of one variable*, Mathematical Surveys, no. 6, Amer. Math. Soc., New York 1951.
3. A. Grothendieck et J. Dieudonné, *Éléments de Géomètrie Algébrique*, Publ. Math. I.H.E.S., France, vols. 8, 11, 24, 28, 32, Paris.
4. A. Grothendieck, *Fondements de la Géométrie Algèbrique*—extraits du Séminaire Bourbaki, 1957–62, Secrétariat Mathématique, Paris.
5. ———, *Séminaire de Géométrie Algébrique* (SGA 2), North Holland Publ. Co., Amsterdam, 1968.
6. ———, *Local cohomology*, Lecture Notes in Mathematics, vol. 41, Springer-Verlag, New York, 1967.
7. T. Kambayashi, M. Miyanishi, and M. Takeuchi, *Unipotent algebraic groups*, Lecture Notes in Mathematics, vol. 414, Springer-Verlag, New York, 1974.
8. T. Kambayashi and M. Miyanishi, *On forms of the affine line over a field*, Lectures in Mathematics, Kyoto University, vol. 10, Kinokuniya Book-Store, Ltd., Tokyo, 1977.
9. M. Maruyama, *On classification of ruled surfaces*, Lectures in Mathematics, Kyoto University, vol. 3, Kinokuniya Book-Store, Ltd., Tokyo, 1970.
10. M. Miyanishi, *On unirational quasielliptic surfaces*, to appear.
11. M. Nagata, *A remark on the unique factorization theorem*, J. Math. Soc. Japan, vol. 9 (1957), pp. 143–145.
12. H. Bass, E. H. Connell, and D. L. Wright, *Locally polynomial algebras are symmetric algebras*, Bull. Amer. Math. Soc., vol. 82 (1976), pp. 719–720.

Northern Illinois University
De Kalb, Illinois
Osaka University
Toyonaka, Osaka, Japan