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On flat modules over commutative rings

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It is well-known that if R is a Noetherian ring or a local ring, then every finitely generated flat R-module is projective (cf. (1), (3), R need not be commutative.). It is also known that if R is a commutative integral domain, the same conclusion holds (cf. (5) Appendix).

In the present paper, a fairly general sufficient condition for a commutative ring R to the effect that the same conclusion holds, will be given as Theorem 2. It will include all the mentioned results as far as they concern commutative rings. This will be deduced from a more general result, Theorem 1, which is obtained by a homological method as used in (4). We shall add another proof of Theorem 2, which is independent from homological method. Finally we shall examine if the converse of Theorem 2 is true. We could not decide this problem, but proved that this is true in some special cases.

Throughout this paper a ring means a commutative ring with unit element. A local ring means a ring with only one maximal ideal and a semi-local ring means a ring with a finite number of maximal ideals.

Let R be a ring, M be an R-module and S be a multiplicatively closed subset of R. Then the quotient ring and the quotient module of R, M with respect to S are denoted by R_S, M_S, respectively. If S is the complementary set of a prime ideal $\mathfrak{p}$ in R, then we shall use $R_{\mathfrak{p}}$, $M_{\mathfrak{p}}$ instead of R_S, M_S. We shall denote by T the set of all non-zero divisors of R. Then the quotient ring R_T of R with respect to T will be called the total quotient ring of R and denoted by K.

An R-module M is called a torsion-free module if, whenever $tu=0$, $u \in M$, $t \in T$, we have $u=0$. On the other hand an R-module M is called a divisible module if for any $t \in T$, $u \in M$ there is an element v of M with $u=tv$.

The other notations and terminologies are the same as in (1).

1. We begin with

LEMMA 1. *Let R be a ring with the total quotient ring K and M be a finitely generated torsion-free R-module such that M_T is K-projective. Then there exists a finitely generated free R-module F such that $F \supset M$ and $(F/M)_T$ is K-projective.*

PROOF. Since M is torsion-free, we may regard M as an R-submodule of M_T. As M_T is K-projective, there exists a finitely generated free K-module $\bar{F}$ such that M_T is the direct summand of it. Let $u_1, u_2, \cdots, u_t$ be a base of M over R and $v_1, v_2, \cdots, v_s$ be a free base of $\bar{F}$ over K. Then we have $u_i = \sum_{j=1}^{s} q_{ij} v_j$, $q_{ij} \in K, 1 \leqq i \leqq t$. By choosing suitably a non-zero divisor λ of R, we have $\lambda q_{ij} \in R$ for all i and j, and obtain

$$u_i = \sum_{j=1}^{s} (\lambda q_{ij})(\lambda^{-1} v_j), \quad 1 \leqq i \leqq t .$$

Let F be the module generated over R by $\lambda^{-1}v_1, \cdots, \lambda^{-1}v_s$. Then F is a finitely generated free R-module containing M. Since $(F/M)_T \cong \bar{F}/M_T$, $(F/M)_T$ is K-projective. Thus F satisfies our requirements.

LEMMA 2. *Let R be a ring with the total quotient ring K and M, F be flat R-modules such that $M \subset F$ and $(F/M)_T$ is a flat K-module. Then we have* $\mathrm{Tor}_1^R(N, F/M) = (0)$ *for any torsion-free R-module N.*

PROOF. As is easily seen, we have $(\mathrm{Tor}_1^R(N, F/M))_T \cong \mathrm{Tor}_1^K(N_T, (F/M)_T)$. Since $(F/M)_T$ is K-flat, we have $\mathrm{Tor}_1^K(N_T, (F/M)_T) = (0)$ and so $(\mathrm{Tor}_1^R(N, F/M))_T = (0)$. Therefore $\mathrm{Tor}_1^R(N, F/M)$ is a torsion R-module. On the other hand, since N is torsion-free and M is flat, we have the exact sequence

$$(0) \longrightarrow N \otimes_R M \longrightarrow K \otimes_R N \otimes_R M \longrightarrow .$$

Then $N \otimes_R M$ is also torsion-free. As F is flat, $\mathrm{Tor}_1^R(N, F/M)$ is an R-submodule of $N \otimes_R M$. Therefore $\mathrm{Tor}_1^R(N, F/M)$ is torsion-free. Thus we must have $\mathrm{Tor}_1^R(N, F/M) = (0)$.

Now we give

THEOREM 1. *Let R be a ring with the total quotient ring K and M be a finitely generated R-module. Then M is R-projective if and only if M is R-flat and M_T is K-projective.*

The only if part of this theorem is obvious. Hence we have only to consider the if part, which can be proved by using the same method as in the proof of (c)$\Rightarrow$(a) in (4) Theorem 2 namely as follows.

We begin with a general remark.

Let M, N, L be modules over a ring R. We define the R-homomorphism

$$\sigma : \quad \mathrm{Hom}_R(N, L) \otimes_R M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), L)$$

by $\sigma(f \otimes u)(g) = f(g(u))$, $u \in M, f \in \mathrm{Hom}_R(N, L), g \in \mathrm{Hom}_R(M, N)$. The following facts are well known.

(1) If M is a finitely generated projective R-module, then σ is an isomorphism ((**1**) VI. 5.2).

(2) If M is a finitely generated R-module and L is R-injective, then σ is an epimorphism ((4) Lemma 2).

LEMMA 3. *Let M be a finitely generated flat R-module, and N be a divisible R-module. If there exists a finitely generated flat R-module F such that $M \subset F$ and $(F/M)_T$ is a flat K-module, then σ is a monomorphism.*

PROOF. The exact sequence $(0) \to M \to F \to F/M \to (0)$ induces the following commutative diagram with the exact top row:

$$\begin{array}{ccccc} \mathrm{Tor}_1^R(\mathrm{Hom}_R(N, L), F/M) & \longrightarrow & \mathrm{Hom}_R(N, L) \otimes_R M & \longrightarrow & \mathrm{Hom}_R(N, L) \otimes_R F \\ & & \downarrow \sigma & & \downarrow \sigma \\ & & \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), L) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(F, N), L)\,. \end{array}$$

Since N is divisible, $\mathrm{Hom}_R(N, L)$ is torsion-free. Then, by Lemma 2, we have $\mathrm{Tor}_1^R(\mathrm{Hom}_R(N, L), F/M) = (0)$. As the homomorphism σ of the right hand side is an isomorphism by (1), σ of the left hand side is a monomorphism.

The proof of the if part of Theorem 1. Suppose that M is flat and M_T is K-projective. By Lemma 1, then, M satisfies the conditions in Lemma 3. Now let X be an R-module. Then there exists an injective R-module N containing X. If we put $Y = N/X$, then we have the exact sequence

$$(0) \longrightarrow X \longrightarrow N \longrightarrow Y \longrightarrow (0)\,.$$

This yields the following exact sequence

$$\longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, Y) \longrightarrow \mathrm{Ext}_R^1(M, X) \longrightarrow (0)\,.$$

Let L be an injective R-module. Then we have also the exact sequence

$$(0) \longrightarrow \mathrm{Hom}_R(Y, L) \longrightarrow \mathrm{Hom}_R(N, L) \longrightarrow \mathrm{Hom}_R(X, L) \longrightarrow (0)\,.$$

From these exact sequences we derive the following commutative diagram with the exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Tor}_1^R(\mathrm{Hom}_R(X, L), M) & \longrightarrow & \mathrm{Hom}_R(Y, L) \otimes_R M & \longrightarrow & \mathrm{Hom}_R(N, L) \otimes_R M \\ & & & & \downarrow \sigma & & \downarrow \sigma \\ (0) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Ext}_R^1(M,X),L) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M,Y),L) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(M,N),L)\,. \end{array}$$

Since L, N are injective and Y is divisible, both σ's are isomorphisms by (2) and Lemma 3. As M is flat, we have $\mathrm{Tor}_1^R(\mathrm{Hom}_R(X, L), M) = (0)$. So we have also $\mathrm{Hom}_R(\mathrm{Ext}_R^1(M, X), L) = (0)$. Since L is an arbitrary injective R-module, this implies $\mathrm{Ext}_R^1(M, X) = (0)$. This shows that M is projective. Thus the if part is proved.

LEMMA 4. *Let M be an (finitely generated) R-module. Then M is flat if and only if, for any maximal ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-flat (free).*

PROOF. See (1), VII, Ex. 10 and 11, and (3) Prop. 9.

COROLLARY. *Let R be a ring with the total quotient ring K and $\mathfrak{a}$ be a* *finitely generated ideal of R. If $\mathfrak{a}$ is flat and $\mathfrak{a}K$ is generated by an idempotent of K, then $\mathfrak{a}$ is the direct summand of an invertible ideal of R.*

PROOF. By Theorem 1, $\mathfrak{a}$ is projective. If $\mathfrak{a}$ contains a non-zero divisor of R, it can be shown by using an analogous method as in (**1**), VII, 3.2 that $\mathfrak{a}$ is invertible. Hence we may suppose that $\mathfrak{a}$ is a non-zero ideal which does not contain any non-zero divisor of R. By our assumption we can put $\mathfrak{a}K = eK$ for an idempotent e of K different from 1 and 0. Put $\mathfrak{a}' = \mathfrak{a}K \cap R$ and $\mathfrak{b} = (1-e)K \cap R$. Then we have $\mathfrak{a} \subset \mathfrak{a}'$ and $\mathfrak{a}'K = eK$. Suppose $(\mathfrak{a}', \mathfrak{b})R \neq R$. Then we have a maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{a}'$ and $\mathfrak{b}$. Since $\mathfrak{a}$ is R-projective, $\mathfrak{a}R_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-free by Lemma 4. So we have $\mathfrak{a}R_{\mathfrak{m}} = (0)$ or (a) for a non-zero divisor a of $R_{\mathfrak{m}}$. If $\mathfrak{a}R_{\mathfrak{m}} = (0)$, then there exists an element s of $R - \mathfrak{m}$ such that $s\mathfrak{a} = (0)$, as $\mathfrak{a}$ is finitely generated. Then we have $s = \alpha(1-e)$, $\alpha \in K$. Hence $s \in (1-e)K \cap R = \mathfrak{b} \subset \mathfrak{m}$. This is a contradiction. Thus $\mathfrak{a}R_{\mathfrak{m}}$ is generated by a non-zero divisor of $R_{\mathfrak{m}}$. Since $\mathfrak{a}\mathfrak{b} = (0)$, we have $\mathfrak{b}R_{\mathfrak{m}} = (0)$. As $\mathfrak{b}K = (1-e)K$, we can find an element b of $\mathfrak{b}$ such that $b = \beta(1-e)$ for a unit β of K. Since $\mathfrak{b}R_{\mathfrak{m}} = (0)$, there is an element t of $R - \mathfrak{m}$ such that $tb = 0$. Then we have $t\mathfrak{b} = (0)$. Hence $t \in eK$. Since $t \in eK \cap R = \mathfrak{a}' \subset \mathfrak{m}$, this is also a contradiction. Thus $(\mathfrak{a}', \mathfrak{b})\ R = R$. Hence $\mathfrak{a}'$ is the direct summand of R. Since $\mathfrak{a}'K = eK$, we may assume $e \in R$. Put $\mathfrak{c} = (\mathfrak{a}, 1-e)$. Then, as $\mathfrak{a}$ is projective, $\mathfrak{c}$ is also projective. Since $\mathfrak{c}$ contains obviously a non-zero divisor of R, $\mathfrak{c}$ is invertible by the preceding remark. Since $\mathfrak{a}(1-e) = (0)$, $\mathfrak{a}$ is the direct summand of an invertible ideal $\mathfrak{c}$ of R. This completes our proof.

2. We first give

LEMMA 5. *A finitely generated flat R-module M is projective if and only if, when φ is an epimorphism of a finitely generated free R-module F on M, the kernel of φ is finitely generated.*

PROOF. For example, see (**2**), Cor. to Prop. 2.2.

Now we obtain

THEOREM 2. *Let R be a ring such that, for a multiplicatively closed subset S consisting of non-zero divisors of R, R_S is semi-local. Then any finitely generated flat R-module is projective.*

PROOF. Let M be a finitely generated flat R-module. First we assume that R itself is semi-local. Now we have the exact sequence with a finitely generated free R-module F:

$$(0) \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow (0).$$

This yields the exact sequence as $R_{\mathfrak{m}}$-modules for any maximal ideal $\mathfrak{m}$ of R

$$(0) \longrightarrow N_{\mathfrak{m}} \longrightarrow F_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow (0).$$

By Lemma 4 $M_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-projective. Then $N_{\mathfrak{m}}$ is a direct summand of $F_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-

module, and so $N_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-module. Since R is semi-local, N is also finitely generated over R. So, by Lemma 5, M must be projective.

In the general case M_S is R_S-flat. Since R_S is semi-local, M_S is R_S-projective by the above argument. So M_T is K-projective, as $S\subset T$. Then, by Theorem 1, M must be R-projective. This completes our proof.

A Noetherian ring and an integral domain satisfy, obviously, the condition in Theorem 2. In fact, the total quotient ring of a Noetherian ring is always semi-local and also that of an integral domain is a field.

In the above proof of Theorem 2, we applied Theorem 1 at the last step. We can prove Theorem 2 without applying it, i. e., without using the functors Tor and Ext as follows.

If R is semi-local, then M is projective by the first part of the preceding proof. Hence we may assume that R is not semi-local. Denote by $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ all the maximal ideals of R_S and put $\mathfrak{q}_i=\mathfrak{p}_i\cap R$ for every i. Then $\mathfrak{q}_i$'s are prime ideals of R and we may assume $S=\bigcap_{i=1}^{s}(R-\mathfrak{q}_i)$. Let $\overline{\mathfrak{m}}_i$ be a maximal ideal of R containing $\mathfrak{q}_i$ for every i and put $\bar{S}=\bigcap_{i=1}^{s}(R-\overline{\mathfrak{m}}_i)$. Then $\bar{S}$ is contained in S. Further let $\mathfrak{m}$ be a maximal ideal of R different from $\overline{\mathfrak{m}}_1, \cdots, \overline{\mathfrak{m}}_s$, and put $S_{\mathfrak{m}}=\bar{S}\cap(R-\mathfrak{m})$. Then we have also $S_{\mathfrak{m}}\subset S$. So $R_{S_{\mathfrak{m}}}$ is a subring of K and is a semi-local ring with maximal ideals $\mathfrak{m}R_{S_{\mathfrak{m}}}, \overline{\mathfrak{m}}_1R_{S_{\mathfrak{m}}}, \cdots, \overline{\mathfrak{m}}_sR_{S_{\mathfrak{m}}}$. Since M is flat, we can regard M as contained in $M_{S_{\mathfrak{m}}}$ as an R-submodule. As M is flat, $M_{S_{\mathfrak{m}}}$ is $R_{S_{\mathfrak{m}}}$-flat, and so, as $R_{S_{\mathfrak{m}}}$ is semi-local, $M_{S_{\mathfrak{m}}}$ is $R_{S_{\mathfrak{m}}}$-projective by the first part of the preceding proof.

Now let φ be the epimorphism of a finitely generated free R-module F on M. Then φ can be extended naturally to the $R_{S_{\mathfrak{m}}}$-epimorphism of $F_{S_{\mathfrak{m}}}$ on $M_{S_{\mathfrak{m}}}$, which will be denoted by the same φ. Since $M_{S_{\mathfrak{m}}}$ is $R_{S_{\mathfrak{m}}}$-projective, there exists an $R_{S_{\mathfrak{m}}}$-homomorphism $\psi_{\mathfrak{m}}$ of $M_{S_{\mathfrak{m}}}$ in $F_{S_{\mathfrak{m}}}$ such that $\varphi\psi_{\mathfrak{m}}$ is the identity. Let $u_1, u_2, \cdots, u_t$ be a base of M over R. Then, as $R_{S_{\mathfrak{m}}}\supset R, M_{S_{\mathfrak{m}}}\supset M$ and $F_{S_{\mathfrak{m}}}\supset F$, there exists an element $s_{\mathfrak{m}}$ of $S_{\mathfrak{m}}$ such that $\psi_{\mathfrak{m}}(u_i)=\frac{1}{s_{\mathfrak{m}}}f_i, f_i\in F$, for any i. If we restrict $\psi_{\mathfrak{m}}$ on $s_{\mathfrak{m}}M$, then $\psi_{\mathfrak{m}}$ is the R-homomorphism of $s_{\mathfrak{m}}M$ in F. Denote by $\mathfrak{a}$ the ideal generated by all $s_{\mathfrak{m}}$'s, where $\mathfrak{m}$ runs over all maximal ideals of R different from $\overline{\mathfrak{m}}_1, \cdots, \overline{\mathfrak{m}}_s$. Then $\mathfrak{a}$ is obviously not contained in any maximal ideal of R. Accordingly $\mathfrak{a}=R$. So there exists a finite number of $s_{\mathfrak{m}}$'s, say $s_{\mathfrak{m}_1}, s_{\mathfrak{m}_2}, \cdots, s_{\mathfrak{m}_r}$ such that $1=a_1s_{\mathfrak{m}_1}+\cdots+a_rs_{\mathfrak{m}_r}, a_i\in R$. Set $\psi(u_i)=a_1\psi_{\mathfrak{m}_1}(s_{\mathfrak{m}_1}u_i)+\cdots+a_r\psi_{\mathfrak{m}_r}(s_{\mathfrak{m}_r}u_i)$. Since $\psi_{\mathfrak{m}_i}$ is a homomorphism of $s_{\mathfrak{m}_i}M$ in F for each i, ψ is a homomorphism of M in F. Obviously, $\varphi\psi$ is the identity on M. This shows that M is projective.

COROLLARY. *Let R be a ring satisfying the condition in Theorem 2. Then any finitely generated flat ideal of R is a direct summand of an invertible ideal of R.*

PROOF. Let $\mathfrak{a}$ be a finitely generated flat ideal of R. By Corollary to Theorem 1, it suffices to prove that $\mathfrak{a}K$ is generated by an idempotent of K. To prove this we may assume, without loss of generality, that R itself is semi-local, if we use $\mathfrak{a}$, R insteads of $\mathfrak{a}R_S$, R_S, respectively. Denote by $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ all maximal ideals of R. By Lemma 4, any $\mathfrak{a}R_{\mathfrak{m}_i}$ is $R_{\mathfrak{m}_i}$-free, and so $\mathfrak{a}R_{\mathfrak{m}_i}$ is a principal ideal of $R_{\mathfrak{m}_i}$. Let a_i be an element of $\mathfrak{a}$ for each i such that $\mathfrak{a}R_{\mathfrak{m}_i}=(a_i)R_{\mathfrak{m}_i}$. Now we choose an element r_i of R for each i which is not contained in $\mathfrak{m}_i$ but contained in $\mathfrak{m}_1\cap\cdots\cap\mathfrak{m}_{i-1}\cap\mathfrak{m}_{i+1}\cap\cdots\cap\mathfrak{m}_t$. If we put $a=\sum_{i=1}^{t} r_i a_i$, then we have $\mathfrak{a}R_{\mathfrak{m}_i}=(a_i)R_{\mathfrak{m}_i}=(a)R_{\mathfrak{m}_i}$ for any i. Let b be an element of $\mathfrak{a}$. As $\mathfrak{a}R_{\mathfrak{m}_i}=(a)R_{\mathfrak{m}_i}$ for any i, we can find an element s_i of $R-\mathfrak{m}_i$ such that $s_i b\in(a)$. Put $\mathfrak{c}=\{c\,; cb\in(a), c\in R\}$. Then $\mathfrak{c}$ is an ideal of R which is not contained in any $\mathfrak{m}_i$. Hence $\mathfrak{c}=R$, i.e., $b\in(a)$. This shows that $\mathfrak{a}$ is generated by a in R. Furthermore put $\mathfrak{r}=\{r\,; ra=0, r\in R\}$. As $\mathfrak{a}$ is R-projective by Theorem 2, $\mathfrak{r}$ is a direct summand of R. Therefore $\mathfrak{r}$ is generated by an idempotent e' of R. If we put $d=a+e'$ and $e=1-e'$, then we have $a=de$. Obviously d is a non-zero divisor of R and e is an idempotent of R. Since d is a unit in K, we have $\mathfrak{a}K=(a)K=(e)K$. This shows that $\mathfrak{a}K$ is generated by an idempotent e of K. This completes our proof.

3. Theorem 2 says that from the following condition (I) for a ring R follows (II). (I) There is a multiplicatively closed subset S consisting of non-zero divisors in R such that R_S is semi-local. (II) Any finitely generated flat R-module is projective.

We did not succeed in proving the converse (II)→(I) in general but could prove this in some special cases.

First we remark that, if there is an infinite number of idempotents in R, then there exists a finitely generated flat R-module which is not projective. In fact, then, there exists an infinite number of orthogonal idempotents $\{e_i\}$ in R. Let $\mathfrak{a}$ be the ideal generated by all e_i's. Then $\mathfrak{a}$ is not finitely generated and, for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{a}R_{\mathfrak{m}}$ coincides with $R_{\mathfrak{m}}$ or (0). If we put $M=R/\mathfrak{a}$, then $M_{\mathfrak{m}}$ also coincides with $R_{\mathfrak{m}}$ or (0), for any maximal ideal $\mathfrak{m}$ of R, and therefore M is flat. It is obvious that M is generated by only one element. But, since $\mathfrak{a}$ is not finitely generated, M is not projective. Thus M satisfies our requirements.

By this remark, if a ring R satisfies the condition (II) then R has only a finite number of idempotents. Here we shall show that the conjecture is true for rings satisfying some additional conditions.

(A). Let R be a ring such that $R_{\mathfrak{m}}$ is an integral domain for any maximal ideal $\mathfrak{m}$ of R. For such ring R our conjecture is true. In fact, suppose that any finitely generated flat R-module is projective. Then it is sufficient to show that the total quotient ring K of R is semi-local. Put $\mathfrak{p}_{\mathfrak{m}}=\{a\,;as=0$ for some s of $R-\mathfrak{m}, a\in R\}$ for any maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{m}$, $\mathfrak{m}'$ be different maximal ideals of R. Then, if $\mathfrak{m}'\supset\mathfrak{p}_{\mathfrak{m}}$, then $\mathfrak{p}_{\mathfrak{m}}R_{\mathfrak{m}'}=(0)$ and if $\mathfrak{m}'\not\supset\mathfrak{p}_{\mathfrak{m}}$, then $\mathfrak{p}_{\mathfrak{m}}R_{\mathfrak{m}'}=R_{\mathfrak{m}'}$. If we put $M=R/\mathfrak{p}_{\mathfrak{m}}$, then $M_{\mathfrak{m}'}$ coincides with $R_{\mathfrak{m}'}$ or (0). According to Lemma 4 M is R-flat. Therefore, by our assumption, M must be projective. This shows also that $\mathfrak{p}_{\mathfrak{m}}$ is generated by an idempotent in R. By the preceding remark, there is only a finite number of idempotents in R. From this it follows that there is only a finite number of $\mathfrak{p}_{\mathfrak{m}}$'s different from each other. This shows that R is expressible as a direct sum of a finite number of integral domains. Consequently K is semi-simple, i.e., it is semi-local.

(B). Let R be a ring with the total quotient ring K such that any prime ideal $\mathfrak{p}_k$ of K is maximal in K and any maximal ideal of R contains only one of $\mathfrak{p}_k\cap R$'s. For such ring R our conjecture is true. Suppose that any finitely generated flat R-module is projective. Put $\mathfrak{a}_{\mathfrak{m}}=\{a\,;as=0$ for some s of $R-\mathfrak{m}$, $a\in R\}$ for any maximal ideal $\mathfrak{m}$ of R. Since any maximal ideal $\mathfrak{m}$ of R contains only one minimal prime ideal $\mathfrak{p}_{\mathfrak{m}}=\mathfrak{p}_k\cap R, \mathfrak{a}_{\mathfrak{m}}$ is also contained in only one minimal prime ideal $\mathfrak{p}_{\mathfrak{m}}$. Hence, if two maximal ideals $\mathfrak{m},\mathfrak{m}'$ of R contain the same minimal prime ideal, then we have $\mathfrak{a}_{\mathfrak{m}}=\mathfrak{a}_{\mathfrak{m}'}$ and if they contain the different minimal prime ideals, then we have $\mathfrak{a}_{\mathfrak{m}}\not\subset\mathfrak{m}'$ and $\mathfrak{a}_{\mathfrak{m}'}\not\subset\mathfrak{m}$. Then it can be shown as in (A) that $\mathfrak{a}_{\mathfrak{m}}$ is generated by an idempotent of R and that there exists only a finite number of $\mathfrak{a}_{\mathfrak{m}}$'s different from each other. This shows that there exists only a finite number of maximal ideals in K, i.e., K is semi-local.

If we replace "finitely generated flat R-module" by "finitely generated flat ideal of R" in the condition (II), our conjecture is false. In fact, we can easily show an example of a ring which is regular but non-semi-simple. More generally we obtain the following

PROPOSITION. *Let R be a ring with the total quotient ring K in which any prime ideal is maximal. Then any finitely generated flat ideal of R is a direct summand of an invertible ideal of R.*

PROOF. Let $\mathfrak{a}$ be a finitely generated flat ideal of R. Then $\mathfrak{a}K$ is clearly K-flat. Hence $\mathfrak{a}K_{\mathfrak{m}}$ is $K_{\mathfrak{m}}$-free for any maximal ideal $\mathfrak{m}$ of K by Lemma 4. Since any non-zero divisor of $K_{\mathfrak{m}}$ is a unit of $K_{\mathfrak{m}}$, $\mathfrak{a}K_{\mathfrak{m}}$ must coincide with $K_{\mathfrak{m}}$ or (0). If we put $\mathfrak{c}=\{c\,;c\mathfrak{a}=0, c\in K\}$, then an ideal $(\mathfrak{a},\mathfrak{c})K$ of K is not contained in any maximal ideal of K and so we have $(\mathfrak{a},\mathfrak{c})K=K$. This shows that $\mathfrak{a}K$ is generated by an idempotent of K. Then, according to Corollary to Theorem 1, $\mathfrak{a}$ is a direct summand of an invertible ideal of R.

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