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On Fractional Fourier Transform Moments

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Abstract—Based on the relation between the ambiguity function represented in a quasipolar coordinate system and the fractional power spectra, the fractional Fourier transform (FT) moments are introduced. Important equalities for the global second order fractional FT moments are derived, and their applications for signal analysis are discussed. The connection between the local moments and the angle derivative of the fractional power spectra is established. This permits us to solve the phase retrieval problem if only two close fractional power spectra are known.

Index Terms—Ambiguity function, fractional Fourier transform, phase retrieval.

I. INTRODUCTION

OFTEN the application of the different mixed time frequency distributions depends on how informative their moments are and how easily they can be measured or calculated. In this paper, based on the connection between the ambiguity function (AF) and the fractional Fourier transform (FT), we introduce the fractional moments. These moments are related to the fractional power spectra and therefore can be easily measured for example in optics. The application of the fractional moments for signal analysis and the phase retrieval problem is discussed.

II. AMBIGUITY FUNCTION IN QUASI-POLAR COORDINATES AND FRACTIONAL POWER SPECTRA

Let us first recall the important connection between the AF and the fractional FT. We define the AF $A_f(x, u)$ of a signal $f(x)$ as, cf. [1]

$$A_f(x, u) = \int_{-\infty}^{\infty} f(x' + \tfrac{1}{2}x) f^*(x' - \tfrac{1}{2}x) \exp(-i2\pi u x') \, dx'. \tag{1}$$

The fractional FT of function $f(x)$ can be written in the form [2]

$$R^\alpha[f(x)](u) = F_\alpha(u) = \int_{-\infty}^{\infty} K(\alpha, x, u) f(x) \, dx \tag{2}$$

where the kernel $K(\alpha, x, u)$ is given by

$$K(\alpha, x, u) = \frac{\exp(i\tfrac{1}{2}\alpha)}{\sqrt{i \sin \alpha}} \exp\left(i\pi \frac{(x^2 + u^2)\cos\alpha - 2ux}{\sin\alpha}\right). \tag{3}$$

Note that in particular, $F_0(u) = f(u)$, $F_\pi(u) = f(-u)$, and that $F_{\pi/2}(u)$ corresponds to a normal FT.

It is well known (see for example [2]–[4]) that the fractional FT corresponds to a rotation of the AF as well as of the Wigner distribution at the position-frequency plane (x, u). The rotation can be described by introducing the quasipolar coordinates

$$\begin{aligned} x &= R\cos\alpha \\ u &= R\sin\alpha \end{aligned} \tag{4}$$

where $R \in (-\infty, \infty)$ and $\alpha \in [0, \pi)$, in which coordinates we denote the AF as $\tilde{A}_f(R, \alpha) = A_f(R\cos\alpha, R\sin\alpha)$. A simple relationship between the AF $\tilde{A}_f(R, \alpha)$ in this coordinate system and the fractional power spectra $|F_\alpha(x)|^2$ can be derived [5], [6]

$$\tilde{A}_f(R, \alpha - \tfrac{1}{2}\pi) = \int_{-\infty}^{\infty} |F_\alpha(x)|^2 \exp(i2\pi Rx) \, dx \tag{5}$$

from which relation we conclude that the fractional power spectrum $|F_\alpha(x)|^2$ is the FT of the AF. Note that this relationship is very important for the experimental determination of the AF in optics, where the fractional power spectra related to intensity distributions can be measured by a simple optical setup [5], [6], and [7]–[9].

III. GLOBAL FRACTIONAL FT MOMENTS

In this section, we elaborate on (5) and relate the derivatives of the AF $\tilde{A}_f(R, \alpha - (\tfrac{1}{2})\pi)$ in the origin (i.e., for $R = 0$) to the fractional FT moments.

For the zero order moment E, we have

$$E = \int_{-\infty}^{\infty} |F_\alpha(x)|^2 \, dx = \tilde{A}_f(R, \alpha - \tfrac{1}{2}\pi)\Big|_{R=0} = A_f(0, 0). \tag{6}$$

Note that the zero order moment E represents the signal's energy and that, in accordance with Parseval's theorem for a unitary transformation, it does not depend on α.

For the (normalized) first order moments m_α, we have

$$\begin{aligned} m_\alpha &= \frac{1}{E} \int_{-\infty}^{\infty} |F_\alpha(x)|^2 \, x \, dx \\ &= \frac{1}{E} \frac{1}{2\pi i} \frac{\partial \tilde{A}_f\left(R, \alpha - \tfrac{1}{2}\pi\right)}{\partial R}\Bigg|_{R=0}. \end{aligned} \tag{7}$$

Note that the moments m_α are related to the centers of gravity of the fractional power spectra and that they are determined by the first order derivative of the AF in the direction $\alpha - (\tfrac{1}{2})\pi$. We

remark that (7) is a generalization of the two well known special cases [1] for $\alpha = (\frac{1}{2})\pi$ and $\alpha = \pi$

$$\left.\frac{\partial \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R}\right|_{R=0, \alpha=\pi/2} = \left.\frac{\partial A_f(x, u)}{\partial x}\right|_{x=0, u=0}$$

$$= 2\pi i \int_{-\infty}^{\infty} |F_{\pi/2}(u)|^2 u \, du,$$

$$\left.\frac{\partial \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R}\right|_{R=0, \alpha=\pi} = \left.\frac{\partial A_f(x, u)}{\partial u}\right|_{x=0, u=0}$$

$$= 2\pi i \int_{-\infty}^{\infty} |f(-x)|^2 x \, dx.$$

From the relationship

$$\frac{\partial \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R} = \frac{\partial A_f(x, u)}{\partial x} \frac{\partial x}{\partial R} + \frac{\partial A_f(x, u)}{\partial u} \frac{\partial u}{\partial R}$$

$$= \frac{\partial A_f(x, u)}{\partial x} \sin\alpha - \frac{\partial A_f(x, u)}{\partial u} \cos\alpha$$

we have

$$m_\alpha = m_0 \cos\alpha + m_{\pi/2} \sin\alpha. \tag{8}$$

We conclude that the sum of the squares of the centers of gravity in the position domain and the Fourier domain is invariant under fractional FT

$$m_\alpha^2 + m_{\alpha+\pi/2}^2 = m_0^2 + m_{\pi/2}^2. \tag{9}$$

The (normalized) second order moments w_α, defined by

$$w_\alpha = \frac{1}{E} \int_{-\infty}^{\infty} |F_\alpha(x)|^2 x^2 \, dx$$

$$= \frac{1}{E} \left(\frac{1}{2\pi i}\right)^2 \left.\frac{\partial^2 \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R^2}\right|_{R=0} \tag{10}$$

are related to the effective widths in the fractional FT domain and are determined by the second order derivative of the AF in the direction $\alpha - (\frac{1}{2})\pi$.

On the analogy of the relationship

$$\left.\frac{\partial^2 A_f(x, u)}{\partial x \partial u}\right|_{x=0, u=0}$$

$$= \pi i \int_{-\infty}^{\infty} \left[\frac{\partial F_{\pi/2}(u)}{\partial u} F_{\pi/2}^*(u) - F_{\pi/2}(u)\frac{\partial F_{\pi/2}^*(u)}{\partial u}\right] u \, du$$

$$= \pi i \int_{-\infty}^{\infty} \left[\frac{\partial f(x)}{\partial x} f^*(x) - f(x)\frac{\partial f^*(x)}{\partial x}\right] (-x) \, dx,$$

we introduce the *mixed* second order derivative of the AF, where we have a first order derivative in the direction $\alpha - (\frac{1}{2})\pi$ combined with a first order derivative in the direction α

$$\left.\frac{\partial^2 \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R \partial R_\perp}\right|_{R=0}$$

$$= \pi i \int_{-\infty}^{\infty} \left[\frac{\partial F_\alpha(x)}{\partial x} F_\alpha^*(x) - F_\alpha(x)\frac{\partial F_\alpha^*(x)}{\partial x}\right] x \, dx. \tag{11}$$

$R_\perp$ is a local coordinate for a given angle coordinate orthogonal to R. The (normalized) mixed second order moments μ_α, associated with the mixed second order derivative, are now defined as

$$\mu_\alpha = \frac{\pi i}{E} \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \left[\frac{\partial F_\alpha(x)}{\partial x} F_\alpha^*(x) - F_\alpha(x)\frac{\partial F_\alpha^*(x)}{\partial x}\right] x \, dx. \tag{12}$$

From the relationship

$$\frac{\partial^2 \tilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R^2} = \frac{\partial^2 A_f(x, u)}{\partial x^2} \sin^2\alpha + \frac{\partial^2 A_f(x, u)}{\partial u^2} \cos^2\alpha$$

$$- \frac{\partial^2 A_f(x, u)}{\partial x \partial u} \sin 2\alpha$$

we have

$$w_\alpha = w_0 \cos^2\alpha + w_{\pi/2} \sin^2\alpha - \mu_0 \sin 2\alpha \tag{13}$$

which is periodic in α with period π. Equation (13) is in accordance with the relationships between second order moments of signals whose AF's are related through a canonical transformation described by a real, symplectic 2×2 $ABCD$ matrix, as described in [1]. In the particular case of a fractional FT corresponding to a rotation of the AF, we thus have [9], [10]

$$\begin{bmatrix} w_\alpha & \mu_\alpha \\ \mu_\alpha & w_{\alpha+\pi/2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} w_0 & \mu_0 \\ \mu_0 & w_{\pi/2} \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

from which we also get, besides (13)

$$\mu_\alpha = \frac{1}{2}(w_0 - w_{\pi/2}) \sin 2\alpha + \mu_0 \cos 2\alpha. \tag{14}$$

In general, all second order moments w_α and μ_α can be obtained from any three second order moments w_α taken for three different angles α from the region $[0, \pi)$. We have, for instance, $\mu_0 = (\frac{1}{2})(w_0 + w_{\pi/2}) - w_{\pi/4}$. This implies that the corresponding three fractional power spectra define all second order moments.

From (13), it follows that the sum of the signal widths in the position domain and the Fourier domain is invariant under fractional Fourier transformation

$$w_\alpha + w_{\alpha+\pi/2} = w_0 + w_{\pi/2}. \tag{15}$$

Regarding μ_α, we conclude from (14) that

$$\mu_\alpha + \mu_{\alpha+\pi/2} = 0. \tag{16}$$

Moreover, from (13), we have that

$$w_{\alpha-\pi/4} - w_{\alpha+\pi/4}$$

$$= (w_0 - w_{\pi/2}) \left[\cos^2\left(\alpha - \frac{1}{4}\pi\right) - \sin^2\left(\alpha - \frac{1}{4}\pi\right)\right]$$

$$+ 2\mu_0 \cos 2\alpha$$

$$= (w_0 - w_{\pi/2}) \sin 2\alpha + 2\mu_0 \cos 2\alpha$$

and with (14), we conclude that

$$\mu_\alpha = \tfrac{1}{2}(w_{\alpha-\pi/4} - w_{\alpha+\pi/4}). \tag{17}$$

In particular, the mixed second order moment μ_0 is the difference of the signal width in the fractional domains rotated at the angles $\pm(\tfrac{1}{4})\pi$ and therefore can be calculated if the fractional power spectra $|F_{\pm\pi/4}(x)|^2$ are known.

Let us now find those fractional domains where, for certain criteria, the signal could be represented in a more compact form. First, we will look for the domain with the smallest signal width. From (13), it is easy to see that the first derivative of w_α with respect to the angle α equals zero if

$$\tan 2\alpha = -\frac{2\mu_0}{w_0 - w_{\pi/2}}. \tag{18}$$

Due to the invariance relationship (15), the solution of this equation corresponds to the domains with the smallest w_α and largest $w_{\alpha+\pi/2}$ or vice versa. Note also that the mixed second order moment, μ_α equals zero in those fractional domains where the signal width w_α takes its extremal value.

Second, let us find the fractional domain for which the product $w_\alpha w_{\alpha+\pi/2}$ takes its extremal value. From (13), we have

$$w_\alpha w_{\alpha+\pi/2} = w_0 w_{\pi/2} + \tfrac{1}{4}\left[\left(w_0 - w_{\pi/2}\right)^2 - 4\mu_0^2\right]\sin^2 2\alpha$$
$$+ \tfrac{1}{2}\mu_0\left(w_0 - w_{\pi/2}\right)\sin 4\alpha \tag{19}$$

which expression is periodic in α with period $(\tfrac{1}{2})\pi$, and the derivative of $w_\alpha w_{\alpha+\pi/2}$ with respect to α vanishes if

$$\tan 4\alpha = \frac{4\mu_0\left(w_0 - w_{\pi/2}\right)}{4\mu_0^2 - \left(w_0 - w_{\pi/2}\right)^2}. \tag{20}$$

It is easy to see that the product $w_\alpha w_{\alpha+\pi/2}$ has a maximum and a minimum in each period $(\tfrac{1}{2})\pi$, with a difference of $(\tfrac{1}{4})\pi$ between them. Note that due to the uncertainty principle, $w_\alpha w_{\alpha+\pi/2} \geq (\tfrac{1}{4})$.

Let us now consider some particular cases.

Suppose that the widths of a signal in the position and in the Fourier domain are the same, $w_0 = w_{\pi/2}$. In that case, we have

$$w_\alpha = w_0 - \mu_0 \sin 2\alpha$$
$$\mu_\alpha = \mu_0 \cos 2\alpha. \tag{21}$$

Then the angles $\alpha = (\tfrac{1}{4})\pi + n(\tfrac{1}{2})\pi$ correspond to the zeros of the mixed second order moment and to the extrema of the signal width [see (18)]. The extrema of the product $w_\alpha w_{\alpha+\pi/2}$ occur for the angles $\alpha = n(\tfrac{1}{4})\pi$ [see (20)]. In particular, the self-Fourier functions, which are eigenfunctions of the FT, i.e., $F_\alpha(x) = \exp(in\pi/2)F_{\alpha+\pi/2}(x)$, belong to this class of signals. In that particular case, we have that $w_\alpha = w_{\alpha+\pi/2} = w_0$ and $\mu_\alpha = 0$ for any angle α, which implies that for the self-Fourier functions, the width w_α and the product $w_\alpha w_{\alpha+\pi/2}$ are invariant under propagation through the fractional FT system.

Suppose now that $\mu_0 = 0$, then

$$w_\alpha = w_0 \cos^2\alpha + w_{\pi/2}\sin^2\alpha$$
$$\mu_\alpha = \tfrac{1}{2}(w_0 - w_{\pi/2})\sin 2\alpha. \tag{22}$$

In this case, the width w_α and the product $w_\alpha w_{\alpha+\pi/2}$ take their extremal values at angles $\alpha = n(\tfrac{1}{2})\pi$ and $\alpha = n(\tfrac{1}{4})\pi$, respectively.

IV. LOCAL FRACTIONAL FT MOMENTS

Instead of the global moments, we now consider local moments that are related to the local spatial frequency in the different fractional FT domains.

One can easily express the well known expression [11] for the local spatial frequency $U_0(r)$ at the position r

$$U_0(r) = -\frac{1}{2\pi i}\frac{1}{|f(r)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0}\exp(i2\pi ru)\,du \tag{23}$$

in terms of the local moments of the fractional power spectra. Indeed, using the relationship

$$\left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0} = \frac{\partial \tilde{A}_f(R,\alpha-\tfrac{1}{2}\pi)}{\partial R}\left.\frac{\partial R}{\partial x}\right|_{x=0}$$
$$+ \frac{\partial \tilde{A}_f(R,\alpha-\tfrac{1}{2}\pi)}{\partial \alpha}\left.\frac{\partial \alpha}{\partial x}\right|_{x=0}$$
$$= \frac{\partial \tilde{A}_f(R,\alpha-\tfrac{1}{2}\pi)}{\partial R}\left.\frac{x}{\sqrt{x^2+u^2}}\right|_{x=0}$$
$$+ \frac{\partial \tilde{A}_f(R,\alpha-\tfrac{1}{2}\pi)}{\partial \alpha}\left.\frac{-u}{x^2+u^2}\right|_{x=0}$$
$$= -\frac{1}{u}\left.\frac{\partial \tilde{A}_f(-u,\alpha-\tfrac{1}{2}\pi)}{\partial \alpha}\right|_{\alpha=0}$$

we have, after substituting from (5)

$$\left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0} = -\frac{1}{u}\int_{-\infty}^{\infty}\left.\frac{\partial |F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0}$$
$$\cdot\exp(-i2\pi ux)\,dx$$

and thus

$$U_0(r) = \frac{1}{2|F_0(r)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial |F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0}$$
$$\times\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{u}\exp[i2\pi u(r-x)]\,du\right)dx. \tag{24}$$

Finally, with $(1/\pi i)\int_{-\infty}^{\infty}(1/u)\exp[i2\pi u(r-x)]\,du = \mathrm{sgn}(r-x)$, we get

$$U_0(r) = \frac{1}{2|F_0(r)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial |F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0}\mathrm{sgn}(r-x)\,dx \tag{25}$$

a relationship that can easily be generalized to

$$U_\beta(r) = \frac{1}{2|F_\beta(r)|^2}\int_{-\infty}^{\infty}\left.\frac{\partial |F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=\beta}\mathrm{sgn}(r-x)\,dx. \tag{26}$$

We remark that, with $F_\beta(r) = |F_\beta(r)|\exp[i\varphi_\beta(r)]$, the local spatial frequency $U_\beta(r)$ is related to the phase $\varphi_\beta(r)$ of the fractional FT through

$$U_\beta(r) = \frac{d\varphi_\beta(r)}{dr}. \tag{27}$$

In general, the complex-valued fractional FT $F_\beta(r)$, and in particular the signal $f(r) = F_0(r)$, can be completely reconstructed except for a constant phase shift and the possible occurence of an additional π phase shift from its intensity distribution $|F_\beta(r)|^2$ and its local spatial frequency $U_\beta(r)$. Since the latter quantity is determined by the derivative of the fractional power spectra, see (26), only two fractional power spectra for close angles suffice to solve the phase retrieval problem.

We conclude that moments of phase-space distributions like the ones for the Wigner distribution, for instance, which are frequently used in signal processing, can be obtained from knowledge of the fractional power spectra. Introducing fractional FT moments might then be helpful, for example, in the search for an appropriate fractional domain, i.e., a proper choice for α, in which filtering operations will be performed. In the special case of noise that is equally distributed in the phase plane, for instance, the fractional domain with the smallest signal width w_α is then evidently the most preferred choice.

V. CONCLUSION

Based on the relation between the AF represented in a quasipolar coordinate system and the fractional power spectra, we have introduced the fractional FT moments.

Some important equalities for the global second order fractional FT moments have been derived. It was shown how to find the optimal angles α for certain criteria (minimal signal width w_α or minimal product $w_\alpha w_{\alpha+\pi/2}$) from the analysis of the global fractional FT moments. Some particular examples have been considered.

The connection between the local fractional FT moments and the angle derivative of the fractional power spectra has been established. This opens a way for the signal phase reconstruction from the knowledge of only two nearest fractional power spectra.

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