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ON FOURTH ORDER BOUNDARY VALUE PROBLEMS ARISING IN BEAM ANALYSIS

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Abstract. We consider a general fourth order nonlinear ordinary differential equation together with two point boundary conditions which occur in the deflection of a beam rigidly fastened at left and simply fastened at right. For this general boundary value problem, we provide necessary and sufficient conditions for the existence and uniqueness of the solutions. We also obtain upper estimates on the length of interval so that Newton's method converges quadratically to the unique solutions. Some of our results are the best possible in their frame.

1. Introduction. Imposing some ideal conditions, the deflection of a beam rigidly fastened at left and simply fastened at right leads to a fourth order differential equation (in simplest form) [4, 7-9, 11, 19, 22]

$$x'''' = \alpha^4 x \quad (\alpha > 0) \tag{1.1}$$

together with the two point boundary conditions

$$x(a) = A, \;\; x'(a) = B, \;\; x(b) = C, \;\; x''(b) = D. \tag{1.2}$$

The problem (1.1), (1.2) has a unique solution if and only if

$$\cosh\alpha(b-a)\sin\alpha(b-a) - \cos\alpha(b-a)\sinh\alpha(b-a) = p \neq 0 \tag{1.3}$$

and can be expressed in terms of elementary functions

$$\begin{aligned} x(t) = A\cosh\alpha(t-a) + \frac{B}{\alpha}\sinh\alpha(t-a) + \beta\,(\cos\alpha(t-a) - \cosh\alpha(t-a)) \\ + \gamma\,(\sin\alpha(t-a) - \sinh\alpha(t-a)) \end{aligned} \tag{1.4}$$

where the constants β and γ appear as

$$\begin{aligned} \beta = \frac{1}{2p}\Big[\Big(2A\cosh\alpha(b-a) + \frac{2B}{\alpha}\sinh\alpha(b-a) - \frac{D}{\alpha^2} - C\Big)\sin\alpha(b-a) \\ + (\frac{D}{\alpha^2} - C)\sinh\alpha(b-a)\Big], \end{aligned}$$

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$$\gamma = \frac{-1}{2p}\left[\left(2A\cosh\alpha(b-a) + \frac{2B}{\alpha}\sinh\alpha(b-a) - \frac{D}{\alpha^2} - C\right)\cos\alpha(b-a) + (\frac{D}{\alpha^2} - C)\cosh\alpha(b-a)\right].$$

If $p = 0$, and any one of the constants A, B, C or D is nonzero, then (1.1), (1.2) has no solution, whereas an infinite number of solutions if $A = B = C = D = 0$; these solutions can be written as

$$x(t) = c\left[\sinh\alpha(b-a)\sin\alpha(b-t) - \sin\alpha(b-a)\sinh\alpha(b-t)\right] \tag{1.5}$$

where c is an arbitrary constant.

Motivated with the above problem (1.1), (1.2) in this paper, we shall consider the following fourth-order nonlinear differential equation

$$x'''' = f(t, x, x', x'', x''') \tag{1.6}$$

together with the boundary conditions (1.2). The function f is assumed to be continuous in all of its arguments. We begin with a series of lemmas in Section 2, which are repeatedly needed later. One important contribution here is the inequalities (2.3) which are best possible. In Section 3. we shall provide necessary and sufficient conditions for the existence and uniqueness of the solutions of the boundary value problem (1.6), (1.2). These results are quite sharp and give at least rough lower as well as upper estimates on the solutions; i.e., the regions in which the solutions exist. In general, even if it is known that (1.6), (1.2) has a unique solution, it is not possible to find it explicitly. Faced with this difficulty, we resort to numerical methods, and for this, we note that shooting type methods proposed in our earlier work [2,4] can be used directly or after converting it into its equivalent first order system. Further, in Section 3, our results in Theorem 3.8 and Remark 3.4 provide upper estimates on the length of the interval $(b-a)$ so that Picard's iterative method converges to the solution of (1.6), (1.2). For this, explicit error estimates are also obtained. Newton's method (also known as Quasilinearization [3]) is used in Section 4 to construct the solution of our problem (1.6), (1.2). The quadratic convergence of the method is proved, and once again we obtain explicit error estimates. Unfortunately, our results in Sections 3 and 4 are not the best possible for the general differential equation (1.6). In Section 5, we assume that the function f is independent of x', x'' and x''', i.e.,

$$x'''' = f(t, x) \tag{1.7}$$

and obtain several results for the boundary value problem (1.7), (1.2) which are best possible. An important contribution here is our Theorem 5.5, which besides proving the existence of a solution of (1.7), (1.2) also provides two sided monotonic convergence of the Picard method. Obviously, from the computational point of view, monotone convergence has superiority over ordinary convergence and several monotonic iterative schemes for different problems have been developed and analyzed in the recent monograph [15]. Finally, we remark that the existence, uniqueness and constructive methods for fourth order boundary value problems have attracted the attention of several workers [1, 4-6, 10, 12-14, 16-18, 20, 21], particularly because they occur in a wide variety of applications.

2. Preliminary results.

Lemma 2.1. *The Green's function $g(t,s)$ of the boundary value problem*

$$x'''' = 0 \tag{2.1}$$

$$x(a) = x'(a) = x(b) = x''(b) = 0 \tag{2.2}$$

can be written as

$$g(t,s) = \frac{1}{12(b-a)^3}\begin{cases} (s-a)^2(b-t)\big[(s-a)(b-t)^2 + 3(b-a)^2(b-s) - 3(b-a)(b-t)^2\big], \\ \qquad a \le s \le t, \\ (t-a)^2(b-s)\big[(t-a)(b-s)^2 + 3(b-a)^2(b-t) - 3(b-a)(b-s)^2\big], \\ \qquad t \le s \le b, \end{cases}$$

and

(1) $g(t,s) \ge 0$ *for all* $(t,s) \in [a,b] \times [a,b]$

(2) $\int_a^b |g^{(i)}(t,s)|\,ds \le c_i(b-a)^{4-i};\quad i = 0,1,2,3$

where

$$c_0 = \frac{39 + 55\sqrt{33}}{65536},\quad c_1 = \frac{1}{48},\quad c_2 = \frac{1}{8},\quad c_3 = \frac{5}{8}.$$

Proof: The proof involves computation which is tedious, though elementary.

Corollary 2.2. *Let $x(t) \in C^{(4)}[a,b]$, satisfying (2.2). Then, the following inequalities hold*

$$|x^{(i)}(t)| \le c_i(b-a)^{4-i} \max_{a \le t \le b} |x''''(t)|;\quad i = 0,1,2,3. \tag{2.3}$$

Proof: Any such function can be written as

$$x(t) = \int_a^b g(t,s)x''''(s)\,ds$$

and hence

$$|x^{(i)}(t)| \le \Big(\int_a^b |g^{(i)}(t,s)|\,ds\Big) \max_{a \le t \le b} |x''''(t)|.$$

Remark 2.1. In (2.3), the constants c_i; $i = 0,1,2,3$ are the best possible as they are exact for the function

$$x(t) = \frac{1}{48}(t-a)^2(b-t)[3(b-a) - 2(t-a)]$$

and only for this function upto a constant factor.

Lemma 2.3. *The unique polynomial $P_3(t)$ of degree 3, satisfying (1.2) can be written as*

$$\begin{aligned} P_3(t) =& \frac{(b-t)}{2(b-a)^3}[3(b-a)^2 - (b-t)^2]A + \frac{(t-a)(b-t)}{2(b-a)^2}[2(b-a) - (t-a)]B \\ &+ \frac{(t-a)^2}{2(b-a)^3}[3(b-a) - (t-a)]C - \frac{(t-a)^2(b-t)}{4(b-a)}D \end{aligned} \tag{2.4}$$

and

$$|P_3(t)| \le |A| + \frac{1}{3\sqrt{3}}(b-a)|B| + |C| + \frac{1}{27}(b-a)^2|D|$$

$$|P_3'(t)| \le \frac{1}{(b-a)}\left[\frac{3}{2}|A| + (b-a)|B| + \frac{3}{2}|C| + \frac{11}{9}(b-a)^2|D|\right]$$

$$|P_3''(t)| \le \frac{1}{(b-a)^2}\left[3|A| + 3(b-a)|B| + 3|C| + (b-a)^2|D|\right]$$

$$|P_3'''(t)| \le \frac{1}{(b-a)^3}\left[3|A| + 3(b-a)|B| + 3|C| + (b-a)^2|D|\right].$$

Lemma 2.4. *For the differential equation*

$$x'''' = \lambda x \tag{2.5}$$

together with the boundary conditions (2.2), the following hold:

(1) $\lambda \le 0$ *is not an eigenvalue*

(2) $\lambda = \lambda_n > 0$ *is an eigenvalue provided*

$$\tan \lambda_n^{1/4}(b-a) = \tanh \lambda_n^{1/4}(b-a)$$

(3) *the eigenfunction* $\phi_1(t)$ *corresponding to the first eigenvalue* λ_1 *is of fixed sign in* (a, b) *and can be expressed as*

$$\phi_1(t) = \sinh \lambda_1^{1/4}(b-a)\sin \lambda_1^{1/4}(b-t) - \sin \lambda_1^{1/4}(b-a)\sinh \lambda_1^{1/4}(b-t) \ge 0$$

(4) $$\int_a^b |g(t,s)|\,|\phi_1(s)|\,ds = \int_a^b g(t,s)\phi_1(s)\,ds = \frac{1}{\lambda_1}|\phi_1(t)| = \frac{1}{\lambda_1}\phi_1(t),$$

$$\lambda_1(b-a)^4 = 237.721069\ldots.$$

Proof: Let $\lambda^* < 0$ be an eigenvalue and $\phi^*(t)$ be the corresponding eigenfunction, then

$$\phi^*(t)\phi''''^*(t) = \lambda^*[\phi^*(t)]^2,$$

and integration by parts provides

$$\int_a^b |\phi''^*(t)|^2\,dt = \lambda^* \int_a^b |\phi''^*(t)|^2\,dt$$

which is possible only if $\phi''^*(t) = 0$ almost everywhere. But then, $\phi^*(t) \equiv 0$ from the boundary conditions (2.2).

The rest of the parts follow by direct computation (cf. see introduction).

Remark 2.2. Since the function

$$\int_a^b |g(t,s)|\,ds = \int_a^b g(t,s)\,ds = \frac{1}{48}(t-a)^2(b-t)[3(b-a) - 2(t-a)]$$

as well as $\phi_1(t)$ is nonnegative and have the same zeros, there exists a smallest possible constant m such that

$$\int_a^b |g(t,s)|\,ds = m|\phi_1(t)|. \tag{2.6}$$

Lemma 2.5. [4] *Let $(E, \leq)$ be a partially ordered space and $x_0 \leq y_0$ be two elements of E. $[x_0, y_0]$ denotes the interval $\{x \in E : x_0 \leq x \leq y_0\}$. Let $T : [x_0, y_0] \to E$ be an isotone operator ($T(x) \leq T(y)$, whenever $x \leq y$) and let it possess the properties*

(i) $x_0 \leq T(x_0)$

(ii) *the (nondecreasing) sequence $\{T^n(x_0)\}$ where $T^0(x_0) = x_0$, $T^{n+1}(x_0) = T[T^n(x_0)]$ for each $n = 0, 1, \dots$ is well defined, i.e., $T^n(x_0) \leq y_0$ for each natural n*

(iii) *the sequence $\{T^n(x_0)\}$ has $\sup x \in E$, i.e., $T^n(x_0) \uparrow x$*

(iv) $T^{n+1}(x_0) \uparrow T(x)$

(i)' $T(y_0) \leq y_0$

(ii)' *the (nonincreasing) sequence $\{T^n(y_0)\}$ is well defined, i.e., $T^n(y_0) \geq x_0$ for each natural n*

(iii)' *the sequence $\{T^n(y_0)\}$ has $\inf y \in E$, i.e., $T^n(y_0) \downarrow y$*

(iv)' $T^{n+1}(y_0) \downarrow T(y)$.

Then, $x = T(x)$ and for any other fixed point $z \in [x_0, y_0]$ of T, $x \leq z$ is true. (Then, $y = T(y)$ and for any other fixed point $z \in [x_0, y_0]$ of T, $z \leq y$ is valid). Moreover, if T possesses both properties (i) and (i)', then the sequences $\{T^n(x_0)\}$, $\{T^n(y_0)\}$ are well defined and if further, T has the properties (iii), (iii)' and (iv), (iv)', then

$$x_0 \leq T(x_0) \leq \cdots \leq T^n(x_0) \leq \cdots \leq x \leq y \leq \cdots \leq T^n(y_0) \leq \cdots \leq T(y_0) \leq y_0$$

and $x = T(x)$, $y = T(y)$, also any other fixed point $z \in [x_0, y_0]$ of T satisfies $x \leq z \leq y$.

Lemma 2.6. [4] *Let $M > 0$ and $\{x_n(t)\}$ be a sequence of functions in $C^{(4)}[a, b]$ such that $|x_n(t)| \leq M$ and $|x_n''''(t)| \leq M$ for all n. Then, there exists a subsequence $\{x_{n(j)}(t)\}$ such that $\{x_{n(j)}^{(i)}(t)\}$ converges uniformly on $[a, b]$ for each i, $0 \leq i \leq 3$.*

3. Existence and uniqueness.

Theorem 3.1. *Suppose that*

(i) *$K_i > 0$, $0 \leq i \leq 3$ are given real numbers and let Q be the maximum of $|f(t, x_0, x_1, x_2, x_3)|$ on the compact set: $[a, b] \times D_0$, where*

$$D_0 = \{(x_0, x_1, x_2, x_3) : |x_i| \leq 2K_i, \quad 0 \leq i \leq 3\}$$

(ii) $(b - a) \leq (K_i/Qc_i)^{1/4-i}$, $0 \leq i \leq 3$

(iii) $\max_{a \leq t \leq b} |P_3^{(i)}(t)| \leq K_i$, $0 \leq i \leq 3$.

Then, the boundary value problem (1.6), (1.2) has a solution in D_0.

Proof: We begin with the observation that the boundary value problem (1.6), (1.2) is equivalent to the following Fredholm type of integral equation.

$$x(t) = P_3(t) + \int_a^b g(t, s) f(s, x(s), x'(s), x''(s), x'''(s))\, ds\,. \tag{3.1}$$

Next, we define the set

$$B[a, b] = \left\{ x(t) \in C^{(3)}[a, b] : \|x^{(i)}\| = \max_{a \leq t \leq b} |x^{(i)}(t)| \leq 2K_i, \quad 0 \leq i \leq 3 \right\}.$$

It is easy to verify that $B[a,b]$ is a closed convex subset of the Banach space $C^{(3)}[a,b]$. Consider an operator $T : C^{(3)}[a,b] \to C^{(4)}[a,b]$ as follows:

$$(Tx)(t) = P_3(t) + \int_a^b g(t,s)f(s,x(s),x'(s),x''(s),x'''(s))\,ds. \tag{3.2}$$

Obviously, any fixed point of (3.2) is a solution of (1.6), (1.2).

Let $x(t) \in B[a,b]$, then $(Tx)(t) - P_3(t)$ satisfies the conditions of Corollary 2.2 and

$$(Tx)''''(t) - P_3''''(t) = (Tx)''''(t) = f(t,x(t),x'(t),x''(t),x'''(t))$$

thus

$$\max_{a\le t\le b} |(Tx)''''(t)| \le Q.$$

Hence, from Corollary 2.2, it follows that

$$\|(Tx)^{(i)} - P_3^{(i)}\| \le c_i(b-a)^{4-i}Q, \quad 0 \le i \le 3$$

which also implies that

$$\begin{aligned}\|(Tx)^{(i)}\| &\le \|P_3^{(i)}\| + c_i(b-a)^{4-i}Q \\ &\le K_i + K_i = 2K_i, \quad 0 \le i \le 3.\end{aligned} \tag{3.3}$$

Thus, T maps $B[a,b]$ into itself. Further, the inequalities (3.3) imply that the sets $\{(Tx)^{(i)}(t) : x(t) \in B[a,b]\}$, $0 \le i \le 3$ are uniformly bounded and equicontinuous on $[a,b]$. Hence, $\overline{TB}[a,b]$ is compact follows from the Ascoli-Arzela theorem. The Schauder fixed point theorem is applicable and a fixed point of T in D_0 exists.

Corollary 3.2. *Let the conditions of Theorem 3.1 be satisfied. Then, for arbitrarily given* $\epsilon > 0$ *there is a solution* $x(t)$ *of (1.6), (1.2) such that* $|x^{(i)}(t) - P_3^{(i)}(t)| \le \epsilon$, $0 \le i \le 3$ *provided* $(b-a)$ *is sufficiently small.*

Proof: Let $x(t)$ be a solution of the problem (1.6), (1.2) and $x(t) \in B[a,b]$, then

$$|x^{(i)}(t) - P_3^{(i)}(t)| \le c_i(b-a)^{4-i}Q, \quad 0 \le i \le 3.$$

Thus, if

$$(b-a) \le \left(\frac{\epsilon}{c_iQ}\right)^{1/4-i}, \quad 0 \le i \le 3$$

the Corollary follows.

Corollary 3.3. *Let the conditions (i), (ii) of Theorem 3.1 be satisfied, and let* $\phi(t) \in C^{(3)}[a,b]$ *be a given function. Then, the differential equation (1.6), together with*

$$x(a) = \phi(a), \;\; x'(a) = \phi'(a), \;\; x(b) = \phi(b), \;\; x''(b) = \phi''(b) \tag{3.4}$$

has a solution, if

$$\sum_{j=1}^{3} M_j(b-a)^{j-i} \le K_i, \quad 0 \le i \le 3 \tag{3.5}$$

where

$$M_j = \max_{a \le t \le b} |\phi^{(j)}(t)|.$$

Proof: We need to verify that the condition (iii) of Theorem 3.1 is satisfied. For this, in $P_3(t)$ we let $A = \phi(a)$, $B = \phi'(a)$, $C = \phi(b)$ and $D = \phi''(b)$, and note that $\phi(a) - P_3(a) = \phi'(a) - P_3'(a) = \phi(b) - P_3(b) = \phi''(b) - P_3''(b) = 0$. Hence, by Rolle's theorem, there exist points $t_1 \in (a,b)$, $t_2 \in (a,t_1)$ and $t_3 \in (t_2,b)$ such that $\phi'(t_1) - P_3'(t_1) = \phi''(t_2) - P_3''(t_2) = \phi'''(t_3) - p_3'''(t_3) = 0$. Thus, we find that

$$|P_3'''(t)| = |P_3'''(t_3)| = |\phi'''(t_3)| \le \max_{a \le t \le b} |\phi'''(t)| = M_3. \tag{3.6}$$

Next, since

$$-P_3''(t) = -\phi''(b) + \int_t^b P_3'''(s)\, ds$$

an application of (3.6) gives

$$|P_3''(t)| \le M_2 + M_3(b-a). \tag{3.7}$$

Similarly, from

$$P_3'(t) = \phi'(a) + \int_a^t P_3''(s)\, ds$$

and (3.7), it follows that

$$|P_3'(t)| \le M_1 + M_2(b-a) + M_3(b-a)^2. \tag{3.8}$$

Finally, using (3.8) in

$$P_3(t) = \phi(a) + \int_a^t P_3'(s)\, ds$$

we get

$$|P_3(t)| \le M_0 + M_1(b-a) + M_2(b-a)^2 + M_3(b-a)^3. \tag{3.9}$$

Corollary 3.4. *Assume that the function $f(t, x_0, x_1, x_2, x_3)$ on $[a,b] \times R^4$ satisfies the following condition*

$$|f(t, x_0, x_1, x_2, x_3)| \le L + \sum_{i=0}^{3} L_i |x_i|^{\alpha_i} \tag{3.10}$$

where L, L_i, $0 \le i \le 3$ are nonnegative constants, and $0 \le \alpha_0, \alpha_1, \alpha_2, \alpha_3 < 1$. Then, for any function $\phi(t) \in C^{(3)}[a,b]$ the boundary value problem (1.6), (3.4) has a solution.

Proof: For $x(t) \in B[a,b]$ the condition (3.10) implies that

$$\left|f(t, x(t), x'(t), x''(t), x'''(t)))\right| \le L + \sum_{i=0}^{3} L_i (2K_i)^{\alpha_i} = Q_1, \text{ say.}$$

Now, Corollary 3.3 is applicable by choosing k_i, $0 \le i \le 3$ sufficiently large so that

$$c_i(b-a)^{4-i} Q_1 \le K_i, \quad 0 \le i \le 3$$

and

$$\sum_{j=i}^{3} M_j (b-a)^{j-i} \le K_i, \quad 0 \le i \le 3.$$

Corollary 3.5. *Assume that there exists a positive real valued function $\psi(y_0, y_1, y_2, y_3)$ defined for $y_i \geq 0$, $0 \leq i \leq 3$ which is nondecreasing in each variable and such that*

$$|f(t, x_0, x_1, x_2, x_3)| \leq \psi(|x_0|, |x_1|, |x_2|, |x_3|) \tag{3.11}$$

for all (t, x_0, x_1, x_2, x_3) on $[a, b] \times R^4$. If

$$\sum_{i=0}^{3} \frac{y_i}{\psi(y_0, y_1, y_2, y_3)} \to \infty \quad \text{as} \quad \sum_{i=0}^{3} y_i \to \infty, \tag{3.12}$$

then for any function $\psi(t) \in C^{(3)}[a, b]$ the boundary value problem (1.6), (3.4) has a solution.

Proof: From (3.12), for sufficiently large y_i it follows that

$$\frac{y_i}{\psi(y_0, y_1, y_2, y_3)} \geq c_i(b-a)^{4-i}, \quad 0 \leq i \leq 3$$

and hence, from (3.11) we have

$$\begin{aligned} & c_i(b-a)^{4-i} \max_{a \leq t \leq b} |f(t, x(t), x'(t), x''(t), x'''(t))| \\ & \leq c_i(b-a)^{4-i} \max_{a \leq t \leq b} \psi\left(|x(t)|, |x'(t)|, |x''(t)|, |x'''(t)|\right) \\ & \leq |x^{(i)}(t)|, \quad 0 \leq i \leq 3. \end{aligned}$$

The rest of the proof is similar to that of Corollary 3.4. In fact, condition (3.10) is a particular case of (3.11) for which (3.12) is readily satisfied.

Remark 3.1. Theorem 3.1 is a local existence theorem, whereas Corollary 3.4, as well as Corollary 3.5 does not require any condition on the length of the interval or the boundary conditions. As one should expect from the problem (1.1), (1.2) in Section 1, if in (3.10) $\alpha_i = 1$, $0 \leq i \leq 3$ then the solution will exist only when the length of the interval and/or the boundary conditions are restricted. This is the content of our next result.

Theorem 3.6. *Suppose that the function $f(t, x_0, x_1, x_2, x_3)$ on $[a, b] \times D_1$ satisfies the following condition*

$$|f(t, x_0, x_1, x_2, x_3)| \leq L + \sum_{i=0}^{3} L_i |x_i| \tag{3.13}$$

where

$$D_1 = \left\{ (x_0, x_1, x_2, x_3) : |x_i| \leq c_i(b-a)^{4-i} \frac{L + \ell}{1 - \theta} + \max_{a \leq t \leq b} |P_3^{(i)}(t)|, \ 0 \leq i \leq 3 \right\}$$

and

$$\theta = \sum_{i=0}^{3} L_i c_i (b-a)^{4-i} < 1 \tag{3.14}$$

$$\ell = \max_{a \leq t \leq b} \sum_{i=0}^{3} L_i |P_3^{(i)}(t)|. \tag{3.15}$$

Then, the boundary value problem (1.6), (1.2) has a solution in D_1.

Proof: Let $y(t) = x(t) - P_3(t)$, so that the boundary value problem is equivalent to the following:

$$y''''(t) = f\big(t, y(t) + P_3(t), y'(t) + P_3'(t), y''(t) + P_3''(t), y'''(t) + P_3'''(t)\big) \tag{3.16}$$

$$y(a) = y'(a) = y(b) = y''(b) = 0. \tag{3.17}$$

Define S as the set of functions four times differentiable on $[a, b]$ and satisfying (3.17). If we introduce in S the norm $\|y\| = \max_{a\le t\le b}|y''''(t)|$ ($y''''(t) \equiv 0$ implies $y(t) \equiv 0$), then S becomes a Banach space. We define a mapping $T : S \to S$ as follows

$$(Ty)(t) = \int_a^b g(t,s) f\big(s, y(s)+P_3(s), y'(s)+P_3'(s), y''(s)+P_3''(s), y'''(s)+P_3'''(s)\big)\, ds \tag{3.18}$$

and show that it maps the ball $S_1 = \{y(t) \in S : \|y\| \le (L + \ell)/(1 - \theta)\}$ into itself. For this, let $y(t) \in S_1$ then, from Corollary 2.2, it follows that

$$|y^{(i)}(t)| \le c_i(b-a)^{4-i}\frac{L+\ell}{1-\theta}, \quad 0 \le i \le 3.$$

Thus, we find that

$$\big|y^{(i)}(t) + P_3^{(i)}(t)\big| \le c_i(b-a)^{4-i}\frac{L+\ell}{1-\theta} + \max_{a\le t\le b}|P_3^{(i)}(t)|, \quad 0 \le i \le 3$$

and hence, $\big(y(t) + P_3(t), y'(t) + P_3'(t), y''(t) + P_3''(t), y'''(t) + P_3'''(t)\big) \in D_1$.

Using (3.13) into (3.18) and the definition of norm, we get

$$\begin{aligned}\|Ty\| &\le L + \max_{a\le t\le b}\sum_{i=0}^{3} L_i\big|y^{(i)}(t) + P_3^{(i)}(t)\big| \\ &\le L + \ell + \sum_{i=0}^{3} L_i c_i (b-a)^{4-i}\|y\| \le L + \ell + \theta\frac{L+\ell}{1-\theta} \\ &= \frac{L+\ell}{1-\theta}.\end{aligned}$$

Now, it follows from Schauder's fixed point theorem that T has a fixed point $y(t)$ in S_1. This fixed point $y(t)$ is a solution of (3.16), (3.17) and hence the boundary value problem (1.6), (1.2) has a solution $x(t) = y(t) + P_3(t)$ in D_1.

Theorem 3.7. *Assume that the boundary value problem (1.6), (2.2) has a nontrivial solution $x(t)$ and the condition (3.13) with $L = 0$ is satisfied for all $(t, x_0, x_1, x_2, x_3) \in [a, b] \times D_2$, where*

$$D_2 = \big\{(x_0, x_1, x_2, x_3) : |x_i| \le \mu c_i (b-a)^{4-i}, \quad 0 \le i \le 3\big\}$$

and $\mu = \max_{a\le t\le b}|x''''(t)|$. Then, it is necessary that $\theta \ge 1$.

Proof: Since $x(t)$ is a nontrivial solution of (1.6), (2.2), it is necessary that $\mu \ne 0$. Further, as a consequence of inequalities (2.3), we note that $\big(x(t), x'(t), x''(t), x'''(t)\big) \in D_2$. Thus, it follows that

$$|x''''(t)| \le \sum_{i=0}^{3} L_i|x^{(i)}(t)| \le \sum_{i=0}^{3} L_i c_i (b-a)^{4-i}\mu = \theta\mu.$$

Hence, it is necessary that $\theta \geq 1$.

Remark 3.2. In (3.13), with $L = 0$, at least one of the L_i, $0 \leq i \leq 3$ will not be zero; otherwise, $x(t)$ will coincide with a polynomial of degree less than four and will not be a nontrivial solution of (1.6), (2.2).

Remark 3.3. If (3.13) with $L = 0$ is satisfied, then obviously $x(t) \equiv 0$ is a solution of (1.6), (2.2); if $\theta < 1$ then Theorem 3.7 also guarantees its uniqueness in D_2.

Theorem 3.8. *Suppose that the function $f(t, x_0, x_1, x_2, x_3)$ on $[a, b] \times D_1'$ satisfies the Lipschitz condition*

$$\left|f(t, x_0, x_1, x_2, x_3) - f(t, \bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)\right| \leq \sum_{i=0}^{3} L_i |x_i - \bar{x}_i| \tag{3.19}$$

where D_1' is the same as D_1 with $L = \max_{a \leq t \leq b} |f(t, 0, 0, 0, 0)|$. Then, the boundary value problem (1.6), (1.2) has a unique solution in D_1'.

Proof: Since the condition (3.19) implies (3.13), the existence of a solution follows from Theorem 3.6. To prove the uniqueness, let $x(t)$ and $y(t)$ be two solutions in D_1'. Since the function $z(t) = x(t) - y(t)$ satisfies the conditions of Corollary 2.2, it follows that

$$\begin{aligned} |z''''(t)| &\leq \sum_{i=0}^{3} L_i |x^{(i)}(t) - y^{(i)}(t)| \\ &\leq \sum_{i=0}^{3} L_i c_i (b-a)^{4-i} \times \max_{a \leq t \leq b} |z''''(t)| \\ &= \theta \max_{a \leq t \leq b} |z''''(t)|. \end{aligned}$$

Since $\theta < 1$, the above inequality implies that $z''''(t) \equiv 0$, and from this we immediately have $z(t) \equiv 0$, i.e., $x(t) \equiv y(t)$.

Remark 3.4. The importance of Theorem 3.8 lies with the fact that the sequence $\{y_n(t)\}$ generated by the iterative scheme

$$\begin{aligned} &y_{n+1}(t) = \int_a^b g(t,s) f\big(s, y_n(s) + P_3(s), y_n'(s) + P_3'(s), y_n''(s) + P_3''(s), y_n'''(s) + P_3'''(s)\big)\, ds \\ &y_0(t) = 0; \qquad n = 0, 1, \cdots \end{aligned} \tag{3.20}$$

remains in S_1 and converges to the solution $y(t)$ of the boundary value problem (3.16), (3.17). Also, an error estimate is easily available

$$\|y_n - y\| \leq \theta^n \frac{L + \ell}{1 - \theta}.$$

4. Newton's method. Theorem 3.8, besides proving the existence and uniqueness of the solutions, also provides upper estimates on the length of the interval $(b - a)$ in terms of the Lipschitz constants so that Picard's iterative scheme (3.20) converges to the unique solution of (1.6), (1.2). Here, we shall provide upper estimates on $(b-a)$ so that the sequence $\{x_n(t)\}$ generated from the general Newton's iterative scheme (quasilinearization)

$$x''''_{n+1}(t) = f\big(t, x_n(t), x'_n(t), x''_n(t), x'''_n(t)\big) + \alpha(t)\sum_{i=0}^{3}\left(x^{(i)}_{n+1}(t) - x^{(i)}_n(t)\right)\frac{\partial}{\partial x^{(i)}_n(t)} f\big(t, x_n(t), x'_n(t), x''_n(t), x'''_n(t)\big) \tag{4.1}$$

$$x_{n+1}(a) = A,\ \ x'_{n+1}(a) = B,\ \ x_{n+1}(b) = C,\ \ x''_{n+1}(b) = D,\ \ n = 0, 1, \ldots$$

where $x_0(t)$ is an initial approximation and $\alpha(t)$ is a continuous function on $[a,b]$, converges to the unique solution $x^*(t)$ of (1.6), (1.2).

For this, we need the following:

Definition 4.1. *A function $\bar{x}(t) \in C^{(4)}[a,b]$ is called an approximate solution of (1.6), (1.2) if there exist δ and ϵ nonnegative constants such that*

$$\max_{a\le t\le b}\left|\bar{x}''''(t) - f\big(t, \bar{x}(t), \bar{x}'(t), \bar{x}''(t), \bar{x}'''(t)\big)\right| \le \delta$$

and

$$\max_{a\le t\le b}\left|P_3^{(i)}(t) - \bar{P}_3^{(i)}(t)\right| \le \epsilon c_i (b-a)^{4-i},\quad i = 0, 1, 2, 3$$

where $\bar{P}_3(t)$ is the third degree polynomial satisfying

$$\bar{P}_3(a) = \bar{x}(a),\quad \bar{P}'_3(a) = B,\quad \bar{P}_3(b) = C,\quad \bar{P}''_3(b) = D.$$

This approximate solution $\bar{x}(t)$ can also be expressed as

$$\bar{x}(t) = \bar{P}_3(t) + \int_a^b g(t,s)\left[f\big(s, \bar{x}(s), \bar{x}'(s), \bar{x}''(s), \bar{x}'''(s)\big) + \eta(s)\right] ds \tag{4.2}$$

where

$$\eta(t) = \bar{x}''''(t) - f\big(t, \bar{x}(t), \bar{x}'(t), \bar{x}''(t), \bar{x}'''(t)\big)\quad\text{and}\quad \max_{a\le t\le b}|\eta(t)| \le \delta.$$

In what follows, we shall consider the Banach space $B = C^{(3)}[a,b]$ and for all $x(t) \in B$

$$\|x\| = \max_{a\le j\le 3}\left\{\frac{c_0}{c_j}(b-a)^j \max_{a\le t\le b}|x^{(j)}(t)|\right\}.$$

Theorem 4.1. *With respect to the boundary value problem (1.6), (1.2), we assume that there exists an approximate solution $\bar{x}(t)$ and that*

(i) *the function $f(t, x_0, x_1, x_2, x_3)$ is continuously differentiable with respect to all x_i, $0 \le i \le 3$ on $[a,b] \times D_3$, where*

$$D_3 = \left\{(x_0, x_1, x_2, x_3) : |x_j - \bar{x}^{(j)}(t)| \le N\frac{c_j}{c_0}(b-a)^j,\ \ 0 \le j \le 3\right\}$$

and $N > 0$, a real constant

(ii) *there exist nonnegative constants L_i, $0 \le i \le 3$ such that for all $(t, x_0, x_1, x_2, x_3) \in [a,b] \times D_3$*

$$\left|\frac{\partial}{\partial x_i}(t, x_0, x_1, x_2, x_3)\right| \le L_i$$

(iii) $\theta_1 = (1 + 2\alpha)\theta < 1$, *where* $\alpha = \max_{a\le t\le b}|\alpha(t)|$

(iv) $N_1 = (1-\theta_1)^{-1}(\epsilon + \delta)c_0(b-a)^4 \le N.$

Then, the following hold

(1) *the sequence* $\{x_n(t)\}$ *generated by (4.1) with* $x_0(t) = \bar{x}(t)$ *remains in*

$$S(\bar{x}, N_1) = \Big\{x(t) \in C^{(3)}[a,b] : \|x - \bar{x}\| \le N_1\Big\}$$

(2) *the sequence* $\{x_n(t)\}$ *converges to the unique solution* $x^*(t)$ *of (1.6), (1.2)*

(3) *a bound on the error is given by*

$$\|x^* - x_n\| \le \Big(\frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^n \Big(1 - \frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^{-1} \|x_1 - \bar{x}\| \tag{4.3}$$

$$\le \Big(\frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^n \Big(1 - \frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^{-1} (1-\alpha\theta)^{-1}(\epsilon+\delta)c_0(b-a)^4 . \tag{4.4}$$

Proof: Obviously, $\bar{x}(t) = x_0(t) \in S(\bar{x}, N_1)$. Thus, we need to show that if $x_n(t) \in S(\bar{x}, N_1)$, then $x_{n+1}(t) \in S(\bar{x}, N_1)$. From the definition of norm, $x_n(t) \in S(\bar{x}, N_1)$ implies that

$$\frac{c_0}{c_j}(b-a)^j \big|x_n^{(j)}(t) - \bar{x}^{(j)}(t)\big| \le N_1, \quad 0 \le j \le 3$$

and since $N_1 \le N$, we find that

$$\big|x_n^{(j)}(t) - \bar{x}^{(j)}(t)\big| \le N\frac{c_j}{c_0}(b-a)^j, \quad 0 \le j \le 3$$

i.e., $\big(x_n(t), x_n'(t), x_n''(t), x_n'''(t)\big) \in D_3$. Thus, from (4.1) and (4.2), it follows that

$$x_{n+1}(t) - \bar{x}(t) = P_3(t) - \bar{P}_3(t) + \int_a^b g(t,s)\Big[f\big(s, x_n(s), x_n'(s), x_n''(s), x'''(s)\big)$$
$$+ \alpha(s)\sum_{i=0}^{3}\big(x_{n+1}^{(i)}(s) - x_n^{(i)}(s)\big)\frac{\partial}{\partial x_n^{(i)}(s)} f\big(s, x_n(s), x_n'(s), x_n''(s), x'''(s)\big)$$
$$- f\big(s, \bar{x}(s), \bar{x}'(s), \bar{x}''(s), \bar{x}'''(s)\big) - \eta(s)\Big]\, ds$$

and hence from Corollary 2.2, we find that

$$\big|x_{n+1}^{(j)}(t) - \bar{x}^{(j)}(t)\big| \le \epsilon c_j(b-a)^{4-j} + c_j(b-a)^{4-j} \times$$
$$\max_{a\le t\le b}\Big[\big|f\big(t, x_n(t), x_n'(t), x_n''(t), x_n'''(t)\big) - f\big(t, \bar{x}(t), \bar{x}'(t), \bar{x}''(t), \bar{x}'''(t)\big)\big|$$
$$+ \alpha\sum_{i=0}^{3} L_i\big|x_{n+1}^{(i)}(t) - x_n^{(i)}(t)\big| + \delta\Big]$$
$$\le (\epsilon+\delta)c_j(b-a)^{4-j} + c_j(b-a)^{4-j}\sum_{i=0}^{3} L_i \frac{c_i}{c_0(b-a)^i} \times$$
$$\Big\{\|x_n - \bar{x}\| + \alpha\|x_{n+1} - \bar{x}\| + \alpha\|x_n - \bar{x}\|\Big\}, \quad 0 \le j \le 3.$$

Thus, it follows that

$$\|x_{n+1} - \bar{x}\| \le (\epsilon + \delta)c_0(b-a)^4 + \theta\left[\alpha\|x_{n+1} - \bar{x}\| + (1+\alpha)N_1\right].$$

The above inequality gives

$$\|x_{n+1} - \bar{x}\| \le (1-\theta\alpha)^{-1}[(\epsilon+\delta)c_0(b-a)^4 + (1+\alpha)\theta N_1].$$

Hence, $x_{n+1}(t) \in S(\bar{x}, N_1)$.

Next, from (4.1), we have

$$\begin{aligned} &x_{n+1}(t) - x_n(t) = \\ &\int_a^b g(t,s)\Big[f\big(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)\big) - f\big(s, x_{n-1}(s), x_{n-1}'(s), x_{n-1}''(s), x_{n-1}'''(s)\big) \\ &+ \alpha(s)\sum_{i=0}^{3}\Big\{\big(x_{n+1}^{(i)}(s) - x_n^{(i)}(s)\big)\frac{\partial}{\partial x_n^{(i)}(s)} f\big(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)\big) \\ &- \big(x_n^{(i)}(s) - x_{n-1}^{(i)}(s)\big)\frac{\partial}{\partial x_{n-1}^{(i)}(s)} f\big(s, x_{n-1}(s), x_{n-1}'(s), x_{n-1}''(s), x_{n-1}'''(s)\big)\Big\}\Big]\,ds. \end{aligned} \tag{4.5}$$

Thus, from Corollary 2.2 and the fact that $\{x_n(t)\} \subseteq S(\bar{x}, N_1)$, we have

$$\begin{aligned} &\left|x_{n+1}^{(j)}(t) - x_n^{(j)}(t)\right| \le \\ &c_j(b-a)^{4-j}\max_{a\le t\le b}\left[(1+\alpha)\sum_{i=0}^{3} L_i\left|x_n^{(i)}(t) - x_{n-1}^{(i)}(t)\right| + \alpha\sum_{i=0}^{3} L_i\left|x_{n+1}^{(i)}(t) - x_n^{(i)}(t)\right|\right] \end{aligned}$$

and hence,

$$\|x_{n+1} - x_n\| \le (1+\alpha)\theta\|x_n - x_{n-1}\| + \alpha\theta\|x_{n+1} - x_n\|$$

which is the same as

$$\|x_{n+1} - x_n\| \le \frac{(1+\alpha)\theta}{1-\alpha\theta}\|x_n - x_{n-1}\|.$$

Finally, an easy induction provides

$$\|x_{n+1} - x_n\| \le \Big(\frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^n\|x_1 - \bar{x}\|. \tag{4.6}$$

Since $\theta_1 = (1+2\alpha)\theta < 1$, (4.6) implies that $\{x_n(t)\}$ is a Cauchy sequence and hence converges to some $x^*(t) \in S(\bar{x}, N_1)$. This $x^*(t)$ is indeed the only solution of the boundary value problem (1.6), (1.2) can be verified easily.

The error bound (4.3) follows from (4.6) and the triangle inequality

$$\begin{aligned} \|x_{n+p} - x_n\| &\le \sum_{i=1}^{p}\|x_{n+i} - x_{n+i-1}\| \le \sum_{i=1}^{p}\Big(\frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^{n+i-1}\|x_1 - \bar{x}\| \\ &\le \Big(\frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^n\Big(1 - \frac{(1+\alpha)\theta}{1-\alpha\theta}\Big)^{-1}\|x_1 - \bar{x}\| \end{aligned}$$

and by now taking $p \to \infty$.

Next, from (4.1) and (4.2), we have

$$x_1(t) - x_0(t) = P_3(t) - \bar{P}_3(t) + \int_a^b g(t,s)\Big[\alpha(s)\sum_{i=0}^{3}\big(x_1^{(i)}(s) - x_0^{(i)}(s)\big)\times \frac{\partial}{\partial x_0^{(i)}(s)} f\big(s, x_0(s), x_0'(s), x_0''(s), x_0'''(s)\big) - \eta(s)\Big]\,ds$$

and as earlier, we find

$$\|x_1 - x_0\| \le (\epsilon + \delta)c_0(b-a)^4 + \alpha\theta\|x_1 - x_0\|\,.$$

From the above inequality, we find

$$\|x_1 - x_0\| \le (1-\alpha\theta)^{-1}(\epsilon+\delta)c_0(b-a)^4\,. \tag{4.7}$$

Using (4.7) in (4.3), inequality (4.4) follows.

Remark 4.1. If $\alpha(t) \equiv 0$, then the conclusions of Theorem 4.1 are more informative than Theorem 3.3 as well as Remark 3.4. However, in both the results, the convergence is only linear. In (4.1), the case $\alpha(t) \equiv 1$ gives quadratic convergence, which we prove in our next result.

Theorem 4.2. *Let the conditions of Theorem 4.1 be satisfied and $\alpha(t) \equiv 1$. Further, let $f(t, x_0, x_1, x_2, x_3)$ be continuously twice differentiable with respect to all x_i, $0 \le i \le 3$ on $[a,b] \times D_3$, and for all $(t, x_0, x_1, x_2, x_3) \in [a,b] \times D_3$*

$$\Big|\frac{\partial^2}{\partial x_i \partial x_j} f(t, x_0, x_1, x_2, x_3)\Big| \le L_i L_j k, \quad 0 \le i,\ j \le 3.$$

Then, the following holds:

$$\|x_{n+1} - x_n\| \le \beta\|x_n - x_{n-1}\|^2 \le \frac{1}{\beta}\big(\beta\|x_1 - x_0\|\big)^{2^n} \le \frac{1}{\beta}\Big[\frac{1}{2}(\epsilon+\delta)k\Big(\frac{\theta}{1-\theta}\Big)^2\Big]^{2^n} \tag{4.8}$$

where $\beta = k\theta^2/2(1-\theta)c_0(b-a)^4$. Thus, the convergence is quadratic if

$$\frac{1}{2}(\epsilon+\delta)k\Big(\frac{\theta}{1-\theta}\Big)^2 < 1\,.$$

Proof: In Theorem 4.1, we have proved that $\{x_n(t)\} \subseteq S(\bar{x}, N_1)$, thus, for all n,

$$(x_n(t), x_n'(t), x_n''(t), x_n'''(t)) \in D_3.$$

Furthermore, since f is twice continuously differentiable, we have

$$\begin{aligned} &f\big(t, x_n(t), x_n'(t), x_n''(t), x_n'''(t)\big) - f\big(t, x_{n-1}(t), x_{n-1}'(t), x_{n-1}''(t), x_{n-1}'''(t)\big) \\ &\quad - \sum_{i=0}^{3}\big(x_n^{(i)}(t) - x_{n-1}^{(i)}(t)\big)\frac{\partial}{\partial x_{n-1}^{(i)}(t)} f\big(t, x_{n-1}(t), x_{n-1}'(t), x_{n-1}''(t), x_{n-1}'''(t)\big) \\ &= \frac{1}{2}\Big[\sum_{i=0}^{3}\big(x_n^{(i)}(t) - x_{n-1}^{(i)}(t)\big)\frac{\partial}{\partial p_i(t)}\Big]^2 f\big(t, p_0(t), p_1(t), p_2(t), p_3(t)\big) \end{aligned} \tag{4.9}$$

where $p_i(t)$ lies between $x_{n-1}^{(i)}(t)$ and $x_n^{(i)}(t)$, $0 \le i \le 3$.

Using (4.9) in (4.5) to obtain

$$x_{n+1}(t) - x_n(t) =$$
$$\int_a^b g(t,s)\Big\{ \sum_{i=0}^{3} \big(x_{n+1}^{(i)}(s) - x_n^{(i)}(s)\big) \frac{\partial}{\partial x_n^{(i)}(s)} \times f\big(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)\big)$$
$$+ \frac{1}{2}\Big[\sum_{i=0}^{3} \big(x_n^{(i)}(s) - x_{n-1}^{(i)}(s)\big)\frac{\partial}{\partial p_i(s)}\Big]^2 f\big(s, p_0(s), p_1(s), p_2(s), p_3(s)\big)\Big\}\, ds$$

and now an application of Corollary 3.3 gives

$$\big|x_{n+1}^{(i)}(t) - x_n^{(j)}(t)\big| \le c_j(b-a)^{4-j}\Big[\sum_{i=0}^{3} L_i \frac{c_i}{c_0}(b-a)^{-i}\|x_{n+1} - x_n\|$$
$$+ \frac{1}{2}\Big[\sum_{i=0}^{3} L_i \frac{c_i}{c_0}(b-a)^{-i}\Big]^2 k\|x_n - x_{n-1}\|^2\Big], \quad 0 \le j \le 3$$

and hence,

$$\|x_{n+1} - x_n\| \le \theta\|x_{n+1} - x_n\| + \frac{k\theta^2}{2c_0(b-a)^4}\|x_n - x_{n-1}\|$$

which is the same as

$$\|x_{n+1} - x_n\| \le \beta\|x_n - x_{n-1}\|\,.$$

The second part of the inequality (4.8) follows by an easy induction, whereas the last part is an application of (4.7).

5. Best possible results.

Theorem 5.1. *Suppose that the function $f(t,x_0)$ on $[a,b] \times D_4$ satisfies the following condition*

$$|f(t,x_0)| \le L + L_0|x_0| \tag{5.1}$$

where $L_0 < \lambda_1$ and

$$D_4 = \Big\{x_0 : |x_0| \le \Big(1 - \frac{L_0}{\lambda_1}\Big)^{-1} m\big(L + L_0 \max_{a\le t\le b}|P_3(t)|\big)|\phi_1(t)| + \max_{a\le t\le b}|P_3(t)|\Big\}\,.$$

Then, the boundary value problem (1.7), (1.2) has a solution in D_4.

Proof: Define $U[a,b]$ as the space of continuous functions. If we introduce in $U[a,b]$ the finite norm

$$\|y\| = \sup_{a\le t\le b}\frac{|y(t)|}{|\phi_1(t)|}\,,$$

then it becomes a Banach space. As in Theorem 3.6, we shall show that $T : U[a,b] \to U[a,b]$ defined by

$$(Ty)(t) = \int_a^b g(t,s) f(s, y(s) + P_3(s))\, ds$$

maps

$$U_1[a,b] = \left\{y(t) \in U[a,b] : \|y\| \le (1 - \frac{L_0}{\lambda_0})^{-1} m(L + L_0 \max_{a\le t\le b} |P_3(t)|)\right\}$$

into itself. For this, if $y(t) \in U_1[a,b]$ then it is immediate that $y(t) + P_3(t) \in D_4$ and hence from (5.1), (5.2), Lemma 2.4 and inequality (2.6), it follows that

$$\begin{aligned}|(Ty)(t)| &\le \int_a^b |g(t,s)|\big[L + L_0|y(s) + P_3(s)|\big]\,ds \\ &\le \int_a^b |g(t,s)|\big[L + L_0 \max_{a\le t\le b}|P_3(t)| \\ &\qquad + L_0(1 - \frac{L_0}{\lambda_1})^{-1} m(L + L_0 \max_{a\le t\le b}|P_3(t)|)|\phi_1(s)|\big]\,ds \\ &\le m(L + L_0 \max_{a\le t\le b}|P_3(t)|)\big(1 + \frac{L_0}{\lambda_1}(1 - \frac{L_0}{\lambda_1})^{-1}\big)|\phi_1(t)| \\ &= (1 - \frac{L_0}{\lambda_1})^{-1} m(L + L_0 \max_{a\le t\le b}|P_3(t)|)|\phi_1(t)|\end{aligned}$$

which implies that

$$\|Ty\| \le (1 - \frac{L_0}{\lambda_1})^{-1} m(L + L_0 \max_{a\le t\le b}|P_3(t)|).$$

Remark 5.1. In Theorem 5.1, the inequality $L_0 < \lambda_1$ is the best possible. Indeed, in case of equality $L_0 = \lambda_1$ the boundary value problem $x'''' = L_0 x$, $x(a) = \epsilon\,(\ne 0)$, $x'(a) = x(b) = x''(b) = 0$ has no solution.

Theorem 5.2. *Suppose that the boundary value problem (1.7), (2.2) has a nontrivial solution $x(t)$ and the condition (5.1) with $L = 0$ is satisfied for all $(t, x_0) \in [a,b] \times D_5$, where*

$$D_5 = \left\{x_0 : |x_0| \le \frac{\mu}{48}(t-a)^2(b-t)[3(b-a) - 2(t-a)]\right\}$$

and $\mu = \max_{a\le t\le b}|x''''(t)|$. Then, it is necessary that $L_0 \ge \lambda_1$.

Proof: From the proof of Corollary 2.2, for any function $x(t) \in C^{(4)}[a,b]$ satisfying (2.2), it follows that $x(t) \in D_5$. Now, since $x(t)$ is a nontrivial solution of (1.7), (2.2) we find that $\sup_{a\le t\le b}|x(t)|/|\phi_1(t)|$ exists and is different from zero. Thus, from the integral representation

$$x(t) = \int_a^b g(t,s) f(s, x(s))\,ds$$

and (5.1) and Lemma 2.4, we get

$$\begin{aligned}\frac{|x(t)|}{|\phi_1(t)|} &\le \frac{1}{|\phi_1(t)|}\int_a^b |g(t,s)| L_0 \frac{|x(s)|}{|\phi_1(s)|}|\phi_1(s)|\,ds \\ &\le \frac{L_0}{\lambda_1}\sup_{a\le t\le b}\frac{|x(t)|}{|\phi_1(t)|}\end{aligned}$$

and hence $L_0 \ge \lambda_1$.

Remark 5.2. In Theorem 5.2, the inequality $L_0 \ge \lambda_1$ is the best possible. Indeed, the boundary value problem $x'''' = \lambda_1 x$, (2.2) has nontrivial solutions $x(t) = c\phi_1(t)$, where c is an arbitrary constant.

Theorem 5.3. *Suppose that the function $f(t, x_0)$ on $[a, b] \times D_4'$ satisfies the Lipschitz condition*

$$|f(t, x_0) - f(t, \bar{x}_0)| \leq L_0 |x_0 - \bar{x}_0| \tag{5.3}$$

where D_4' is the same as D_4 with $L = \max_{a \leq t \leq b} |f(t, 0)|$. Then, the boundary value problem (1.7), (1.2) has a unique solution in D_4'.

Proof: The proof employs the ideas of Theorem 3.8 and Theorem 5.1.

Remark 5.3. Once again, in Theorem 5.3, the inequality $L_0 < \lambda_1$ is the best possible. Indeed, in case of equality, existence and/or uniqueness may fail, e.g., see Remarks 5.1 and 5.2.

Remark 5.4. As in Remark 3.4, the importance of Theorem 5.3 lies with the fact that the sequence $\{x_n(t)\}$ generated by the iterative scheme

$$\begin{aligned} x_{n+1}(t) &= P_3(t) + \int_a^b g(t, s) f(s, x_n(s))\, ds \\ x_0(t) &= P_3(t); \quad n = 0, 1, \cdots \end{aligned} \tag{5.4}$$

remains in $U_1[a, b]$ and converges to the solution $x(t)$ of the boundary value problem (1.7), (1.2). Also, and error estimate is easily available

$$\|x_n - x\| \leq (\frac{L_0}{\lambda_1})^n (1 - \frac{L_0}{\lambda_1})^{-1} m (L + L_0 \max_{a \leq t \leq b} |P_3(t)|)\,. \tag{5.5}$$

Further, (5.5) implies that

$$|x_n(t) - x(t)| \leq (\frac{L_0}{\lambda_1})^n (1 - \frac{L_0}{\lambda_1})^{-1} m (L + L_0 \max_{a \leq t \leq b} |P_3(t)|) |\phi_1(t)|, \tag{5.6}$$

which has the property that the right side satisfies the boundary conditions (2.2).

In our next result, we need the existence of a lower and an upper solution of (1.7), (1.2) which are defined as follows: We call a function $\mu(t) \in C^{(4)}[a, b]$ a lower solution of (1.7), (1.2) provided

$$\mu''''(t) \leq f(t, \mu(t)), \quad t \in [a, b] \tag{5.7}$$

$$\mu(a) \leq A,\ \ \mu'(a) \leq B,\ \ \mu(b) \leq C,\ \ \mu''(b) \geq D. \tag{5.8}$$

Similarly, a function $\nu(t) \in C^{(4)}[a, b]$ is called an upper solution of (1.7), (1.2) if

$$\nu''''(t) \geq f(t, \nu(t)), \quad t \in [a, b] \tag{5.9}$$

$$\nu(a) \geq A,\ \ \nu'(a) \geq B,\ \ \nu(b) \geq C,\ \ \nu''(b) \leq D. \tag{5.10}$$

Lemma 5.4. *Let $\mu(t)$ and $\nu(t)$ be lower and upper solutions of (1.7), (1.2), and $P_{3,\mu}(t)$ and $P_{3,\nu}(t)$ be the polynomials of degree 3, satisfying*

$$P_{3,\mu}(a) = \mu(a),\ P_{3\mu}'(a) = \mu'(a),\ P_{3,\mu}(b) = \mu(b),\ P_{3,\mu}''(b) = \mu''(b) \tag{5.11}$$

and

$$P_{3\nu}(a) = \nu(a),\ P_{3,\nu}'(a) = \nu'(a),\ P_{3,\nu}(b) = \nu(b),\ P_{3,\nu}''(b) = \nu''(b) \tag{5.12}$$

respectively. Then, for all $t \in [a, b]$

$$P_{3,\mu}(t) \leq P_3(t) \leq P_{3,\nu}(t). \tag{5.13}$$

Proof: The proof is immediate from the explicit form of $P_3(t)$ obtained in (2.4).

In the space $C[a, b]$, we shall consider the norm $\|x\| = \max_{a \leq t \leq b} |x(t)|$, and introduce a partial ordering as follows: For x, $y \in C[a, b]$ we say that $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [a, b]$.

Theorem 5.5. *With respect to the boundary value problem (1.7), (1.2), we assume that* $f(t, u_0)$ *is nondecreasing in* u_0. *Further, let there exist lower and upper solutions* $x_0(t)$, $y_0(t)$ *such that* $x_0 \leq y_0$. *Then, the sequences* $\{x_n\}$, $\{y_n\}$ *where* $x_n(t)$ *and* $y_n(t)$ *are defined by the iterative schemes*

$$x_{n+1}(t) = P_3(t) + \int_a^b g(t,s) f(s, x_n(s))\, ds \tag{5.14}$$

$$y_{n+1}(t) = P_3(t) + \int_a^b g(t,s) f(s, y_n(s))\, ds \tag{5.15}$$

$n = 0, 1, \ldots,$ *are well defined and* $\{x_n\}$ *converges to an element* $x \in C[a,b]$, $\{y_n\}$ *converges to an element* $y \in C[a,b]$ *(the convergence being in the norm of* $C[a,b]$*). Further,*

$$x_0 \leq x_1 \leq \cdots \leq x_n \cdots \leq x \leq y \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0,$$

$x(t)$ *and* $y(t)$ *are solutions of (1.7), (1.2) and each solution* $z(t)$ *of this problem which is such that* $z \in [x_0, y_0]$ *satisfies* $x \leq z \leq y$.

Proof: First, we shall show that the operator $T: C[a,b] \to C[a,b]$ defined by

$$Tx(t) = P_3(t) + \int_a^b g(t,s) f(s, x(s))\, ds \tag{5.16}$$

is isotone. For this, let x, $y \in C[a,b]$ and $x \leq y$, then from the partial ordering, it follows that $x(s) \leq y(s)$ for all $s \in [a,b]$, and hence from the monotonic property of f, we have $f(s, x(s)) \leq f(s, y(s))$, $s \in [a,b]$. Thus, from the sign property of the Green's function (Lemma 2.1), it follows that

$$g(t,s) f(s, x(s)) \leq g(t,s) f(s, y(s)), \quad s, t \in [a,b].$$

From this, the inequality $T(x) \leq T(y)$ is obvious, and this completes the proof of T being isotone.

Next, since $x_0(t)$ is a lower solution, Lemma 5.4 gives that

$$\begin{aligned} x_0(t) &= P_{3,x_0}(t) + \int_a^b g(t,s) x_0''''(s)\, ds \\ &\leq P_3(t) + \int_a^b g(t,s) f(s, x_0(s))\, ds = Tx_0(t), \end{aligned}$$

i.e., $x_0 \leq T(x_0)$. The inequality $T(y_0) \leq y_0$ can be proved analogously. Thus, the conditions (i) and (i)$'$ of Lemma 2.5 hold and, in conclusion, the sequences $\{T^n(x_0)\}$, $\{T^n(y_0)\}$ are well defined.

Since $T^n(x_0) = T[T^{n-1}(x_0)]$, we have $T^n(x_0) = x_n$ and $T^n(y_0) = y_n$. The sequence $\{x_n(t)\}$ is nondecreasing and bounded from above by $y_0(t)$, $t \in [a,b]$. Similarly, the sequence $\{y_n(t)\}$ is nonincreasing and bounded from below by $x_0(t)$, $t \in [a,b]$. Hence, in conclusion, the sequences $\{x_n(t)\}$, $\{y_n(t)\}$ are uniformly bounded on $[a,b]$.

Now, on using the above monotonicity properties, it is easy to verify that

$$x_0''''(t) \leq f(t, x_0(t)) \leq f(t, y_n(t)) = y_{n+1}''''(t) \leq f(t, y_0(t)) \leq y_0''''(t), \ \ t \in [a,b]$$

for all n. A similar argument holds for the sequence $\{x_n''''(t)\}$. Hence, the sequences $\{x_n''''(t)\}$, $\{y_n''''(t)\}$ are also uniformly bounded on $[a,b]$. Thus, from Lemma 2.6, there exist subsequences $\{x_{n(j)}(t)\}$, $\{y_{n(j)}(t)\}$ which converge uniformly on $[a,b]$. However, since the sequences $\{x_n(t)\}$, $\{y_n(t)\}$ are monotonic, we conclude that the whole sequences $\{x_n(t)\}$, $\{y_n(t)\}$ converge uniformly to some $x(t)$, $y(t)$ such that x, $y \in C[a,b]$, i.e., $T^n(x_0) \uparrow x$ and $T^n(y_0) \downarrow y$.

Finally, since the operator T is continuous, it is obvious that

$$T^{n+1}(x_0) = T[T^n(x_0)] \uparrow T(x) \quad \text{and} \quad T^{n+1}(y_0) = T[T^n(y_0)] \downarrow T(y).$$

Hence, the conditions of Lemma 2.5 are satisfied and the conclusions of Theorem 5.5 follow.

Remark 5.5. In Theorem 5.5, lower and upper solutions $x_0(t)$ and $y_0(t)$ serve as lower and upper bounds for solutions in the interval $[x_0, y_0]$ and these bounds can be improved by the iterative schemes (5.14) and (5.15). The most important characteristic of our result is that if offers effective tools in constructing multiple solutions of (1.7), (1.2).

Remark 5.6. If $f(t,u_0)$ is nondecreasing in u_0, then uniqueness of the solutions of (1.7), (1.2) is not guaranteed, e.g., the boundary value problem (1.1), (1.2) has an infinite number of solutions if $p = 0$. However, if $f(t,u_0)$ is nonincreasing, then the problem (1.7), (1.2) has at most one solution. This is the content of our next result.

Theorem 5.6. *With respect to the boundary value problem (1.7), (1.2), we assume that $f(t,x_0)$ is nonincreasing in x_0. Then, it has at most one solution.*

Proof: We assume that the boundary value problem (1.7), (1.2) has two solutions, say $x(t)$ and $y(t)$ so that

$$x''''(t) - y''''(t) = f(t,x(t)) - f(t,y(t))$$

and hence,

$$(x(t) - y(t))(x''''(t) - y''''(t)) = (x(t) - y(t))(f(t,x(t)) - f(t,y(t))) \le 0$$

where the inequality follows as a consequence of nonincreasing nature of $f(t,x_0)$ in x_0. Now, as in Lemma 2.4, an integration by parts provides that

$$\int_a^b (x''(t) - y''(t))^2\,dt \le 0,$$

which is possible only when $x(t) = y(t)$.

Corollary 5.7. *The linear differential equation*

$$x'''' = f(t)x + g(t), \tag{5.17}$$

together with the boundary conditions (1.2), has a unique solution if

$$\max_{a \le t \le b} f(t) \le 0. \tag{5.18}$$

Proof: For the linear boundary value problem (5.17), (1.2), obviously, the uniqueness implies the existence. Hence, if $f(t) \le 0$ for all $t \in [a,b]$, Theorem 5.6 ensures the uniqueness and as a consequence the existence.

Remark 5.7. From Corollary 5.7, the differential equation

$$x'''' = -\lambda_1 x + 2\lambda_1 \phi_1(t), \tag{5.19}$$

together with the boundary conditions (2.2), has a unique solution $x(t) = \phi_1(t)$. However, the iterative procedure given by

$$x''''_{n+1}(t) = -\lambda_1 x_n(t) + 2\lambda_1 \phi_1(t)$$
$$x_{n+1}(a) = 0,\ x'_{n+1}(a) = 0,\ x_{n+1}(b) = 0,\ x'_{n+1}(b) = 0$$

with $x_0(t) = 0$ oscillates ($x_{2n+1}(t) = 2\phi_1(t)$, $x_{2n}(t) = 0$), and hence Theorem 5.6 does not imply the convergence of the iterative scheme (5.4).

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