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ON FOUR-DIMENSIONAL RIEMANNIAN WARPED PRODUCT MANIFOLDS SATISFYING CERTAIN PSEUDO-SYMMETRY CURVATURE CONDITIONS

BY

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1. Introduction. Let (M, g) be a connected n-dimensional, $n \geq 3$, smooth Riemannian manifold with a not necessarily definite metric g. The manifold (M, g) is said to be pseudo-symmetric ([11]) if its curvature tensor R satisfies at every point of M the following condition:

$(*)$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

It is easy to see that if $(*)$ holds at a point of M then the Weyl conformal curvature tensor C satisfies at this point the condition

$(**)$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

A manifold (M, g) fulfilling $(**)$ at each point of M is called Weyl-pseudo-symmetric ([8]).

As was proved in ([12]), if $n \geq 5$ then $(*)$ and $(**)$ are equivalent at each point at which C is non-zero. In particular, from this result if follows (see also [16]) that for $n \geq 5$ the conditions $R \cdot C = 0$ and $R \cdot R = 0$ are equivalent at each point of (M, g) at which $C \neq 0$. On 4-manifolds, this is not always the case. A suitable example was given in [5] (Lemme 1.1). That example, by a certain modification, also gives rise to an example of a non-pseudo-symmetric manifold satisfying $(**)$ with $R \cdot C$ non-zero (see [10]). Moreover, in [2] an example of a non-pseudo-symmetric conformally flat manifold of dimension $n \geq 4$ was described.

In the present paper we shall prove (Section 4) that $(*)$ and $(**)$ are equivalent at every point of a 4-dimensional warped product manifold at which C does not vanish. From this it follows immediately that the above-mentioned Riemannian manifold obtained in [5] is a non-warped product manifold satisfying $R \cdot C = 0$. It is known that $(*)$ and $(**)$ are equivalent on manifolds isometrically immersed as hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$ (see [3], Corollary).

If $(*)$ holds at a point of M then at this point the following condition is fulfilled:

(∗∗∗) the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent,

where S denotes the Ricci tensor. A manifold (M, g) satisfying (∗∗∗) at every point of M is said to be Ricci-pseudo-symmetric ([14]). So, any pseudo-symmetric manifold is Ricci-pseudo-symmetric. However, the converse fails in general (see [14], [7]). We shall prove (Section 5) that (∗) and (∗∗∗) are equivalent at every point of a 4-dimensional warped product manifold at which the tensor $S - (K/n)g$ does not vanish, where K is the scalar curvature.

Section 2 is concerned with some facts on pseudo-symmetric tensors. We recapitulate the basic formulas about warped products in Section 3. Finally, an analogue of Theorem 1 from [19] is mentioned at the end of that section.

2. Pseudo-symmetric tensors. Let (M, g) be an n-dimensional, $n \geq 3$, Riemannian manifold with a not necessarily definite metric g. We denote by ∇, R, S, C and K the Levi-Cività connection, the curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of (M, g) respectively. For a $(0, k)$-tensor field T on M, $k \geq 1$, we define the tensor fields $R \cdot T$ and $Q(g, T)$ by

$$
\begin{aligned}
&(R \cdot T)(X_1, \ldots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \ldots, X_k) \\
&\quad = -T(R(X, Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, R(X, Y)X_k), \\
&Q(g, T)(X_1, \ldots, X_k; X, Y) = -((X \wedge Y) \cdot T)(X_1, \ldots, X_k) \\
&\quad = T((X \wedge Y)X_1, X_2, \ldots, X_k) + \ldots + T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k)
\end{aligned}
$$

respectively, where $R(X, Y)$ and $X \wedge Y$ are derivations of the algebra of tensor fields on M and $X_1, \ldots, X_k, X, Y \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M. These derivations are extensions of the endomorphisms $R(X, Y)$ and $X \wedge Y$ of $\mathfrak{X}(M)$ defined by

$$
\begin{aligned}
R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\
(X \wedge Y)Z &= g(Z, Y)X - g(Z, X)Y
\end{aligned}
$$

respectively. A $(0, k)$-tensor field T is said to be *pseudo-symmetric* if the tensors $R \cdot T$ and $Q(g, T)$ are linearly dependent at every point of M. In the special case when $R \cdot T$ vanishes on M, the tensor T is called *semi-symmetric.* A $(0, 4)$-tensor field T on M is said to be a *generalized curvature tensor* [18] if

$$
\begin{gathered}
T(X_1, X_2, X_3, X_4) + T(X_1, X_3, X_4, X_2) + T(X_1, X_4, X_2, X_3) = 0, \\
T(X_1, X_2, X_3, X_4) = -T(X_2, X_1, X_3, X_4), \\
T(X_1, X_2, X_3, X_4) = T(X_3, X_4, X_1, X_2),
\end{gathered}
$$

for all $X_i \in \mathfrak{X}(M)$. For a generalized curvature tensor field T we define the *concircular curvature tensor* $Z(T)$ by

$$Z(T) = T - \frac{K(T)}{n(n-1)} G,$$

where $K(T)$ is the scalar curvature of T and G is the generalized curvature tensor defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4).$$

A generalized curvature tensor T is called *trivial* at $x \in M$ ([8]) if $Z(T)$ vanishes at x. Similarly, for a symmetric $(0,2)$-tensor field A we define the tensor $Z(A)$ by

$$Z(A) = A - \frac{\operatorname{tr}(A)}{n} g.$$

A symmetric $(0,2)$-tensor field A is said to be *trivial* at $x \in M$ if $Z(A)$ vanishes at x.

Remark 1 ([2], Lemma 1.1(iii)). Let T be a generalized curvature tensor (resp. a $(0,2)$-symmetric tensor) at a point x of a manifold (M,g). Then the equalities $Z(T) = 0$ and $Q(g,T) = 0$ are equivalent at this point.

If a generalized curvature tensor T (resp. a $(0,2)$-symmetric tensor A) is pseudo-symmetric then $R \cdot T = L_T Q(g,T)$ (resp. $R \cdot A = L_A Q(g,A)$) on $U_T = \{x \in M : Z(T)(x) \neq 0\}$ (resp. on $U_A = \{x \in M : Z(A)(x) \neq 0\}$), where L_T is a function defined on U_T (resp. L_A is a function defined on U_A). The functions L_T and L_A are uniquely determined and called the *associated functions* of the pseudo-symmetric tensors T and A respectively ([8]).

A Riemannian manifold (M,g) is said to be *pseudo-symmetric* if its curvature tensor R is pseudo-symmetric [11]; then

$$R \cdot R = L_R Q(g,R) \tag{1}$$

on U_R. Any semi-symmetric manifold ($R \cdot R = 0$, [20]) is pseudo-symmetric. Examples of non-semi-symmetric pseudo-symmetric manifolds are given in [2], [3], [6] and [11].

(M,g) is said to be *Weyl-pseudo-symmetric* if its Weyl conformal curvature tensor C is pseudo-symmetric [8]; then

$$R \cdot C = L_C Q(g,C) \tag{2}$$

on U_C. Any pseudo-symmetric manifold is Weyl-pseudo-symmetric. The converse fails in general (see Section 1). Note that $U_C = \{x \in M : C(x) \neq 0\}$.

(M,g) is said to be *Ricci-pseudo-symmetric* if its Ricci tensor S is pseudo-symmetric ([14], [7]); then

$$R \cdot S = L_S Q(g,S) \tag{3}$$

on U_S. Of course, any pseudo-symmetric manifold is Ricci-pseudo-symmetric. The converse fails in general (see [14], [7]). The conditions (1) and (3) are equivalent on manifolds with vanishing Weyl conformal curvature tensor C. Namely, we have

LEMMA 1 ([2], Lemma 1.2, [13], Lemma 2). *If C vanishes at $x \in M$ and $n \geq 3$, then at x the following three identities are equivalent to each other*:

$$\left((n-2)\alpha + \frac{K}{n-1}\right)\left(S - \frac{k}{n}g\right) = S^2 - \frac{\operatorname{tr}(S^2)}{n}g\,,$$

$$R \cdot S = \alpha Q(g, S)\,, \qquad R \cdot R = \alpha Q(g, R)\,,$$

where $\alpha \in \mathbb{R}$, $S^2(X,Y) = S(\widetilde{S}(X),Y)$, $S(X,Y) = g(\widetilde{S}(X),Y)$ and $x, y \in \mathfrak{X}(M)$.

LEMMA 2. (i) *If (M,g) is 3-dimensional then C vanishes identically.*

(ii) (cf. [15], p. 48) *Any generalized curvature tensor T at a point x of a 3-dimensional Riemannian manifold (M,g) satisfies*

$$\begin{aligned} T(X_1,X_2,X_3,X_4) &= g(X_1,X_4)A(X_2,X_3) + g(X_2,X_3)A(X_1,X_4) \\ &\quad - g(X_1,X_3)A(X_2,X_4) - g(X_2,X_4)A(X_1,X_3), \end{aligned}$$

for all $X_i \in \mathfrak{X}(M)$, where A is the $(0,2)$-tensor defined by

$$A(X_1,X_2) = \operatorname{Ricc}(T)(X_1,X_2) - \frac{K(T)}{4}g(X_1,X_2),$$

$\operatorname{Ricc}(T)$ *and $K(T)$ being the Ricci tensor and the scalar curvature of T respectively.*

(iii) *Let A be a symmetric $(0,2)$-tensor on a 2-dimensional Riemannian manifold (M,g). Then*

$$\begin{aligned} g(X_1,X_4)A(X_2,X_3) &+ g(X_2,X_3)A(X_1,X_4) - g(X_2,X_4)A(X_1,X_3) \\ &- g(X_1,X_3)A(X_2,X_4) = \operatorname{tr}(A)G(X_1,X_2,X_3,X_4) \end{aligned}$$

on M.

LEMMA 3. *Let A and B be non-zero symmetric $(0,2)$-tensors at a point x of a manifold (M,g). If $Q(A,B) = 0$ at x then $A = \lambda B$, $\lambda \in \mathbb{R} - \{0\}$, at x.*

The proof of this lemma was given in [9] (see the proof of Lemma 3.4).

3. Warped products. Let $(M,\overline{g})$ and $(N,\widetilde{g})$, $\dim M = p$, $\dim N = n-p$, $1 \leq p < n$, be Riemannian manifolds covered by systems of charts $\{\overline{V}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$ respectively. Let F be a positive smooth function on M. The *warped product* $M \times_F N$ of $(M,\overline{g})$ and $(N,\widetilde{g})$ (see [17], [1]) is the Cartesian product $M \times N$ with the metric $g = \overline{g} \oplus F\widetilde{g}$ (more precisely, $\overline{g} \oplus F\widetilde{g} = \Pi_1^*\overline{g} + (F \circ \Pi_1)\Pi_2^*\widetilde{g}$, $\Pi_1 : M \times N \to M$ and $\Pi_2 : M \times N \to N$

being the natural projections). Let $\{\overline{V} \times \widetilde{V};\ u^1 = x^1, \dots,\ u^p = x^p,\ u^{p+1} = y^1, \dots,\ u^n = y^{n-p}\}$ be a product chart for $M \times N$. The local components of the metric $\overline{g} \oplus F\widetilde{g}$ with respect to this chart are

$$g_{rs} = \begin{cases} \overline{g}_{ab} & \text{if } r = a,\ s = b, \\ F\widetilde{g}_{\alpha\beta} & \text{if } r = \alpha,\ s = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Here and below, $a, b, c, d, e, f \in \{1, \dots, p\}$, $\alpha, \beta, \gamma, \delta, \lambda, \mu \in \{p+1, \dots, n\}$ and $r, s, t, u, v, w \in \{1, \dots, n\}$. The local components of the tensors R and S of the metric $\overline{g} \oplus F\widetilde{g}$ which may not vanish identically are

$$R_{abcd} = \overline{R}_{abcd}, \tag{5}$$

$$R_{\alpha ab\beta} = -\tfrac{1}{2} T_{ab} \widetilde{g}_{\alpha\beta}, \tag{6}$$

$$R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta F}{4} \widetilde{G}_{\alpha\beta\gamma\delta}, \tag{7}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \tag{8}$$

$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(\operatorname{tr}(T) + \frac{(n-p-1)\Delta F}{2F}\right)\widetilde{g}_{\alpha\beta}, \tag{9}$$

$$\begin{gathered} T_{ab} = \nabla_b F_a - \frac{1}{2F} F_a F_b, \qquad \operatorname{tr}(T) = \overline{g}^{ab} T_{ab}\,, \\ \Delta F = \overline{g}^{ab} F_a F_b\,, \qquad F_a = \frac{\partial}{\partial x^a}(F)\,. \end{gathered} \tag{10}$$

The scalar curvature K of $\overline{g} \oplus F\widetilde{g}$ satisfies

$$K = \overline{K} + \frac{1}{F}\widetilde{K} - \frac{n-p}{F}\left(\operatorname{tr}(T) + \frac{(n-p-1)\Delta F}{4F}\right). \tag{11}$$

Using (5)–(9) and (11), we obtain the following relations for the local components C_{rstu} of the tensor C of $\overline{g} \oplus F\widetilde{g}$:

$$\begin{aligned} C_{abcd} = \overline{R}_{abcd} &- \frac{1}{n-2}(\overline{g}_{ad}\overline{S}_{bc} - \overline{g}_{ac}\overline{S}_{bd} + \overline{g}_{bc}\overline{S}_{ad} - \overline{g}_{bd}\overline{S}_{ac}) \\ &+ \frac{n-p}{2(n-2)F}(\overline{g}_{ad}T_{bc} - \overline{g}_{ac}T_{bd} + \overline{g}_{bc}T_{ad} - \overline{g}_{bd}T_{ac}) \\ &+ \frac{K}{(n-1)(n-2)}\overline{G}_{abcd}, \end{aligned} \tag{12}$$

$$\begin{aligned} C_{\alpha ab\beta} = &- \frac{1}{n-2}\left(\frac{p-2}{2}T_{ab} + F\overline{S}_{ab}\right)\widetilde{g}_{\alpha\beta} - \frac{1}{n-2}\overline{g}_{ab}\widetilde{S}_{\alpha\beta} \\ &+ \frac{1}{(n-1)(n-2)} \\ \times\Bigg(F\overline{K} + \widetilde{K} &- \frac{(n-2p+1)\operatorname{tr}(T)}{2} + \frac{(p-1)(n-p-1)\Delta F}{4F}\Bigg)\overline{g}_{ab}\widetilde{g}_{\alpha\beta}, \end{aligned} \tag{13}$$

$$(14)\qquad C_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{F}{n-2}(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta} + \widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}) + FP\widetilde{G}_{\alpha\beta\gamma\delta},$$

$$(15)\qquad C_{abc\alpha} = C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0,$$

$$(16)\qquad P = \frac{1}{n-2}\left(\frac{FK}{n-1} + \operatorname{tr}(T) + \frac{(n-2p)\Delta F}{4F}\right).$$

Lemma 4 ([6], Theorem 1). *The curvature tensor R of a warped product $M \times_F N$ satisfies $R\cdot R = LQ(g,R)$ if and only if*

$$(17)\qquad (\overline{R}\cdot\overline{R})_{abcdef} = LQ(\overline{g},\overline{R})_{abcdef}\,,$$

$$(18)\qquad H^f{}_d\overline{R}_{fabc} = \frac{1}{2F}(T_{ac}H_{bd} - T_{ab}H_{cd})\,,$$

$$(19)\qquad H_{ad}\left(\widetilde{R}_{\delta\alpha\beta\gamma} - \frac{\Delta F}{4F}\widetilde{G}_{\delta\alpha\beta\gamma}\right) = -\frac{1}{2}T_{fd}H^f{}_a\widetilde{G}_{\delta\alpha\beta\gamma}\,,$$

$$(20)\qquad (\widetilde{R}\cdot\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \left(LF + \frac{\Delta F}{4F}\right)Q(\widetilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\,,$$

where

$$(21)\qquad H_{ad} = \tfrac{1}{2}T_{ad} + FL\overline{g}_{ad}.$$

Lemma 5 ([6], Corollary 1). *Let* $(M,\overline{g})$, $\dim M \geq 2$ *and* $(N,\widetilde{g})$, $\dim N \geq 2$, *be manifolds of constant curvature. The curvature tensor R of the warped product $M\times_F N$ satisfies $R\cdot R = LQ(g,R)$ if and only if*

$$(22)\qquad \frac{2\overline{K}}{p(p-1)}(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}) = \frac{1}{F}(T_{ac}H_{bd} - T_{ab}H_{cd})\,,$$

$$(23)\qquad H_{ad}\left(\frac{\widetilde{K}}{(n-p)(n-p-1)} - \frac{\Delta F}{4F}\right) = -\frac{1}{2}T_{fd}H^f{}_a\,,$$

Using (4)–(16), (21) and Lemma 2(iii), we obtain

Lemma 6. *The only local components of the Weyl conformal curvature tensor C of a 4-dimensional warped product $M\times_F N$ which are not identically zero are*

$$(24)\qquad C_{\alpha11\beta} = -\frac{1}{2}\overline{g}_{11}\left(\widetilde{S}_{\alpha\beta} - \frac{\widetilde{K}}{3}\widetilde{g}_{\alpha\beta}\right),$$

$$(25)\qquad C_{\alpha\beta\gamma\delta} = \frac{F}{2}(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta} + \widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}) - \frac{F\widetilde{K}}{3}\widetilde{G}_{\alpha\beta\gamma\delta},$$

provided that $\dim M = 1$;

$$(26)\qquad C_{abcd} = \frac{P}{F}G_{abcd}\,,\qquad P = \frac{1}{6}\left(F\overline{K} + \widetilde{K} + \operatorname{tr}(T) - \frac{\Delta F}{2F}\right),$$

$$C_{\alpha\beta\gamma\delta} = \frac{P}{F} G_{\alpha\beta\gamma\delta}, \tag{27}$$

$$C_{\alpha ab\beta} = -\frac{P}{2F} G_{\alpha ab\beta}, \tag{28}$$

provided that $\dim M = 2$;

$$C_{4ab4} = -\frac{1}{2}\widetilde{g}_{44} W_{ab}, \quad W_{ab} = V_{ab} - \frac{\operatorname{tr}(V)}{3}\overline{g}_{ab}, \quad V_{ab} = F\overline{S}_{ab} + \frac{1}{2}T_{ab}, \tag{29}$$

$$C_{abcd} = \frac{1}{2F}(\overline{g}_{ad}W_{bc} + \overline{g}_{bc}W_{ad} - \overline{g}_{ac}W_{bd} - \overline{g}_{bd}W_{ac}), \tag{30}$$

provided that $\dim M = 3$.

From the above lemma the following theorem follows immediately:

THEOREM 1. *Suppose* $\dim M \times_F N = 4$.

(i) *If* $\dim M = 1$, *then* $M \times_F N$ *is conformally flat if and only if*

$$\widetilde{S} = \frac{\widetilde{K}}{3}\widetilde{g}.$$

(ii) *If* $\dim M = 2$, *then* $M \times_F N$ *is conformally flat if and only if*

$$\operatorname{tr}(T) = -\frac{FK}{3}.$$

(iii) *If* $\dim M = 3$, *then* $M \times_F N$ *is conformally flat if and only if*

$$F\overline{S} + \frac{T}{2} = \frac{1}{3}\left(F\overline{K} + \frac{\operatorname{tr}(T)}{2}\right)\overline{g}.$$

Remark 2. (i) Necessary and sufficient conditions for $M \times_F N$, $\dim M \times_F N \geq 4$ and $\dim N \geq 2$, to be conformally flat are given in [19] (Theorem 1).

(ii) An example of a 4-dimensional conformally flat warped product $M \times_F N$, $\dim N = 1$, is described in [2] (Lemma 4.3). The manifold $(M, \overline{g})$ considered in that example is non-semi-symmetric, conformally flat and pseudo-symmetric, but $M \times_F N$ is not pseudo-symmetric.

(iii) The assertion (iii) of Theorem 1 can be easily generalized (by making use of (12)–(16)) as follows: The manifold $M \times_F N$, $\dim M = n-1$, $n \geq 4$, is conformally flat if and only if

$$\overline{C} = 0 \quad \text{and} \quad F\overline{S} + \frac{(n-3)T}{2} = \frac{1}{n-1}\left(F\overline{K} + \frac{(n-3)\operatorname{tr}(T)}{2}\right)\overline{g}$$

on M.

Another consequence of Lemma 6 is

THEOREM 2. *Suppose* $\dim M \times_F N = 4$ *and* $\dim M = 2$. *Then* $C \cdot C = -\frac{P}{2F}Q(g, C)$ *on* $M \times_F N$.

The tensor $C \cdot C$ is defined analogously to the tensor $R \cdot T$ in Section 2. Riemannian manifolds satisfying the condition $C \cdot C = 0$ were considered in [4] (see also [13], Corollary 1).

LEMMA 7 ([7], Theorem 1). *The Ricci tensor S of $M \times_F N$ satisfies $R \cdot S = LQ(g, S)$ if and only if*

$$(\overline{R} \cdot \overline{S})_{abcd} - LQ(\overline{g}, \overline{S})_{abcd} = \frac{n-p}{F}((\overline{R} \cdot H)_{abcd} - LQ(\overline{g}, H)_{abcd}), \tag{31}$$

$$H_{ab}\left(\widetilde{S}_{\alpha\beta} - \frac{1}{2F}\left(\mathrm{tr}(T) + \frac{(n-p-1)\Delta F}{2F}\right) g_{\alpha\beta}\right) = H_{cb}\left(\overline{S}^c{}_a - \frac{n-p}{2F} T^c{}_a\right) g_{\alpha\beta}, \tag{32}$$

$$(\widetilde{R} \cdot \widetilde{S})_{\alpha\beta\gamma\delta} = \left(LF + \frac{\Delta F}{4F}\right) Q(\widetilde{g}, \widetilde{S})_{\alpha\beta\gamma\delta}. \tag{33}$$

LEMMA 8 ([7], Corollary 1). *Let $(M, \overline{g})$, $\dim M \geq 2$ and $(N, \widetilde{g})$, $\dim N \geq 2$, be Einstein manifolds. Then the Ricci tensor S of $M \times_F N$ satisfies $R \cdot S = LQ(g, S)$ if and only if*

$$(\overline{R} \cdot H)_{abcd} = LQ(\overline{g}, H)_{abcd}, \tag{34}$$

$$\frac{F}{n-p}\left(\frac{\overline{K}}{p} - \frac{\widetilde{K}}{(n-p)F} + (n-p)L + \frac{1}{2F}\left(\mathrm{tr}(T) + \frac{(n-p-1)\Delta F}{2F}\right)\right) H_{ab} = H_{ac} H^c{}_b. \tag{35}$$

4. Not conformally flat 4-dimensional warped products satisfying $R \cdot C = LQ(g, C)$

PROPOSITION 1. *Suppose $R \cdot C = LQ(g, C)$ on $M \times_F N$, where $\dim M = 1$ and $\dim N = 3$. If C is non-zero at $x \in M \times_F N$ then $R \cdot R = LQ(g, R)$ at x.*

Proof. From (2) we have

$$(R \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = LQ(g, C)_{\alpha\beta\gamma\delta\lambda\mu},$$

whence, by (7), it follows that

$$(\widetilde{R} \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = \left(LF + \frac{\Delta F}{4F}\right) Q(\widetilde{g}, C)_{\alpha\beta\gamma\delta\lambda\mu}.$$

This, by an application of (25) and contraction with $\widetilde{g}^{\beta\gamma}$, yields

$$(\widetilde{R} \cdot \widetilde{S})_{\alpha\delta\lambda\mu} = \left(LF + \frac{\Delta F}{4F}\right) Q(\widetilde{g}, \widetilde{S})_{\alpha\delta\lambda\mu},$$

which, in view of Lemma 2(i) and Lemma 1, implies (20). Further, the relation $(R \cdot C)_{1\alpha\beta\gamma 1\delta} = LQ(g, C)_{1\alpha\beta\gamma 1\delta}$, in virtue of (6), (15), (24) and (21), turns into

$$H_{11}\left(\frac{1}{F}C_{\delta\alpha\beta\gamma} - \frac{\widetilde{K}}{6}\widetilde{G}_{\delta\alpha\beta\gamma} + \frac{1}{2}(\widetilde{g}_{\gamma\delta}\widetilde{S}_{\alpha\beta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma})\right) = 0.$$

Applying (25) and contracting the resulting equality with $\widetilde{g}^{\gamma\delta}$, we get $H_{11}(\widetilde{S}_{\alpha\beta} - \frac{\widetilde{K}}{3}\widetilde{g}_{\alpha\beta}) = 0$, which, by (24), (25) and the assumption $C(x) \neq 0$, gives $H_{11}(x) = 0$. Now Lemma 4 completes the proof.

PROPOSITION 2. *Suppose* $R \cdot C = LQ(g, C)$ *on* $M \times_F N$, *where* $\dim M = \dim N = 2$. *If* C *is non-zero at* $x \in M \times_F N$ *then* $R \cdot R = LQ(g, R)$ *and* $H = 0$ *at* x.

P r o o f. The relation $(R \cdot C)_{a\alpha\beta\gamma d\delta} = LQ(g, C)_{a\alpha\beta\gamma d\delta}$, by making use of (15), (27), (28), (6) and (21), gives $PH = 0$. Since $C(x) \neq 0$, it follows that $H(x) = 0$. But this, in view of Lemma 5, completes the proof.

PROPOSITION 3. *Suppose* $R \cdot C = LQ(g, C)$ *on* $M \times_F N$, *where* $\dim M = 3$ *and* $\dim N = 1$. *If* C *is non-zero at* $x \in M \times_F N$ *then* $R \cdot R = LQ(g, R)$ *at* x.

P r o o f. From the equality $(R \cdot C)_{4ab4cd} = LQ(g, C)_{4ab4cd}$, by making use of (15) and (29), it follows that

$$(\overline{R} \cdot W)_{abcd} = LQ(\overline{g}, W)_{abcd}. \tag{36}$$

Furthermore, the equality $(R \cdot C)_{4abcd4} = LQ(g, C)_{4abcd4}$, by an application of (6), (15), (29) and (21), yields

$$H^e{}_d C_{eabc} = \frac{1}{2F}(H_{bd}W_{ac} - H_{cd}W_{ab}), \tag{37}$$

whence, by (30), we get

$$\overline{g}_{ab}H^e{}_d W_{ec} - \overline{g}_{ac}H^e{}_d W_{eb} + 2(W_{ab}H_{cd} - W_{ac}H_{bd}) = 0. \tag{38}$$

Contracting this with $\overline{g}^{ad}$ and $\overline{g}^{cd}$ respectively, we obtain

$$H^e{}_a W_{eb} = H^e{}_b W_{ea}, \tag{39}$$

$$H^e{}_b W_{ea} = \tfrac{2}{3}\tau W_{ab} + \tfrac{1}{3}\rho\overline{g}_{ab}, \qquad \rho = H^{ef}W_{ef}, \qquad \tau = \mathrm{tr}(H), \tag{40}$$

respectively. Now (38) takes the form

$$(W_{ab}H_{cd} - W_{ac}H_{bd}) + \tfrac{1}{3}\tau(\overline{g}_{ab}W_{cd} - \overline{g}_{ac}W_{bd}) + \tfrac{1}{6}\rho\overline{G}_{dabc} = 0. \tag{41}$$

Transvecting the above equality with H^{ab} and using (39) and (40) we find

$$H_{cd} = \tfrac{2}{3}\tau^2 W_{cd} + \tfrac{1}{3}\tau\rho\overline{g}_{cd}. \tag{42}$$

From this we have $\rho\overline{R}\cdot H = \frac{2}{3}\tau^2\overline{R}\cdot W$ and, by (36),

$$\overline{R}\cdot H = \tfrac{2}{3}\tau^2 LQ(\overline{g}, W).$$

Applying (42) in the last equality, we get

$$\rho(\overline{R}\cdot H - LQ(\overline{g}, H)) = 0, \tag{43}$$

which, by (21), implies

$$\rho(\overline{R}\cdot T - LQ(\overline{g}, T)) = 0. \tag{44}$$

Thus (36), in virtue of (29) and (44), turns into

$$\rho(\overline{R}\cdot \overline{S} - LQ(\overline{g}, \overline{S})) = 0. \tag{45}$$

We have now the two possibilities: (a) $\rho(x) \neq 0$ and (b) $\rho(x) = 0$.

(a) In this case $\overline{R}\cdot\overline{S} = LQ(\overline{g}, \overline{S})$. Thus, in view of Lemma 1, (17) holds at x. Now, we prove that (18) also holds at x. (42) shows that this is trivial if $\tau(x) = 0$. Suppose that $\tau(x) \neq 0$. First of all, we note that from (41), by transvection with $H^a{}_f$ and an application of (40) and (42) the following relation can be obtained at x:

$$H_{bf}H_{cd} - H_{cf}H_{bd} = 0, \tag{46}$$

whence

$$H_{bf}H^f{}_d = \tau H_{bd}. \tag{47}$$

Next, transvecting the equality $\overline{C}_{eabc} = 0$ with $H^e{}_d$ we obtain

$$\begin{aligned} H^e{}_d\overline{R}_{eabc} = H_{cd}\overline{S}_{ab} - H_{bd}\overline{S}_{ac} + \overline{g}_{ab}\overline{S}_{ec}H^e{}_d - \overline{g}_{ac}\overline{S}_{eb}H^e{}_d \\ - \frac{\overline{K}}{2}(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}). \end{aligned} \tag{48}$$

The formula (42), by making use of (29), can be rewritten in the form

$$F\overline{S}_{ab} = \left(\frac{3}{2}\frac{1}{\tau^2}\rho - 1\right)H_{ab} + \left(FL - \frac{1}{2\tau}\rho\right)\overline{g}_{ab}, \tag{49}$$

which, by transvection with $H^b{}_d$ and the use of (47), yields

$$F\overline{S}_{ea}H^e{}_d = \left(\frac{1}{\tau}\rho - \tau + FL\right)H_{ad}.$$

Applying the last two equalities in (48) we get

$$\begin{aligned} H^e{}_d\overline{R}_{eabc} = \frac{1}{F}\left(\frac{3}{2\tau^2}\rho - 1\right)(H_{cd}H_{ab} - H_{bd}H_{ac}) \\ + \frac{1}{F}\left(\frac{1}{2\tau}\rho - \tau + 2FL - \frac{\overline{K}F}{2}\right)(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}), \end{aligned}$$

which, by (46), reduces to

$$H^e{}_d \overline{R}_{eabc} = \frac{1}{F}\left(\frac{1}{2\tau}\rho - \tau + 2FL - \frac{\overline{K}F}{2}\right)(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}). \tag{50}$$

This, by the Ricci identity, yields

$$\overline{R}\cdot H = \frac{1}{F}\left(\frac{1}{2\tau}\rho - \tau + 2FL - \frac{\overline{K}F}{2}\right)Q(\overline{g}, H). \tag{51}$$

Comparing (51) with (43) we obtain

$$\left(\frac{1}{F}\left(\frac{1}{2\tau}\rho - \tau + 2FL - \frac{\overline{K}F}{2}\right) - L\right)Q(\overline{g}, H) = 0\,,$$

whence

$$\left(\frac{1}{F}\left(\frac{1}{2\tau}\rho - \tau + 2FL - \frac{\overline{K}F}{2}\right) - L\right)\left(H - \frac{\tau}{3}\overline{g}\right) = 0. \tag{52}$$

If $(H - (\tau/3)\overline{g})(x) = 0$, then (49) gives $\overline{S} = (\overline{K}/3)\overline{g}$. But this, by (29), gives $W = 0$ and, consequently, $C(x) = 0$, which is a contradiction. So $(H - (\tau/3)\overline{g})(x) \neq 0$. Applying now (52) in (50) we get

$$H^e{}_d \overline{R}_{eabc} = L(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}). \tag{53}$$

Note that (46) can be expressed in the following form:

$$L(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}) = \frac{1}{2F}(T_{ac}H_{bd} - T_{ab}H_{cd}).$$

Thus (53) turns into (18).

(b) Since ρ vanishes at x, the formula (42) takes the form $\tau W = 0$, whence, by (29), (30) and the assumption $C(x) \neq 0$, we obtain the equality

$$\tau = \frac{\mathrm{tr}(T)}{2} + 3FL = 0 \tag{54}$$

at this point. The tensor W now takes the form

$$W_{ab} = F\overline{S}_{ab} - \frac{F\overline{K}}{3}\overline{g}_{ab} + H_{ab}. \tag{55}$$

The formula (40), by (54), gives

$$H^e{}_a W_{eb} = 0. \tag{56}$$

Thus (38) turns into

$$W_{ab}H_{cd} - W_{ac}H_{bd} = 0, \tag{57}$$

which can be rewritten in the form

$$F(\overline{S}_{ab}H_{cd} - \overline{S}_{ac}H_{bd}) = H_{ac}H_{bd} - H_{ab}H_{cd} + \frac{F\overline{K}}{3}(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}). \tag{58}$$

From (57), by transvection with $H^c{}_e$ and making use of (56), we find

$$H_{cd}H^c{}_e = 0. \tag{59}$$

Further, transvecting (55) with $H^a{}_d$ and applying (56) and (59) we get

$$H^c{}_a\overline{S}_{cd} = \frac{\overline{K}}{3}H_{ad}. \tag{60}$$

Next, transvecting (58) with $\overline{S}_e{}^b$ and using (60) we obtain

$$F\left(\overline{S}_e{}^b\overline{S}_{ab} - \frac{\overline{K}}{3}\overline{S}_{ea}\right)H_{cd}$$
$$= \frac{\overline{K}}{3}(H_{ac}H_{ed} - H_{ea}H_{cd}) + \frac{F\overline{K}}{3}\left(\overline{S}_{ac} - \frac{\overline{K}}{3}\overline{g}_{ac}\right)H_{ed},$$

which, by (58), turns into

$$\left(\overline{S}^2 - \frac{2\overline{K}\,\overline{S}}{3} + \frac{\overline{K}^2}{9}\overline{g}\right)H = 0, \quad \text{or}$$

$$\left(\left(\overline{S}^2 - \frac{\operatorname{tr}(\overline{S}^2)}{3}\overline{g}\right) - \left(\frac{\overline{K}}{6} + \frac{\overline{K}}{2}\right)\left(\overline{S} - \frac{\overline{K}}{3}\overline{g}\right) + \frac{1}{3}\left(\operatorname{tr}(\overline{S}^2) - \frac{\overline{K}^2}{3}\right)\overline{g}\right)H = 0$$

and

$$\left(\left(\overline{S}^2 - \frac{\operatorname{tr}(\overline{S}^2)}{3}\overline{g}\right) - \left(\frac{\overline{K}}{6} + \frac{\overline{K}}{2}\right)\left(\overline{S} - \frac{\overline{K}}{3}\overline{g}\right)\right)H = 0\,. \tag{61}$$

Suppose that $H(x) = 0$. Then, of course, (18) holds at x. The formula (36) turns into $\overline{R}\cdot\overline{S} = LQ(\overline{g},\overline{S})$. But this, in view of Lemma 2(i) and Lemma 1, implies (17). Consider now the case $H(x) \neq 0$. Then (61) gives

$$\overline{S}^2 - \frac{\operatorname{tr}(\overline{S}^2)}{3}\overline{g} = \left(\frac{\overline{K}}{6} + \frac{\overline{K}}{2}\right)\left(\overline{S} - \frac{\overline{K}}{3}\overline{g}\right).$$

But this, in view of Lemma 2(i) and Lemma 1, yields $\overline{R}\cdot\overline{R} = LQ(\overline{g},\overline{R})$ and

$$L = \overline{K}/6. \tag{62}$$

Thus the condition (17) is fulfilled. Finally, the identity (48), in virtue of (60), gives

$$H^e{}_d\overline{R}_{eabc} = H_{cd}\overline{S}_{ab} - H_{bd}\overline{S}_{ac} - \frac{\overline{K}}{6}(\overline{g}_{ab}H_{cd} - \overline{g}_{ac}H_{bd}),$$

which, by (58), turns into

$$FH^e{}_d\overline{R}_{eabc} = H_{ac}H_{bd} - H_{ab}H_{cd} + \frac{F\overline{K}}{6}(\overline{g}_{ab}H_{ac} - \overline{g}_{ac}H_{bd}).$$

This, by making use of (61) and (21), leads to (18). Our proposition is thus proved.

Combining Propositions 1–3 we obtain

Theorem 3. *Suppose* $\dim M \times_F N = 4$. *If at a point of* $M \times_F N$ *the tensor* C *is non-zero, then the relations* $R \cdot C = LQ(g, C)$ *and* $R \cdot R = LQ(g, R)$ *are equivalent at this point.*

The following corollary is a consequence of Theorem 2.

Corollary 1. *Let* $M \times_F N$ *be an analytic not conformally flat 4-dimensional warped product. Then the relations* $R \cdot C = LQ(g, C)$ *and* $R \cdot R = LQ(g, R)$ *are equivalent on* $M \times_F N$.

5. Non-Einstein 4-dimensional warped product satisfying $R{\cdot}S = LQ(g, S)$

Proposition 4. *Suppose* $R{\cdot}S = LQ(g, S)$ *on* $M{\times_F}N$, *where* $\dim M = 1$ *and* $\dim N = 3$. *If* $S - (K/4)g$ *is non-zero at* $x \in M \times_F N$, *then* $R \cdot R = LQ(g, R)$ *at* x.

Proof. The equality (33), in view of Lemma 2(i) and Lemma 1, turns into (20). Further, (32) yields

$$H_{11}\left(\widetilde{S}_{\alpha\beta} - \left(\frac{1}{2F}\Delta F - \operatorname{tr}(T)\right)\widetilde{g}_{\alpha\beta}\right) = 0,$$

whence $H(\widetilde{S} - (\widetilde{K}/3)\widetilde{g}) = 0$. If $H = 0$, then (18) and (19) are satisfied and Lemma 4 completes the proof. If $\widetilde{S} - (\widetilde{K}/3)\widetilde{g} = 0$, then $C = 0$, and our assertion, by Lemma 1, is also true.

Proposition 5. *Suppose* $R \cdot S = LQ(g, S)$ *on* $M \times_F N$, *where* $\dim M = \dim N = 2$. *If* $S - (K/4)g$ *is non-zero at* $x \in M \times_F N$ *then* $R{\cdot}R = LQ(g, R)$ *at* x.

Proof. The relations (34) and (35) take the forms

$$\left(L - \frac{\overline{K}}{2}\right)\left(H - \frac{\operatorname{tr}(H)}{2}\overline{g}\right) = 0\,, \tag{63}$$

$$H_{ca}H^{c}{}_{b} = \rho H_{ab} \tag{64}$$

respectively, where

$$\rho = \frac{F}{2}\left(\frac{\overline{K}}{2} - \frac{\widetilde{K}}{2F} + 2L + \frac{\operatorname{tr}(T)}{2F} + \frac{1}{4F^2}\Delta F\right). \tag{65}$$

In view of Lemma 2(iii), H satisfies the following identity at x:

$$\overline{g}_{bc}H_{ad} + \overline{g}_{ad}H_{bc} - \overline{g}_{ac}H_{bd} - \overline{g}_{bd}H_{ac} = \operatorname{tr}(H)(\overline{g}_{ad}\overline{g}_{bc} - \overline{g}_{ac}\overline{g}_{bd}). \tag{66}$$

Transvecting this with $H^{b}{}_{f}$ and using (64) we get

$$H_{cf}H_{ad} - H_{df}H_{ac} = (\operatorname{tr}(H) - \rho)(\overline{g}_{ad}H_{cf} - \overline{g}_{ac}H_{df})\,, \tag{67}$$

whence, by transvection with H^{cf} and making use of (64) we obtain

$$\rho(\operatorname{tr}(H)-\rho)\left(H-\frac{\operatorname{tr}(H)}{2}\overline{g}\right)=0. \tag{68}$$

Consider two possibilities:

(a) $\left(H-\frac{\operatorname{tr}(H)}{2}\overline{g}\right)(x)\neq 0$,

(b) $\left(H-\frac{\operatorname{tr}(H)}{2}\overline{g}\right)(x)=0$.

(a) In this case we have

$$L=\overline{K}/2, \tag{69}$$

$$\rho(\operatorname{tr}(H)-\rho)=0. \tag{70}$$

(a1) Suppose additionally that $\rho(x)=0$. Then from (64) and (67) it follows that $\operatorname{tr}(H)H_{ae}=0$ and, in consequence,

$$\operatorname{tr}(H)=0. \tag{71}$$

Now (67), by (21), (69) and (71), turns into (22). Further, (71) gives $L=-\operatorname{tr}(T)/(4F)$. Thus, from (65) we obtain $FL=\widetilde{K}/2-\Delta F/(4F)$. But this, together with (64) and (21), yields (23). Now Lemma 5 completes the proof.

(a2) Let $\rho(x)\neq 0$. Then, in virtue of (70), we get

$$\operatorname{tr}(H)=\rho. \tag{72}$$

Applying this, (21) and (69) in (67) we obtain (22). Further, (72) yields

$$\frac{\operatorname{tr}(T)}{4}+FL=\frac{FL}{2}-\frac{\widetilde{K}}{4}+\frac{\Delta F}{8F},$$

whence

$$\operatorname{tr}(H)=FL-\frac{\widetilde{K}}{2}+\frac{\Delta F}{4F}.$$

But this, together with (72) and (64), yields (23). Now Lemma 5 completes the proof in this case.

(b) Now (64) takes the form

$$\operatorname{tr}(H)\left(\frac{\operatorname{tr}(H)}{2}-\rho\right)=0. \tag{73}$$

If $\operatorname{tr}(H)=0$, then also $H=0$ and Lemma 5 completes the proof. If $\operatorname{tr}(H)\neq 0$, then (73) gives $\operatorname{tr}(H)=2\rho$, whence we get $F\overline{K}/2=\widetilde{K}/2-\Delta F/(4F)$. Moreover, the equality $H=\frac{1}{2}\operatorname{tr}(H)\overline{g}$ yields $T=\frac{1}{2}\operatorname{tr}(T)\overline{g}$. Now (8) and (9) lead to $S=(K/4)g$, a contradiction. This completes the proof.

PROPOSITION 6. *Suppose* $R\cdot S = LQ(g,S)$ *on* $M\times_F N$, *where* $\dim M = 3$ *and* $\dim N = 1$. *If* $S-(K/4)g$ *and* C *are non-zero at* $x\in M\times_F N$, *then* $R\cdot R = LQ(g,R)$ *at* x.

Proof. We consider two cases:

(a) $(\overline{S}-(\overline{K}/3)\overline{g})(x) = 0$,
(b) $(\overline{S}-(\overline{K}/3)\overline{g})(x) \neq 0$.

(a) The relation (31) turns into $(L-\overline{K}/6)Q(\overline{g},H) = 0$, which, by Remark 1, yields

$$\left(L-\frac{\overline{K}}{6}\right)\left(H-\frac{\operatorname{tr}(H)}{3}\overline{g}\right) = 0. \tag{74}$$

If $H-(\operatorname{tr}(H)/3)\overline{g} = 0$, then (29) and (30) yield $C = 0$, a contradiction. If $H-(\operatorname{tr}(H)/3)\overline{g}\neq 0$, then from (74) we obtain $L = \overline{K}/6$. Further, (32) takes the form $H_{ca}H^c{}_b = \frac{1}{2}(F\overline{K}+\operatorname{tr}(T))H_{ab}$. Let B be a $(0,4)$-tensor with local components $B_{abcd} = H_{ad}H_{bc}-H_{ac}H_{bd}$. We note that $\operatorname{Ricc}(B) = 0$ and $K(B) = 0$. Thus Lemma 2(ii) implies $B_{abcd} = 0$, whence

$$\frac{\overline{K}}{3}(\overline{g}_{ab}H_{cd}-\overline{g}_{ac}H_{bd}) = \frac{1}{F}(T_{ac}H_{bd}-T_{ab}H_{cd}).$$

But this turns into (18). Now Lemma 4 completes the proof.

(b) We rewrite (32) in the form

$$\begin{aligned} H^2{}_{ab} &= \left(FL+\frac{\operatorname{tr}(T)}{2}\right)H_{ab}+FA_{ab},\\ H^2{}_{ab} &= H_{ca}H^c{}_b, \qquad A_{ab} = \overline{S}_{ca}H^c{}_b. \end{aligned} \tag{75}$$

Now, transvecting the identity $\overline{C}_{ebcd} = 0$ with $H_a{}^e$ and $\overline{S}_a{}^e$ respectively, we obtain

$$\begin{aligned} H_a{}^e\overline{R}_{ebcd} &= H_{ad}\overline{S}_{bc}-H_{ac}\overline{S}_{bd}+\overline{g}_{bc}A_{ad}\\ &\quad -\overline{g}_{bd}A_{ac}-\frac{\overline{K}}{2}(\overline{g}_{bc}H_{ad}-\overline{g}_{bd}H_{ac}),\\ \overline{S}_a{}^e\overline{R}_{ebcd} &= \overline{S}_{ad}\overline{S}_{bc}-\overline{S}_{ac}\overline{S}_{bd}+\overline{g}_{bc}\overline{S}^2{}_{ad}\\ &\quad -\overline{g}_{bd}\overline{S}^2{}_{ac}-\frac{\overline{K}}{2}(\overline{g}_{bc}\overline{S}_{ad}-\overline{g}_{bd}\overline{S}_{ac}) \end{aligned} \tag{76}$$

respectively, where $\overline{S}^2{}_{ad} = \overline{S}_{ea}\overline{S}^e{}_d$. From the last two relations, by symmetrization in a, b and making use of the Ricci identity and (31), it follows that

$$Q\left(\overline{g},F\overline{S}^2-F\left(\frac{\overline{K}}{2}+L\right)\overline{S}-A+\left(\frac{\overline{K}}{2}+L\right)H\right)_{abcd} = -Q(H,\overline{S})_{abcd}. \tag{77}$$

Contracting (77) with $\overline{g}^{ad}$ we get

$$(78)\quad F\overline{S}^2 = \frac{F\operatorname{tr}(\overline{S}^2)}{3}\overline{g} + F\left(\frac{\overline{K}}{2}+L\right)\left(\overline{S}-\frac{\overline{K}}{3}\overline{g}\right) - \frac{\operatorname{tr}(H)}{3}\overline{S} + \frac{\overline{K}}{3}H + \left(A - \frac{\operatorname{tr}(A)}{3}\overline{g}\right) - \left(\frac{\overline{K}}{2}+L\right)\left(H - \frac{\operatorname{tr}(H)}{3}\overline{g}\right).$$

Substituting this in (77) we find

$$(79)\qquad Q\left(\overline{S}-\frac{\overline{K}}{3}\overline{g}, H - \frac{\operatorname{tr}(H)}{3}\overline{g}\right) = 0.$$

We may assume that $H - (\operatorname{tr}(H)/3)\overline{g} \neq 0$. Of course, if $H = 0$, then $R\cdot R = LQ(g,R)$ at x. If $H \neq 0$ and $H - (\operatorname{tr}(H)/3)\overline{g} = 0$, then (75) yields $\overline{S} = (\overline{K}/3)\overline{g}$, a contradiction. Now (79), in view of Lemma 3, implies

$$(80)\qquad \overline{S}-\frac{\overline{K}}{3}\overline{g} = \lambda\left(H - \frac{\operatorname{tr}(H)}{3}\overline{g}\right),\qquad \lambda\in\mathbb{R}-\{0\}.$$

Thus (31) yields $(\lambda F - 1)(\overline{R}\cdot\overline{S} - LQ(\overline{g},\overline{S})) = 0$. Assume that

$$(81)\qquad \lambda F = 1;$$

then (80) gives $A = \frac{1}{3}(\overline{K} - \operatorname{tr}(H)/F)H + H^2/F$. Substituting this into (75) we obtain $(\overline{K} + \operatorname{tr}(T)/F)H = 0$, whence $K + \operatorname{tr}(T)/F = 0$. But the last relation, together with (80), (81), (8), (9) and (11), leads to $S = (K/4)g$, a contradiction. Thus $\lambda F - 1 \neq 0$ and

$$(82)\qquad \overline{R}\cdot\overline{S} = LQ(\overline{g},\overline{S}).$$

Now, in view of Lemma 1, we obtain (17). Furthermore, the equality (31), in virtue of (82), (76)–(78), turns into

$$Q\left(\overline{g}, \left(L+\frac{\overline{K}}{6}\right)H + \frac{\operatorname{tr}(H)}{3}\overline{S} - A\right) = 0,$$

whence, by (80), it follows that

$$A = \frac{\operatorname{tr}(A)}{3}\overline{g} + \left(\frac{\lambda\operatorname{tr}(H)}{3} + L + \frac{\overline{K}}{6}\right)\left(H - \frac{\operatorname{tr}(H)}{3}\overline{g}\right).$$

On the other hand, combining (75) and (80), we find

$$(83)\qquad \left(\frac{1}{\lambda} - F\right)A = \frac{1}{3}\left(\operatorname{tr}(T) + \frac{\overline{K}}{\lambda}\right)H.$$

But the last two relations yield

$$(\lambda\operatorname{tr}(T) - 6L(1-\lambda F) + \overline{K})(1+\lambda F)\left(H - \frac{\operatorname{tr}(H)}{3}\overline{g}\right) = 0,$$

whence

$$(\lambda\,\mathrm{tr}(T) - 6L(1-\lambda F) + \overline{K})(1+\lambda F) = 0.$$

Suppose that $1+\lambda F = 0$. Then (80), (29) and (30) give $C = 0$, a contradiction. Thus we have $\lambda\,\mathrm{tr}(T) - 6L(1-\lambda F) + \overline{K} = 0$, or, equivalently,

$$\frac{1}{\lambda}\left(L - \frac{\overline{K}}{6}\right) = \frac{\mathrm{tr}(H)}{3}\,. \tag{84}$$

Thus (83) takes the form

$$A = 2LH. \tag{85}$$

Substituting this into (75) we obtain $H^2 = \mathrm{tr}(H)H$. Let B be a $(0,4)$-tensor with local components $B_{abcd} = H_{ad}H_{bc} - H_{ac}H_{bd}$. Evidently, $\mathrm{Ricc}(B) = 0$ and $K(B) = 0$. Thus, Lemma 2(ii) implies

$$B_{abcd} = 0. \tag{86}$$

From (80), by (86), we obtain

$$H_{ad}\overline{S}_{bc} - H_{ac}\overline{S}_{bd} = \tfrac{1}{3}(\overline{K} - \lambda\,\mathrm{tr}(H))(\overline{g}_{bc}H_{ad} - \overline{g}_{bd}H_{ac}).$$

Substituting this and (85) into (76) we find

$$H_a{}^e\overline{R}_{ebcd} = \left(2L - \frac{\overline{K}}{6} - \frac{\lambda\,\mathrm{tr}(H)}{3}\right)(\overline{g}_{bc}H_{ad} - \overline{g}_{bd}H_{ac}),$$

whence, by (84), we get $H_a{}^e\overline{R}_{ebcd} = L(\overline{g}_{bc}H_{ad} - \overline{g}_{db}H_{ac})$. But this, by (86), turns into (18). Now Lemma 4 completes the proof.

Combining Propositions 4–6 and Lemma 1 we obtain

THEOREM 4. *Suppose* $\dim M\times_F N = 4$. *If* $S-(K/4)g$ *is non-zero at a point of* $M\times_F N$, *then the relations* $R\cdot S = LQ(g,S)$ *and* $R\cdot R = LQ(g,R)$ *are equivalent at this point.*

This theorem yields

COROLLARY 2. *Let* $M\times_F N$ *be an analytic non-Einstein* 4-*dimensional warped product. Then the relations* $R\cdot S = LQ(g,S)$ *and* $R\cdot R = LQ(g,R)$ *are equivalent on* $M\times_F N$.

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