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On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation

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Abstract

We first show that four fractional integro-differential inclusions have solutions. Also, we show that dimension of the set of solutions for the second fractional integro-differential inclusion problem is infinite dimensional under some different conditions.

MSC: Primary 34A08; secondary 34A60

Keywords: Caputo–Fabrizio fractional derivation; Dimension of the set of solutions; Fractional differential inclusion

1 Introduction

A lot of papers on fractional differential equations (see, for example, [1–18] and the references therein) have been published. As you know, most famous fractional derivations are the Caputo and Riemann–Liouville derivations. In 2015, Caputo and Fabrizio introduced a new fractional derivation without singular kernel [19]. Some researchers published some works about solving different equations including the new derivation (see, for example, [2, 3, 10, 20–25]). Some researchers investigated some results on dimension of the set of solutions for some fractional differential inclusions (see, for example, [26]).

Let $b > 0$, $u \in H^1(0,b)$, and $\zeta \in (0,1)$. As you know, the Caputo–Fabrizio fractional derivative of order ζ is defined by

$$^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) = \frac{(2-\zeta)M(\zeta)}{2(1-\zeta)}\int_0^t \exp\left(\frac{-\zeta}{1-\zeta}(t-s)\right)u'(s)\,ds,$$

where $t \geq 0$ and $M(\zeta)$ is a normalization constant depending on ζ such that $M(0) = M(1) = 1$ [19]. Losada and Nieto showed that $^{\mathrm{CF}}\mathcal{I}^{\zeta}u(t) = \frac{2(1-\zeta)}{(2-\zeta)M(\zeta)}u(t) + \frac{\zeta}{(2-\zeta)M(\zeta)}\int_0^t u(s)\,ds$ [27]. Also, they showed that $M(\zeta) = \frac{2}{2-\zeta}$ [27]. Hence, the fractional Caputo–Fabrizio derivative of order ζ is given by $^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) = \frac{1}{1-\zeta}\int_0^t \exp(-\frac{\zeta}{1-\zeta}(t-s))u'(s)\,ds$, when $t \geq 0$ and $0 < \zeta < 1$ [27]. If $n \geq 1$ and $\zeta \in (0,1)$, then the fractional derivative $^{\mathrm{CF}}\mathcal{D}^{\zeta+n}$ of order $n+\zeta$ is defined by $^{\mathrm{CF}}\mathcal{D}^{\zeta+n}u := {}^{\mathrm{CF}}\mathcal{D}^{\zeta}(\mathcal{D}^n u(t))$ [27]. Let $u, v \in H^1(0,1)$ and $\zeta \in (0,1)$. If $u^{(s)}(0) = 0$ for all $s = 1, 2, \ldots, n$, then $^{\mathrm{CF}}\mathcal{D}^{\zeta}(^{\mathrm{CF}}\mathcal{D}^{n}(u(t)) = {}^{\mathrm{CF}}\mathcal{D}^{n}(^{\mathrm{CF}}\mathcal{D}^{\zeta}(u(t))$. Also, we have $\lim_{\zeta\to 0}{}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) = u(t) - u(0)$, $\lim_{\zeta\to 1}{}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) = u(t)'$, and $^{\mathrm{CF}}\mathcal{D}^{\zeta}(\lambda u(t) + \gamma v(t)) =$

$\lambda^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) + \gamma^{\mathrm{CF}}\mathcal{D}^{\zeta}v(t)$ [27]. It has been proved that the unique solution for the problem ${}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) = v(t)$ with boundary condition $u(0) = c$ is given by $u(t) = c + a_{\zeta}(v(t) - v(0)) + b_{\zeta}\int_0^t v(s)\,ds$, where $a_{\zeta} = \frac{2(1-\zeta)}{(2-\zeta)M(\zeta)} = 1 - \zeta$ and $b_{\zeta} = \frac{2\zeta}{(2-\zeta)M(\zeta)} = \zeta$ ([19] and [27]). Note that $v(0) = 0$. Suppose that $u, v \in C_{\mathbb{R}}[0,1]$, $u(0) = 0$, and there is a real constant L such that $|u(t) - v(t)| \leq L$ for all $t \in [0,1]$. Recently, Baleanu, Mousalou, and Rezapour proved that $|{}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t) - {}^{\mathrm{CF}}\mathcal{D}^{\zeta}v(t)| \leq \frac{1}{(1-\zeta)^2}L$ for all $t \in [0,1]$ [10]. This leads to $|{}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t)| \leq (\frac{1}{(1-\zeta)^2})L$ for all $t \in [0,1]$ whenever $u \in C_{\mathbb{R}}[0,1]$ and $|u(t)| \leq L$ for some $L \geq 0$ and all $t \in [0,1]$ with $u(0) = 0$ [10]. Also, they showed that $|{}^{\mathrm{CF}}\mathcal{I}^{\zeta}u(t) - {}^{\mathrm{CF}}\mathcal{I}^{\zeta}v(t)| \leq L$ for all $t \in [0,1]$ [10] and so $|{}^{\mathrm{CF}}\mathcal{I}^{\zeta}u(t)| \leq L$ for all $t \in [0,1]$ whenever $u \in C_{\mathbb{R}}[0,1]$ with $|u(t)| \leq L$ for some $L \geq 0$ and all $t \in [0,1]$. For some more necessary definitions, see [1].

Let $u \in C_{\mathbb{R}}[0,d]$, $d > 0$ and $\zeta \in (0,1)$. The extended fractional Caputo–Fabrizio derivation of order ζ is defined by [11]

$$\begin{aligned} {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) &= \frac{B(\zeta)}{1-\zeta}\big(u(t) - u(0)\big)\exp\left(\frac{-\zeta}{1-\zeta}t\right) \\ &\quad + \frac{\zeta B(\zeta)}{(1-\zeta)^2}\int_0^t \big(u(t) - u(s)\big)\exp\left(\frac{-\zeta}{1-\zeta}(t-s)\right)ds. \end{aligned}$$

If $u(0) = 0$, then we have ${}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) = \frac{B(\zeta)}{1-\zeta}u(t) - \frac{\zeta B(\zeta)}{(1-\zeta)^2}\int_0^t \exp(-\frac{\zeta}{1-\zeta}(t-s))u(s)\,ds$ [11].

Lemma 1 ([11]) *Let $u \in H^1(0,b)$, $b > 0$, and $\zeta \in (0,1)$. Then ${}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) = {}^{\mathrm{CF}}\mathcal{D}^{\zeta}u(t)$. If $u \in C_{\mathbb{R}}[0,b]$, then $\lim_{\zeta\to 0}{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) = u(t) - u(0)$.*

Lemma 2 ([11]) *Let $0 < \zeta < 1$. Then a solution for the problem ${}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) = v(t)$ with boundary condition $u(0) = 0$ is given by $u(t) = a_{\zeta}v(t) + b_{\zeta}\int_0^t v(s)\,ds$.*

Lemma 3 ([11]) *Let $u, v \in C_{\mathbb{R}}[0,1]$. If there is a real constant L such that $|u(t) - v(t)| \leq L$ for all $t \in [0,1]$, then $|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}v(t)| \leq \frac{(2-\zeta)B(\zeta)}{(1-\zeta)^2}L$ for all $t \in [0,1]$. If $u(0) = v(0)$, then $|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}v(t)| \leq \frac{B(\zeta)}{(1-\zeta)^2}L$.*

This result implies that $|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}u(t)| \leq \frac{(2-\zeta)B(\zeta)}{(1-\zeta)^2}L$ for all $t \in [0,1]$ whenever $u \in C_{\mathbb{R}}[0,1]$ with $|u(t)| \leq L$ for some $L \geq 0$ and all $t \in [0,1]$.

We need the following results.

Lemma 4 ([28]) *Suppose that $\mathcal{Y}$ is a Banach space, $\mathcal{F} : I \times \mathcal{Y} \to \mathcal{P}_{cp,cv}(\mathcal{Y})$ is an L^1-Caratheodory multivalued and ϵ is a linear continuous mapping from $L^1(I,\mathcal{Y})$ to $C(I,\mathcal{Y})$. Then the mapping $\epsilon \circ S_{\mathcal{F}} : C(I,\mathcal{Y}) \to \mathcal{P}_{cp,cv}C(I,\mathcal{Y})$ defined by $(\epsilon \circ S_{\mathcal{F}})(y) = \epsilon(S_{\mathcal{F},y})$ is a closed graph mapping in $C(I,\mathcal{Y}) \times C(I,\mathcal{Y})$.*

Theorem 5 ([29]) *Assume that Y is a Banach space, D is a closed and convex subset of Y, and W is an open subset of D with $0 \in W$. If $\mathcal{F} : \bar{W} \to P_{cp,c}(D)$ is an upper semi-continuous compact map, then either $\mathcal{F}$ has a fixed point in $\bar{W}$ or there is $x \in \partial W$ and $\delta \in (0,1)$ such that $x \in \delta\mathcal{F}(x)$.*

Theorem 6 ([30]) *Suppose that $(\mathcal{Y}, d)$ is a complete metric space. If $\mathcal{G} : \mathcal{Y} \to P_{cl}(\mathcal{Y})$ is a contraction, then $\mathcal{G}$ has a fixed point.*

Theorem 7 ([31]) *Assume that $\mathcal{Y}$ is a Banach space, $\mathcal{E} \in P_{bd,cl,cv}(\mathcal{Y})$ and $\mathcal{F}, \mathcal{G} : \mathcal{E} \to P_{cp,cv}(\mathcal{Y})$ are two multivalued operators. If $\mathcal{F}y + \mathcal{G}y \subset \mathcal{E}$ for all $y \in \mathcal{E}$, $\mathcal{F}$ is a contraction and $\mathcal{G}$ is an upper semi-continuous compact map, then there is $y \in \mathcal{E}$ such that $y \in \mathcal{F}y + \mathcal{G}y$.*

Theorem 8 ([32]) *Assume that $\mathcal{Y}$ is a Banach algebra, $D \in \mathcal{P}_{bd,cl,cv}(\mathcal{Y})$ and $\mathcal{F}_1 : D \to \mathcal{P}_{cl,cv,bd}(\mathcal{Y})$ and $\mathcal{F}_2 : D \to \mathcal{P}_{cp,cv}(\mathcal{Y})$ are two set-valued maps such that $\mathcal{F}_1$ is Lipschitz with a Lipschitz constant δ, $\mathcal{F}_2$ is upper semi-continuous and compact, $\mathcal{F}_1 x \mathcal{F}_2 x$ is a convex subset D for all $x \in D$ and $\mathcal{N}\delta < 1$, where $\mathcal{N} = \|\mathcal{F}_2(D)\| = \sup\{\|\mathcal{F}_2 x\| : x \in D\}$. Then there is $y \in D$ such that $y \in \mathcal{F}_1 y \mathcal{F}_2 y$.*

Lemma 9 ([26]) *Let $\mathcal{A}$ mapping $[0,1]$ into $\mathcal{P}_{cp,cv}(\mathbb{R})$ be measurable such that the Lebesgue measure of the set $\{t : \dim \mathcal{A}(t) < 1\}$ is zero. Then there are arbitrarily many linearly independent measurable selections $y_1(\cdot), \ldots, y_m(\cdot)$ of $\mathcal{A}$.*

Theorem 10 ([26]) *Let $\mathcal{H}$ be a nonempty closed convex subset of a Banach space $\mathcal{Y}$ and $\mathcal{F} : \mathcal{H} \to \mathcal{P}_{cp,cv}(\mathcal{H})$ be a δ-contraction. If $\dim \mathcal{F}(x) \geq m$ for all $x \in \mathcal{H}$, then $\dim Fix(\mathcal{F}) \geq m$.*

2 Main results

Consider the Banach space $\mathcal{X} = C(I)$ of real-valued continuous functions on $I = [0,1]$ via the norm $\|x\| = \sup_{t\in I} |x(t)|$. Assume that $\zeta, \iota : [0,1] \times [0,1] \to [0,\infty)$ are two continuous maps such that $\sup|\int_0^t \iota(t,s)\,ds| < \infty$ and $\sup|\int_0^t \zeta(t,s)\,ds| < \infty$. Consider the maps ϕ and φ defined by $(\phi w)(t) = \int_0^t \zeta(t,s)w(s)\,ds$ and $(\varphi w)(t) = \int_0^t \iota(t,s)w(s)\,ds$. Suppose that $\eta(t) \in L^\infty(I)$ with $\eta^* = \sup_{t\in I} |\eta(t)|$. Put $\zeta_0 = \sup|\int_0^t \zeta(t,s)\,ds|$ and $\iota_0 = \sup|\int_0^t \iota(t,s)\,ds|$. First, we are going to investigate the fractional integro-differential inclusion

$$ {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta} x(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t)\bigr), \tag{1} $$

with boundary condition $x(0) = 0$, where $\zeta, \beta_1, \ldots, \beta_m \in (0,1)$.

We say that a function $x \in \mathcal{X}$ is a solution for problem (1) whenever there exists a function $f \in C(I)$ such that

$$ f(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t)\bigr) $$

for almost all $t \in I$ and $x(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds$.

Theorem 11 *Let $\mathcal{F} : I \times \mathbb{R}^{m+3} \to P_{cp,cv}(\mathbb{R})$ be a Caratheodory multivalued map such that*

$$ \begin{aligned} \bigl\|\mathcal{F}(t, x_1, x_2, x_3, y_1, \ldots, y_m)\bigr\|_p &= \sup\bigl\{|y| : y \in \mathcal{F}(t, x_1, x_2, x_3, y_1, \ldots, y_m)\bigr\} \\ &\leq \eta(t)\left(|x_1| + |x_2| + |x_3| + \sum_{i=1}^m |y_i|\right) \end{aligned} $$

for all $t \in I$, $x_i, y_j \in \mathbb{R}$, $1 \leq i \leq 3$ and $1 \leq j \leq m$. If $\eta^*(1 + \zeta_0 + \iota_0 + \sum_{i=1}^m \frac{B(\beta_i)}{(1-\beta_i)^2}) \leq 1$, then inclusion (1) has one solution.

Proof For $x \in \mathcal{X}$, define a selection set of $\mathcal{F}$ at $x \in \mathcal{X}$ by

$$ \begin{aligned} S_{\mathcal{F},x} := \bigl\{ f \in L^1(I,R) : f(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), \\ {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t)\bigr) \text{ for all } t \in I \bigr\}. \end{aligned} $$

Since $\mathcal{F}$ is a Caratheodory multifunction, by using Theorem 1.3.5 in [33], we get $S_{\mathcal{F},x}$ is nonempty. Define an operator $\Omega\colon \mathcal{X} \to P(\mathcal{X})$ by $\Omega(x) = \{g \in \mathcal{X} : \text{there exists } f \in S_{\mathcal{F},x} \text{ such that } g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds \text{ for all } t \in I\}$. We show that the operator Ω satisfies the hypothesis of Theorem 5. First, we show that $\Omega(x)$ is convex for all $\in \mathcal{X}$.

Let $g_1, g_2 \in \Omega(x)$ and $w \in [0,1]$. Choose $f_1, f_2 \in S_{\mathcal{F},x}$ such that $g_i(t) = a_\zeta f_i(t) + b_\zeta \int_0^t f_i(s)\,ds$ for all $t \in I$. Then we have

$$\bigl[wg_1 + (1-w)g_2\bigr](t) = a_\zeta \bigl(wf_1 + (1-w)f_2\bigr)(t) + b_\zeta \int_0^t \bigl(wf_1 + (1-w)f_2\bigr)(s)\,ds$$

for all $t \in I$. Since $\mathcal{F}$ has convex values, it is easy to check that $S_{\mathcal{F},x}$ is convex, and so $wg_1 + (1-w)g_2 \in \Omega(x)$. Now, we show that Ω maps bounded sets into bounded subsets. Let $\mathcal{B}_r = \{x \in \mathcal{X} : \|x\| \le r\}$, $x \in \mathcal{B}_r$, and $g \in \Omega(x)$. Choose $f \in S_{\mathcal{F},x}$ such that

$$\begin{aligned}
\bigl|g(t)\bigr| &\le a_\zeta \bigl|f(t)\bigr| + b_\zeta \int_0^t \bigl|f(s)\bigr|\,ds \le a_\zeta \eta(t)\bigl(|x| + \bigl|\varphi(x)\bigr| + \bigl|\phi(x)\bigr| \\
&\quad + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t)\bigr| + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2}x(t)\bigr| + \cdots + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m}x(t)\bigr|\bigr) \\
&\quad + b_\zeta \int_0^t \bigl(|x| + \bigl|\varphi(x)\bigr| + \bigl|\phi(x)\bigr| \\
&\quad + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(s)\bigr| + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2}x(s)\bigr| + \cdots + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m}x(s)\bigr|\bigr)\eta(s)\,ds \\
&\le a_\zeta \eta^* \left(r + \zeta_0 r + \iota_0 r + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2} r \right) \\
&\quad + b_\zeta \eta^* \left(r + \zeta_0 r + \iota_0 r + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2} r \right) \\
&= \eta^* \cdot r \cdot \left(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2} \right)(a_\zeta + b_\zeta) \le r.
\end{aligned}$$

Thus, $\|g\| = \max_{t\in I} |g(t)| \le r$. This implies that Ω maps bounded sets into bounded sets in $\mathcal{X}$. Now, we show that Ω maps bounded sets of $\mathcal{X}$ into equi-continuous sets. Let $t_1, t_2 \in I$ with $t_1 < t_2$, $x \in \mathcal{B}_r$ and $g \in \Omega(x)$. Then we have

$$\begin{aligned}
\bigl|g(t_2) - g(t_1)\bigr| &= \left| a_\zeta f(t_2) + b_\zeta \int_0^{t_2} f(s)\,ds - a_\zeta f(t_1) - b_\zeta \int_0^{t_1} f(s)\,ds \right| \\
&\le a_\zeta \bigl|f(t_2) - f(t_1)\bigr| + b_\zeta \int_{t_1}^{t_2} \bigl|f(s)\bigr|\,ds \\
&\le r\left(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2} \right)\bigl(\eta(t_2) - \eta(t_1)\bigr)(a_\zeta + b_\zeta).
\end{aligned}$$

Hence, the right-hand side of the inequality tends to zero (independent on $x \in \mathcal{B}_r$) as $t_2 \to t_1$. This implies that $\Omega\colon \mathcal{X} \to P(\mathcal{X})$ is a compact multivalued map by using the Arzela–Ascoli theorem. We show that Ω has a closed graph. Let $x_n \to x_*$, $g_n \in \Omega(x_n)$ for all n and $g_n \to g_*$. It is sufficient to prove that $g_* \in \Omega(x_*)$. Since $g_n \in \Omega(x_n)$ for all n, there exist $f_n \in S_{\mathcal{F},x_n}$ such that $g_n(t) = a_\zeta f_n(t) + b_\zeta \int_0^t f_n(s)\,ds$ for all $t \in I$. Thus, we have to show that there exist $f_* \in S_{\mathcal{F},x_*}$ such that $g_*(t) = a_\zeta f_*(t) + b_\zeta \int_0^t f_*(s)\,ds$ for all $t \in I$. Consider the linear continuous operator $\theta\colon L^1(I,\mathbb{R}) \to \mathcal{X}$ defined by $f \mapsto \theta(f)(t)$, where $\theta(f)(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds$ for all $t \in I$. Since θ is a linear continuous map, by using Lemma 4 we get $\theta \circ S_{\mathcal{F}}$ is a closed graph operator. Note that $g_n \in \theta \circ S_{\mathcal{F}}(x_n)$ for all n. Since $x_n \to x_*$ and $g_n \to g_*$, there exists $f_* \in S_{\mathcal{F}}(x_*)$ such that $g_*(t) = a_\zeta f_*(t) + b_\zeta \int_0^t f_*(s)\,ds$ for all $t \in I$. For $\lambda \in (0,1)$ and $x \in \lambda\Omega(x)$, there exists $f \in S_{\mathcal{F},x}$ such that $x(t) = a_\zeta \lambda f(t) + b_\zeta \int_0^t \lambda f(s)\,ds$ for all $t \in I$. Hence,

$$\left|x(t)\right| \le \lambda(a_\zeta + b_\zeta)\eta^* \cdot \left(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x\|.$$

Thus, $\|x\| = \max_{t\in I}|x(t)| \le \lambda\|x\|$. Put $\mathcal{W} = \{x \in \mathcal{X}, \|x\| < r(1 + \zeta_0 + \iota_0 + \sum_{i=1}^m \frac{B(\beta_i)}{(1-\beta_i)^2})\}$. Note that the operator $\Omega\colon \overline{\mathcal{W}} \to P_{cp,cv}(\mathcal{X})$ is upper semi-continuous and compact. In view of the choice of $\mathcal{W}$, there is no $x \in \partial\mathcal{W}$ such that $x \in \lambda\Omega(x)$ for some $\lambda \in (0,1)$. Hence, by using Theorem 5, Ω has a fixed point $x \in \overline{\mathcal{W}}$ which is a solution for problem (1). This completes the proof. □

Now consider the Banach space $\mathcal{X} = C(I)$ via the norm

$$\|x\| = \max_{t\in I}\left|x(t)\right| + \sum_{i=1}^{m} \max_{t\in I}\left|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}x(t)\right| + \sum_{j=1}^{n} \max_{t\in I}\left|{}^{\mathrm{CF}}\mathcal{I}^{\gamma_j}x(t)\right|.$$

Here, we investigate the fractional integro-differential inclusion

$$\begin{aligned}{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}x(t) \in \mathcal{F}\bigl(&t, x(t), (\phi x)(t), (\varphi x)(t),\\ &{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2}x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m}x(t),\\ &{}^{\mathrm{CF}}\mathcal{I}^{\gamma_1}x(t), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_2}x(t), \ldots, {}^{\mathrm{CF}}\mathcal{I}^{\gamma_n}x(t)\bigr),\end{aligned} \tag{2}$$

with boundary condition $x(0) = 0$, where $\zeta, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n \in (0,1)$. Similar to the last case, we say that a function $x \in C(I,\mathbb{R})$ is a solution for problem (2) whenever there exists a function $f \in L^1(I)$ such that

$$\begin{aligned}f(t) \in \mathcal{F}\bigl(&t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2}x(t), \ldots,\\ &{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m}x(t)\bigr), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_1}x(t), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_2}x(t), \ldots, {}^{\mathrm{CF}}\mathcal{I}^{\gamma_n}x(t))\end{aligned}$$

for almost all $t \in I$ and $x(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds$ for all $t \in I$.

Theorem 12 *Assume that $\mathcal{F}\colon I \times \mathbb{R}^{m+n+3} \to P_{cv,cp}(\mathbb{R})$ is a multifunction such that the map $t \to \mathcal{F}(t, x_1, x_2, \ldots, x_{3+m+n})$ is measurable for all $x_1, x_2, \ldots, x_{m+n+3} \in \mathbb{R}$, the map $t \to d_H(0, \mathcal{F}(t, 0, \ldots, 0))$ is integrably bounded for almost all $t \in I$ and*

$$\begin{aligned}&H_d\bigl(\mathcal{F}(t, x_1, x_2, x_3, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n),\\ &\quad\mathcal{F}\bigl(t, x_1', x_2', x_3', y_1', y_2', \ldots, y_m', z_1', z_2', \ldots, z_n'\bigr)\bigr)\\ &\quad\le \eta(t)\left(\left|x_1 - x_1'\right| + \left|x_2 - x_2'\right| + \left|x_3 - x_3'\right| + \sum_{i=1}^{m}\left|y_i - y_i'\right| + \sum_{j=1}^{n}\left|z_j - z_j'\right|\right)\end{aligned}$$

for all $t \in I$ and all $x_1, x_2, x_3, x'_1, x'_2, x'_3, y_1, \dots, y_m, y'_1, \dots, y'_m, z_1, \dots, z_n, z'_1, \dots, z'_n \in \mathbb{R}$. If $\Delta \le 1$, then the inclusion problem (2) *has at least one solution, where*

$$\Delta = \eta^* \left(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\left(1 + n + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\right).$$

Proof By using the assumptions of Theorem III-6 in [34], we conclude that $\mathcal{F}$ admits a measurable selection $f: I \to \mathbb{R}$. Since $\mathcal{F}$ is integrable bounded, $f \in L^1(I, \mathbb{R})$ and so $S_{\mathcal{F},x}$ is nonempty for all $x \in \mathcal{X}$, where

$$\begin{aligned} S_{\mathcal{F},x} = \Bigl\{ f \in L^1(I,R) : f(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t),\\ {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \dots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_1} x(t), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_2} x(t), \dots,\\ {}^{\mathrm{CF}}\mathcal{I}^{\gamma_n} x(t)\bigr) \text{ for all } t \in I \Bigr\}. \end{aligned}$$

Define the operator $\Omega: \mathcal{X} \to P(\mathcal{X})$ by

$$\begin{aligned} \Omega(x) = \Bigl\{ g \in \mathcal{X} : \text{ there exists } f \in S_{\mathcal{F},x} \text{ such that}\\ g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds \text{ for all } t \in I \Bigr\}. \end{aligned}$$

First, we show that $\Omega(x) \in P_{cl}(\mathcal{X})$ for all $x \in \mathcal{X}$. Let $g_n \in \Omega(x)$ for all $n \ge 0$ and $g_n \to g_*$ for some $g \in \mathcal{X}$. For each n, choose $f_n \in S_{\mathcal{F},x}$ such that $g_n(t) = a_\zeta f_n(t) + b_\zeta \int_0^t f_n(s)\,ds$ for all $t \in I$. Since $\mathcal{F}$ has compact values, there is a subsequence of f_n that converges to f in $L^1(I, \mathbb{R})$. Thus, $f \in S_{\mathcal{F},x}$ and $g_n(t) \to g_*(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds$ for all $t \in I$. This implies that $g_* \in \Omega$. Now, we show that there exists $\epsilon < 1$ such that $H_d(\Omega(x), \Omega(y)) \le \epsilon \|x - y\|$ for all $x, y \in \mathcal{X}$. Let $x, y \in \mathcal{X}$ and $g_1 \in \Omega(x)$. Choose $f_1 \in S_{\mathcal{F},x}$ such that $g_1(t) = a_\zeta f_1(t) + b_\zeta \int_0^t f_1(s)\,ds$ for all $t \in I$. Consider the multifunction $\tilde{\mathcal{F}}$ defined by

$$\begin{aligned} \tilde{\mathcal{F}}\bigl(t, x(t)\bigr) = \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \dots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t),\\ {}^{\mathrm{CF}}\mathcal{I}^{\gamma_1} x(t), {}^{\mathrm{CF}}\mathcal{I}^{\gamma_2} x(t), \dots, {}^{\mathrm{CF}}\mathcal{I}^{\gamma_n} x(t)\bigr). \end{aligned}$$

Then we have

$$\begin{aligned} H_d\bigl(\tilde{\mathcal{F}}\bigl(t, x(t)\bigr), \tilde{\mathcal{F}}\bigl(t, y(t)\bigr)\bigr) \le \eta(t) \Biggl(\bigl|x(t) - y(t)\bigr| + \bigl|(\phi x)(t) - (\phi y)(t)\bigr|\\ + \bigl|(\varphi x)(t) - (\varphi y)(t)\bigr|\\ + \sum_{i=1}^{m} \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} x(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} y(t)\bigr|\\ + \sum_{j=1}^{n} \bigl|{}^{\mathrm{CF}}\mathcal{I}^{\gamma_j} x(t) - {}^{\mathrm{CF}}\mathcal{I}^{\gamma_j} y(t)\bigr| \Biggr) \end{aligned}$$

for almost $t \in I$. Hence, there exists $w_t \in \tilde{\mathcal{F}}(t, y(t))$ such that

$$\begin{aligned} |f_1(t) - w_t| \le \eta(t)\Bigg(&|x(t) - y(t)| + |(\phi x)(t) - (\phi y)(t)| + |(\varphi x)(t) - (\varphi y)(t)| \\ &+ \sum_{i=1}^{m} \big|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} x(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} y(t)\big| \\ &+ \sum_{j=1}^{n} \big|{}^{\mathrm{CF}}\mathcal{I}^{\gamma_j} x(t) - {}^{\mathrm{CF}}\mathcal{I}^{\gamma_j} y(t)\big|\Bigg) := M_t \end{aligned}$$

for almost $t \in I$. Define $V: I \to P(\mathbb{R})$ by $V(t) = \{u \in \mathbb{R} : |f_1(t) - u| \le M_t\}$ for all $t \in I$. By using Theorem III-41 in [34], we get V is measurable. Since $t \mapsto V(t) \cap \tilde{\mathcal{F}}(t, y(t))$ is measurable (Proposition III-4 in [34]), we can choose $f_2 \in S_{\mathcal{F},y}$ such that $|f_1(t) - f_2(t)| \le M_t$ for almost all $t \in I$. Define $g_2 \in \Omega(y)$ by $g_2(t) = a_\zeta f_2(t) + b_\zeta \int_0^t f_2(s)\,ds$ for all $t \in I$. Then we have

$$\begin{aligned} \|g_1 - g_2\| &= \max_{t\in I}|g_1(t) - g_2(t)| + \sum_{i=1}^{m} \max_{t\in I}\big|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} g_1(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} g_2(t)\big| \\ &\quad + \sum_{i=1}^{n} \max_{t\in I}\big|{}^{\mathrm{CF}}\mathcal{I}^{\gamma_i} g_1(t) - {}^{\mathrm{CF}}\mathcal{I}^{\gamma_i} g_2(t)\big| |g_1(t) - g_2(t)| \\ &\le a_\zeta |f_1(t) - f_2(t)| + b_\zeta \int_0^t |f_1(s) - f_2(s)|\,ds \\ &\le \eta(t)\Bigg(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg)(a_\zeta + b_\zeta)\|x - y\|, \end{aligned}$$

and so

$$\begin{aligned} &\big|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} g_1(t) - {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i} g_2(t)\big| \\ &\quad\le \frac{B(\beta_i)}{(1-\beta_i)^2}|g_1(t) - g_2(t)| \\ &\quad\le \eta(t)\frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg)(a_\zeta + b_\zeta)\|x - y\|. \end{aligned}$$

Thus,

$$\begin{aligned} \big|{}^{\mathrm{CF}}\mathcal{I}^{\gamma_i} g_1(t) - {}^{\mathrm{CF}}\mathcal{I}^{\gamma_i} g_2(t)\big| &\le |g_1(t) - g_2(t)| \\ &\le \eta(t)\Bigg(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg)(a_\zeta + b_\zeta)\|x - y\|, \end{aligned}$$

and so

$$\begin{aligned} \|g_1 - g_2\| \le \eta^*&\Bigg(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg) \\ &\times \Bigg(1 + n + \sum_{i=1}^{m} \frac{B(\beta_i)}{(1-\beta_i)^2}\Bigg)\|x - y\| = \Delta\|x - y\|. \end{aligned}$$

Hence, $H_d(\Omega(x),\Omega(y)) \leq \Delta\|x-y\|$. Since $\Delta < 1$, Ω is a closed-valued contraction. By using Theorem 6, Ω has a fixed point which is a solution for the inclusion problem (2). □

Consider the Banach space $\mathcal{X} = \{x : x, {}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_i}x \in C(I,\mathbb{R})\}$ endowed with the norm $\|x\| = \max_{t\in I}|x(t)| + \max_{t\in I}|{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_i}x(t)|$. Here, we review the inclusion problem

$$\begin{aligned}{}_{N}^{\mathrm{CF}}\mathcal{D}^{\zeta}x(t) &\in \mathcal{F}\big(t,x(t),(\phi x)(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_2}x(t),\dots,{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\big)\\ &\quad + \mathcal{G}\big(t,x(t),(\varphi x)(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_1}x(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_2}x(t),\dots,{}^{\mathrm{CF}}\mathcal{I}^{\beta_n}x(t)\big)\end{aligned} \tag{3}$$

with boundary condition $x(0) = 0$, where $\zeta,\beta_1,\dots,\beta_n \in (0,1)$. Define the set of the selections of $\mathcal{F}$ and $\mathcal{G}$ at x by

$$\begin{aligned}S_{\mathcal{F},x} = \big\{&v \in L^1[0,1] : v(t) \in \mathcal{F}\big(t,x(t),(\phi x)(t),\\ &{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_2}x(t),\dots,{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\big) \text{ for almost all } t\in I\big\}\end{aligned}$$

and

$$\begin{aligned}S_{\mathcal{G},x} = \big\{&v \in L^1[0,1] : v(t) \in \mathcal{G}\big(t,x(t),(\varphi x)(t),\\ &{}^{\mathrm{CF}}\mathcal{I}^{\beta_1}x(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_2}x(t),\dots,{}^{\mathrm{CF}}\mathcal{I}^{\beta_n}x(t)\big) \text{ for almost all } t\in I\big\}.\end{aligned}$$

We suppose that $S_{\mathcal{F},x} \neq \emptyset$ and $S_{\mathcal{G},x} \neq \emptyset$ for all $x \in \mathcal{X}$. A function $x \in C(I,\mathbb{R})$ is a solution for problem (3) whenever there exist two functions $f \in H^1(I)$ and $f' \in H^1(I)$ such that

$$f(t) \in \mathcal{F}\big(t,x(t),(\phi x)(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_2}x(t),\dots,{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\big)$$

and $f' \in \mathcal{G}(t,x(t),(\varphi x)(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_1}x(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_2}x(t),\dots,{}^{\mathrm{CF}}\mathcal{I}^{\beta_n}x(t))$ for almost all $t \in I$ and

$$x(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds + a_\zeta f'(t) + b_\zeta \int_0^t f'(s)\,ds$$

for all $t \in I$.

Theorem 13 *Let $\mathcal{F} : I \times \mathbb{R}^{n+2} \to P_{cp,cv}(\mathbb{R})$ be a multifunction and $\mathcal{G} : I \times \mathbb{R}^{n+2} \to P_{cp,cv}(\mathbb{R})$ be a Caratheodory set-valued map. Assume that there exist continuous functions $p, m : I \to (0,\infty)$ and $\eta(t) \in L^\infty(I)$ such that $t \vdash \mathcal{F}(t,y_1,\dots,y_{n+2})$ is measurable,*

$$\begin{aligned}&\big\|\mathcal{F}\big(t,x(t),(\phi x)(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t),{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_2}x(t),\dots,{}_{N}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\big)\big\| \leq m(t),\\ &\big\|\mathcal{G}\big(t,x(t),(\varphi x)(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_1}x(t),{}^{\mathrm{CF}}\mathcal{I}^{\beta_2}x(t),\dots,{}^{\mathrm{CF}}\mathcal{I}^{\beta_n}x(t)\big)\big\| \leq p(t),\end{aligned}$$

and

$$H_d\big(\mathcal{F}(t,y_1,\dots,y_{n+2}),\mathcal{F}\big(t,y'_1,\dots,y'_{n+2}\big)\big) \leq \eta(t)\sum_{i=1}^{n+2}\big(|y_i - y'_i|\big)$$

for all $t \in I$, $x \in \mathcal{X}$ and $y_1,\dots,y_{n+2}, y'_1,\dots,y'_{n+2} \in \mathbb{R}$. If $L = \eta^*(1 + \sum_{i=1}^n \frac{B(\beta_i)}{(1-\beta_i)^2})(1 + \zeta_0 + \sum_{i=1}^n \frac{B(\beta_i)}{(1-\beta_i)^2}) < 1$, then the inclusion problem (3) has at least one solution.

Proof Put $\mathcal{Y} = \{x \in \mathcal{X} : \|x\| \leq M\}$, where $M = (1 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})(\|p\|_\infty + \|m\|_\infty)$. One can check that $\mathcal{Y}$ is a closed, bounded, and convex subset of $\mathcal{X}$. Define the multivalued operators $\mathcal{A}, \mathcal{B} : \mathcal{Y} \to P(\mathcal{X})$ by

$$\mathcal{A}x := \left\{x \in \mathcal{X} : \text{ there is } v \in S_{\mathcal{F},x} \text{ such that } x(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds \text{ for all } t \in I\right\}$$

and $\mathcal{B}x := \{x \in \mathcal{X} : \text{ there is } v \in S_{\mathcal{G},x} \text{ such that } x(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds \text{ for all } t \in I\}$. Note that problem (3) is equivalent to the inclusion fixed point problem $x \in \mathcal{A}x + \mathcal{B}x$. Also, the operator $\mathcal{A}$ is equivalent to the composition $\theta \circ S_{\mathcal{F}}$, where θ is the continuous linear operator on $L^1(0,1)$ into $\mathcal{X}$ defined by $\theta v(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$. Let $x \in \mathcal{Y}$ and $\{v_n\}_{n\geq 1}$ be a sequence in $S_{\mathcal{F},x}$. Then $v_n(t) \in \mathcal{F}(t, x(t), (\phi x)(t), {}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t), {}^{\mathrm{CF}}\mathcal{D}^{\beta_2}x(t), \ldots, {}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t))$ for almost $t \in I$. Since

$$\mathcal{F}\bigl(t, x(t), (\phi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2}x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_n}x(t)\bigr)$$

is compact for all $t \in I$, there is a convergent subsequence of $\{v_n(t)\}$ (call it again $\{v_n(t)\}$) such that it converges in measure to some $v(t) \in S_{\mathcal{F},x}$ for almost all $t \in I$. Since θ is continuous, $\theta v_n(t) \to \theta v(t)$ pointwise on I. In order to show that the convergence is uniform, we show that $\{\theta v_n\}$ is an equi-continuous sequence. Let $\tau < t \in I$. Then we have

$$\bigl|\theta v_n(t) - \theta v_n(\tau)\bigr| \leq a_\zeta \bigl|v_n(t) - v_n(\tau)\bigr| + b_\zeta \int_\tau^t \bigl|v_n(s)\bigr|\,ds.$$

Since the right-hand of the above inequality tends to 0 as $t \to \tau$, the sequence $\{\theta v_n\}$ is equi-continuous. Now, by using the Arzela–Ascoli theorem, there is a uniformly convergent subsequence of $\{v_n\}$ (we show it again by $\{v_n\}$) such that $\theta v_n \to \theta v$. Note that $\theta v \in \theta(S_{\mathcal{F},x})$. Hence, $\mathcal{A}x = \theta(S_{\mathcal{F},x})$ is compact for all $x \in \mathcal{Y}$. Now, we show that $\mathcal{A}x$ is convex for all $x \in \mathcal{Y}$. Let $u, u' \in \mathcal{A}x$. Choose $v, v' \in S_{\mathcal{F},x}$ such that $u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$ and $u'(t) = a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds$ for almost all $t \in I$. Let $0 \leq \lambda \leq 1$. Then we have

$$\bigl(\lambda u + (1-\lambda)u'\bigr)(t) = a_\zeta \bigl(\lambda v(t) + (1-\lambda)v'(t)\bigr) + b_\zeta \int_0^t \bigl(\lambda v(s) + (1-\lambda)v'(s)\bigr)\,ds.$$

Since $\mathcal{F}$ is convex-valued, $\lambda u + (1-\lambda)u' \in \mathcal{A}x$. Similarly, we can show that $\mathcal{B}$ is compact and convex-valued. Here, we show that $\mathcal{A}y + \mathcal{B}y \subset \mathcal{Y}$ for all $y \in \mathcal{Y}$. Let $y \in \mathcal{Y}$, $u \in \mathcal{A}y$, and $u' \in \mathcal{B}y$. Choose $v \in S_{\mathcal{F},y}$ and $v' \in S_{\mathcal{G},y}$ such that $u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$ and $u'(t) = a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds$ for almost all $t \in I$. Hence,

$$\bigl|u(t) + u'(t)\bigr| \leq a_\zeta \bigl(\bigl|v(t)\bigr| + \bigl|v'(t)\bigr|\bigr) + b_\zeta \int_0^t \bigl(\bigl|v(s)\bigr| + \bigl|v'(s)\bigr|\bigr)\,ds,$$

and so

$$\begin{aligned}\bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(t) + {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u'(t)\bigr| &\leq \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(t)\bigr| + \bigl|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u'(t)\bigr| \\ &\leq \frac{a_\zeta B(\beta_i)}{(1-\beta_i)^2}\bigl(p(t) + m(t)\bigr) \\ &\quad + \frac{b_\zeta B(\beta_i)}{(1-\beta_i)^2}\bigl(\|p\|_\infty + \|m\|_\infty\bigr)\end{aligned}$$

for $1 \le i \le n$. This implies that

$$\max_{t\in I}\left|u(t)+u'(t)\right| \le a_\zeta\left(\|p\|_\infty+\|m\|_\infty\right)+b_\zeta\left(\|p\|_\infty+\|m\|_\infty\right)=\|p\|_\infty+\|m\|_\infty$$

and

$$\begin{aligned}\max_{t\in I}\left|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(t)+{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u'(t)\right| &\le \frac{a_\zeta B(\beta_i)}{(1-\beta_i)^2}\left(\|p\|_\infty+\|m\|_\infty\right)\\ &\quad+\frac{b_\zeta B(\beta_i)}{(1-\beta_i)^2}\left(\|p\|_\infty+\|m\|_\infty\right)\\ &=\frac{B(\beta_i)(\|p\|_\infty+\|m\|_\infty)}{(1-\beta_i)^2}.\end{aligned}$$

Thus, $\|u+u'\| \le (1+\sum_{i=1}^{n}(\frac{B(\beta_i)}{(1-\beta_i)^2}))(\|p\|_\infty+\|m\|_\infty)=M$. Now, we show that the operator $\mathcal{B}$ is compact on $\mathcal{Y}$. To do this, we prove that $\mathcal{B}(\mathcal{Y})$ is uniformly bounded and equicontinuous in $\mathcal{X}$. Let $u\in\mathcal{B}(\mathcal{Y})$ be arbitrary. Choose $v\in S_{\mathcal{G},x}$ such that $u(t)=a_\zeta v(t)+b_\zeta\int_0^t v(s)\,ds$ for some $x\in\mathcal{Y}$. Hence,

$$\begin{aligned}\left|u(t)\right| &\le a_\zeta\left|v(t)\right|+b_\zeta\int_0^t\left|v(s)\right|ds\left|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(t))\right|\\ &\le a_\zeta\left|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}v(t)\right|+b_\zeta\int_0^t\left|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}v(s)\right|ds\\ &\le\frac{B(\beta_i)(a_\zeta+b_\zeta)}{(1-\beta_i)^2}p(t)\\ &=\frac{B(\beta_i)}{(1-\beta_i)^2}p(t).\end{aligned}$$

Thus, $\max_{t\in I}|u(t)| \le (a_\zeta+b_\zeta)\|p\|_\infty=\|p\|_\infty$ and $\max_{t\in I}|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u_i(t)| \le \frac{B(\beta_i)}{(1-\beta_i)^2}\|p\|_\infty$ for $i=1,\ldots,n$, and so $\|u\| \le (1+\sum_{i=1}^{n}\frac{B(\beta_i)}{(1-\beta_i)^2})\|p\|_\infty$. Here, we show that $\mathcal{B}$ maps $\mathcal{Y}$ to equicontinuous subsets of $\mathcal{X}$. Let $t,\tau\in I$ with $\tau<t$, $x\in\mathcal{Y}$ and $u\in\mathcal{B}x$. Choose $v\in S_{\mathcal{G},x}$ such that $u(t)=a_\zeta v(t)+b_\zeta\int_0^t v(s)\,ds$ for all. Then we have

$$\left|u(t)-u(\tau)\right| \le a_\zeta\left(v(t)-v(\tau)\right)+b_\zeta\int_\tau^t v(s)\,ds \le a_\zeta\left(v(t)-v(\tau)\right)+b_\zeta(t-\tau)\|p\|_\infty$$

and $|{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(t)-{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_i}u(\tau)| \le \frac{B(\beta_i)}{(1-\beta_i)^2}|u(t)-u(\tau)|$. Since the right-hand of the inequality tends to 0 as $t\to\tau$, by using the Arzela–Ascoli theorem, we get $\mathcal{B}$ is compact. Now, we show that $\mathcal{B}$ has a closed graph. Let $x_n\in\mathcal{Y}$ and $u_n\in\mathcal{B}(x_n)$ for all n with $x_n\to x_0$ and $u_n\to u_0$. We show that $u_0\in\mathcal{B}(x_0)$. For each n, choose $v_n\in S_{\mathcal{G},x_n}$ such that $u_n(t)=a_\zeta v_n(t)+b_\zeta\int_0^t v_n(s)\,ds$ for all $t\in I$. Again, consider the continuous linear operator $\theta: L^1(0,1)\to\mathcal{X}$ defined by $\theta(v)(t)=a_\zeta v(t)+b_\zeta\int_0^t v(s)\,ds$. By using Lemma 4, $\theta o S_{\mathcal{G}}$ is a closed graph operator. Since $u_n\in\theta(S_{\mathcal{G},x_n})$ for all n and $x_n\to x_0$, there exists $v_0\in S_{\mathcal{G},x_0}$ such that $u_0(t)=a_\zeta v_0(t)+b_\zeta\int_0^t v_0(s)\,ds$. Hence, $u_0\in\mathcal{B}(x_0)$. This implies that $\mathcal{B}$ has a closed graph, and so $\mathcal{B}$ is upper semi-continuous. Now, we show that $\mathcal{A}$ is a contraction multifunction. Let $x,y\in\mathcal{X}$

and $u \in \mathcal{A}y$. Choose $v \in S_{\mathcal{F},y}$ such that $u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$ for all $t \in I$. Since

$$\begin{aligned} &H_d\bigl(\mathcal{F}\bigl(t, x(t), (\phi x)(t), {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t), \ldots, {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\bigr), \\ &\quad \mathcal{F}\bigl(t, y(t), (\phi y)(t), {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_1}y(t), \ldots, {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_n}y(t)\bigr)\bigr) \\ &\quad \le \eta(t)\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x - y\| \end{aligned}$$

for almost all $t \in I$, there exists $w \in \mathcal{F}(t, x(t), (\phi x)(t), {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t), \ldots, {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t))$ such that $|v(t) - w| \le \eta(t)(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})\|x - y\|$ for almost all $t \in I$. Consider the multifunction $U : I \to 2^{\mathbb{R}}$ defined by

$$U(t) = \left\{ w \in \mathbb{R} : \bigl|v(t) - w\bigr| \le \eta(t)\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x - y\| \text{ for almost all } t \in I \right\}.$$

Since v and $\eta(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})$ are measurable, we get

$$U(\cdot) \cap \mathcal{F}\bigl(t, x(\cdot), (\phi x)(\cdot), {}^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(\cdot), \ldots, {}^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(\cdot)\bigr)$$

is a measurable multifunction. Choose

$$v'(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_1}x(t), \ldots, {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_n}x(t)\bigr)$$

such that $|v(t) - v'(t)| \le \eta(t)(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})\|x - y\|$ and $u'(t) = a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds$ for all $t \in I$. Since $|u(t) - u'(t)| \le a_\zeta (v(t) - v'(t)) + b_\zeta \int_0^t (v(s) - v'(s))\,ds$ and

$$\bigl|{}_N^{\mathrm{CF}}\mathcal{D}^{\beta_i}u(t) - {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_i}u'(t)\bigr| \le \frac{B(\beta_i)}{(1-\beta_i)^2}\bigl|u(t) - u'(t)\bigr|,$$

we get

$$\begin{aligned} \max_{t\in I}\bigl|u(t) - u'(t)\bigr| &\le a_\zeta \eta^*\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x - y\| \\ &\quad + b_\zeta \eta^*\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x - y\| \\ &= \eta^*\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2}\right)\|x - y\| \end{aligned}$$

and

$$\max_{t\in I}\bigl|{}_N^{\mathrm{CF}}\mathcal{D}^{\beta_i}u(t) - {}_N^{\mathrm{CF}}\mathcal{D}^{\beta_i}u'(t)\bigr| \le \eta^*\frac{B(\beta_i)}{(1-\beta_i)^2}\left(1 + \zeta_0 + \sum_{i=1}^{n} \frac{1}{(1-\beta_i)^2}\right)\|x - y\||$$

for $1 \le i \le n$. Hence, $\|u - u'\| \le \eta^*(1 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})(1 + \zeta_0 + \sum_{i=1}^{n} \frac{B(\beta_i)}{(1-\beta_i)^2})\|x - y\|$. This implies that $H_d(\mathcal{A}x, \mathcal{A}y) \le L\|x - y\|$. Now, by using Theorem 7, the inclusion fixed point problem $x \in \mathcal{A}x + \mathcal{B}x$ has a solution which is a solution for the inclusion problem (3). □

Now, we are ready to investigate the fractional integro-differential inclusion

$$
\begin{aligned}
&{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta}\left(\frac{x(t)}{g(t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n}x(t))}\right)\\
&\quad\in \mathcal{G}\bigl(t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_k}x(t)\bigr)
\end{aligned}
\tag{4}
$$

with boundary condition $u(0) = 0$, where $\zeta,\zeta_1,\dots,\zeta_n,\beta_1,\dots,\beta_k \in (0,1)$, $g : I \times \mathbb{R}^{n+3} \to \mathbb{R}\backslash\{0\}$ is continuous and $\mathcal{G} : I \times \mathbb{R}^{k+3} \to \mathcal{P}(\mathbb{R})$ is a multifunction. We say that $x \in \mathcal{X}$ is a solution for problem (4) whenever it satisfies the boundary conditions and there exists $v \in S_{\mathcal{G},x}$ such that

$$
\begin{aligned}
x(t) &= g\bigl(t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n}x(t)\bigr)\\
&\quad\times\left(a_\zeta v(t) + b_\zeta\int_0^t v(s)\,ds\right),
\end{aligned}
$$

where

$$
\begin{aligned}
S_{\mathcal{G},x} = \bigl\{&v \in L^1[0,1] : v(t) \in \mathcal{G}\bigl(t,x(t),(\phi x)(t),(\varphi x)(t),\\
&{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_k}x(t)\bigr) \text{ for almost all } t \in I\bigr\}.
\end{aligned}
$$

Theorem 14 *Suppose that* $\mathcal{G} : I \times \mathbb{R}^{k+3} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ *is a Caratheodory set-valued map,* $g : J \times \mathbb{R}^{n+3} \to \mathbb{R}\backslash\{0\}$ *is a bounded continuous map with upper bound K and there are continuous functions* $p,m : J \to (0,\infty)$ *such that* $\|\mathcal{G}(t,x_1,x_2,\dots,x_{k+3})\| \le m(s)$ *and*

$$
\bigl|g(t,x_1,x_2,\dots,x_{n+3}) - g(t,y_1,y_2,\dots,y_{n+3})\bigr| \le \eta(t)\sum_{i=1}^{n+3}|x_i - y_i|
$$

for all $t \in I$. *If* $\eta^*(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{n}\frac{B(\beta_i)}{(1-\beta_i)^2}) \cdot K \cdot \|m\|_\infty < 1$, *then the inclusion problem* (4) *has a solution.*

Proof Put $S = \{x \in \mathcal{X} : \|x\| \le L\}$, where $L = K\|m\|_\infty$. It is clear that S is a convex, closed, and bounded subset of the Banach space $\mathcal{X}$. Define $\mathcal{A},\mathcal{B} : S \to \mathcal{P}(\mathcal{X})$ by

$$
\mathcal{A}x(t) = g\bigl\{t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n}x(t)\bigr\}
$$

and

$$
\mathcal{B}x(t) = \left\{u \in \mathcal{X} : \text{ there is } v \in S_{\mathcal{G},x} \text{ such that } u(t) = a_\zeta v(t) + b_\zeta\int_0^t v(s)\,ds \text{ for all } t \in I\right\}.
$$

Thus, the problem of fractional differential inclusions is equivalent to the inclusion problem $x \in \mathcal{A}(x)\mathcal{B}(x)$. Consider the operator $\mathcal{B} = \theta \circ S_{\mathcal{G}}$, where θ is the continuous linear operator on $L^1(I)$ into $\mathcal{X}$ defined by $\theta v(s) = a_\zeta v(t) + b_\zeta\int_0^t v(s)\,ds$. Let $x \in S$ be arbitrary and $\{v_n\}$ be a sequence in $S_{\mathcal{G},x}$. Then $v_n(t) \in \mathcal{G}(t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_k}x(t))$ for almost $t \in I$. Since

$$
\mathcal{G}\bigl(t,x(t),(\phi x)(t),(\varphi x)(t),{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t),\dots,{}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_k}x(t)\bigr)
$$

is compact for all $t \in I$, there is a convergent subsequence of $\{v_n(t)\}$ (show it by $\{v_n(t)\}$ again) to some $v \in S_{\mathcal{G},x}$. Note that $\theta v_n(t) \to \theta v(t)$ pointwise on I because θ is continuous. Now, we show that $\{\theta v_n\}$ is an equi-continuous sequence. Let $\tau < t \in I$. Then we have $|\theta v_n(t) - \theta v_n(\tau)| \leq a_\zeta |v_n(t) - v_n(\tau)| + b_\zeta \int_\tau^t |v_n(s)|\,ds$. Thus, the sequence $\{\theta v_n\}$ is equi-continuous because the right-hand of the inequality tends to 0 as $t \to \tau$. Hence, it has a uniformly convergent subsequence by using the Arzela–Ascoli theorem. Choose a subsequence of $\{v_n\}$ (we show it again by $\{v_n\}$) such that $\theta v_n \to \theta v$. Hence, $\theta v \in \theta(S_{\mathcal{G},x})$ and so $\mathcal{B} = \theta(S_{\mathcal{G},x})$ is compact for all $x \in S$. Here, we prove that $\mathcal{B}x$ is convex for all $x \in S$. Let $x \in S$ and $u, u' \in \mathcal{B}x$. Choose $v, v' \in S_{\mathcal{G},x}$ such that $u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$ and $u'(t) = a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds$ for almost all $t \in I$. Let $0 \leq \lambda \leq 1$. Then we have

$$\lambda u(t) + (1-\lambda)u'(t) = a_\zeta \bigl(\lambda v(t) + (1-\lambda)v'(t)\bigr) + b_\zeta \int_0^t \bigl(\lambda v(s) + (1-\lambda)v'(s)\bigr)\,ds.$$

Since $\mathcal{G}$ is convex-valued, $\lambda u + (1-\lambda)u' \in \mathcal{B}x$. It is clear that $\mathcal{A}$ is bounded, closed, and convex-valued. We show that $\mathcal{A}x\mathcal{B}x$ is a convex subset of S for all $x \in S$. Let $x \in S$ and $u, u' \in \mathcal{A}x\mathcal{B}x$. Choose $v, v' \in S_{\mathcal{G},x}$ such that

$$u(t) = g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n} x(t)\bigr) \times \left(a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds\right),$$

and

$$\begin{aligned} u'(t) &= g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n} x(t)\bigr) \\ &\quad \times \left(a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds\right) \end{aligned}$$

for almost all $t \in I$. Hence,

$$\begin{aligned} \lambda u(t) + (1-\lambda)u'(t) &= g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), \\ &\quad {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n} x(t)\bigr) \\ &\quad \times \biggl[a_\zeta \bigl(\lambda v(t) + (1-\lambda)v'(t)\bigr) \\ &\quad + b_\zeta \int_0^t \bigl(\lambda v(s) + (1-\lambda)v'(s)\bigr)\,ds\biggr]. \end{aligned}$$

Note that $\lambda u + (1-\lambda)u' \in \mathcal{A}x\mathcal{B}x$ because $\mathcal{G}$ is convex-valued. Hence, $\mathcal{A}x\mathcal{B}x$ is a convex subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. However, we have

$$\begin{aligned} \bigl|u(t)\bigr| &= \left|g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n} x(t)\bigr) \times \left(a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds\right)\right| \\ &\leq K(a_\zeta + b_\zeta)\|m\|_\infty = L < 1 \end{aligned}$$

for all $t \in I$, and so $u \in S$ and $\mathcal{A}x\mathcal{B}x$ is a convex subset of S for all $x \in S$. Now, we show that the operator $\mathcal{B}$ is compact. It is enough to prove that $\mathcal{B}(S)$ is uniformly bounded and equi-continuous. Let $u \in \mathcal{B}(S)$. Choose $v \in S_{\mathcal{G},x}$ such that

$$u(t) = g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n} x(t)\bigr) \times \left(a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds\right)$$

for some $x \in S$. Since $|u(t)| \leq K(a_\zeta + b_\zeta)\|m\|_\infty$, $\|u\|_\infty = \max_{t\in I}|u(t)| \leq K(a_\zeta + b_\zeta)\|m\|_\infty$. Now, we prove that $\mathcal{B}$ maps S to equi-continuous subsets of $\mathcal{X}$. Let $t, \tau \in J$ with $\tau < t$, $x \in S$, and $u \in \mathcal{B}x$. Choose $v \in S_{\mathcal{G},x}$ such that $u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$. Then we have

$$\bigl|u(t) - u(\tau)\bigr| \leq a_\zeta \bigl|v(t) - v(\tau)\bigr| + b_\zeta \int_\tau^t \bigl|v(s)\bigr|\,ds.$$

Note that the right-hand side of this inequality tends to 0 as $t \to \tau$. By using the Arzela–Ascoli theorem, we get $\mathcal{B}$ is compact. Here, we show that $\mathcal{B}$ has a closed graph. Let $x_n \in S$ and $u_n \in \mathcal{B}x_n$ for all n with $x_n \to x'$ and $u_n \to u'$. We show that $u' \in \mathcal{B}x'$. For each n, choose $v_n \in S_{\mathcal{G},x_n}$ such that $u_n(t) = a_\zeta v_n(t) + b_\zeta \int_0^t v_n(s)\,ds$ for all $t \in J$. Again, consider the continuous linear operator $\theta : L^1(I) \to \mathcal{X}$ such that $\theta(v)(t) = u(t) = a_\zeta v(t) + b_\zeta \int_0^t v(s)\,ds$. By using Lemma 4, $\theta \circ S_{\mathcal{G}}$ is a closed graph operator. Since $x_n \to x'$ and $u_n \in \theta(S_{\mathcal{G},x_n})$ for all n, there is $v' \in S_{\mathcal{G},x'}$ such that $u'(s) = a_\zeta v'(t) + b_\zeta \int_0^t v'(s)\,ds$. Hence, $u' \in \mathcal{B}x'$. Thus, $\mathcal{B}$ has a closed graph and so $\mathcal{B}$ is upper semi-continuous. Finally note that

$$\begin{aligned} H(\mathcal{A}x, \mathcal{A}y) &= \|\mathcal{A}x - \mathcal{A}y\| \\ &= \max_{t\in I}\Bigl|g\bigl(t, x(t), (\phi x)(t), (\varphi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1}x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n}x(t)\bigr) \\ &\quad - g\bigl(t, y(t), (\phi y)(t), (\varphi y)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_1}y(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\zeta_n}y(t)\bigr)\bigr)\Bigr| \\ &\leq \max_{t\in I}\bigl|\eta(t)\bigr|\left(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{n}\frac{B(\zeta_i)}{(1-\zeta_i)^2}\right)\bigl|x(t) - y(t)\bigr| \\ &= \eta^*(\left(1 + \zeta_0 + \iota_0 + \sum_{i=1}^{n}\frac{B(\zeta_i)}{(1-\zeta_i)^2}\right)\|x - y\|_\infty \end{aligned}$$

for all $x, y \in \mathcal{X}$. Now, by using Theorem 8, the inclusion problem $x \in \mathcal{A}x\mathcal{B}x$ has a solution which is a solution for problem (4). □

In this part, we show that the set of solutions for the second fractional integro-differential inclusion problem is infinite dimensional under some conditions. First we prove the next result.

Lemma 15 *Suppose that $m \in L^1(I, \mathbb{R}^+)$, $\mathcal{F} : I \times \mathbb{R}^{m+n+3} \to \mathcal{P}_{cv,cp}(\mathbb{R})$ is a multivalued map such that the map $t \vdash f(t, x_1, x_2, \ldots, x_{3+m+n})$ is measurable and*

$$\bigl\|\mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3})\bigr\| = \sup\bigl\{|f| : f \in \mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3})\bigr\} \leq m(t)$$

for almost all $t \in I$ and $\in x_1, x_2, \ldots, x_{m+n+3} \in \mathbb{R}$. Define $\Phi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\Phi(x) = \left\{g \in \mathcal{X} : \text{there is } f \in S_{\mathcal{F},x} \text{ such that } g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds \text{ for all } t \in I\right\}.$$

Then $\Phi(x) \in \mathcal{P}_{cp.cv}(\mathcal{X})$ for all $x \in \mathcal{X}$.

Proof Note that $\Phi = \theta \circ S_{\mathcal{F}}$, where $\theta : L^1(I, \mathbb{R}) \to \mathcal{X}$ is the continuous linear map defined by $\theta g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds$. Let $x \in \mathcal{X}$ and $\{g_n\}$ be a sequence in $S_{\mathcal{F},x}$. Then we have

$$g_n(t) \in \mathcal{F}\bigl(t, x(t), (\phi x)(t), (\psi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1}x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m}x(t), {}^{cF}\mathcal{I}^{\gamma_1}x(t), \ldots, {}^{cF}\mathcal{I}^{\gamma_n}x(t)\bigr)$$

for almost $t \in I$. Since

$$\mathcal{F}\big(t, x(t), (\phi x)(t), (\psi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t), {}^{cF}\mathcal{I}^{\gamma_1} x(t), \ldots, {}^{cF}\mathcal{I}^{\gamma_n} x(t)\big)$$

is compact for all $t \in I$, there is a convergent subsequence of $\{g_n(t)\}$ (show it by $\{g_n(t)\}$) which converges to some $g \in S_{\mathcal{F},x}$. Note that $\theta g_n(t) \to \theta g(t)$ pointwise on I because θ is continuous. Here, we prove that $\{\theta g_n\}$ is an equi-continuous sequence. Let $\tau < t \in I$. Then we have $|\theta g_n(t) - \theta g_n(\tau)| = a_\zeta(f(t) - f(\tau)) + b_\zeta \int_\tau^t f(s)\,ds$. Note that the sequence $\{\theta g_n\}$ is equi-continuous because the right-hand side of the inequality tends to zero when $\tau \to t$. Thus, there is a uniformly convergent subsequence of $\{g_n\}$ (show it by $\{g_n\}$ again) such that $\theta g_n \to \theta g$ (we use the Arzela–Ascoli theorem). This implies that $\theta g \in \theta(S_{\mathcal{F},x})$. Hence, $\Phi x = \theta(S_{\mathcal{F},x})$ is compact for all $x \in \mathcal{X}$. Now, we show that Φx is convex for each $x \in \mathcal{X}$. Let $g, g' \in \Phi x$. Choose $f, f' \in S_{\mathcal{F},x}$ such that $g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds)$ and $g'(t) = a_\zeta f'(t) + b_\zeta \int_0^t f'(s)\,ds)$ for almost all $t \in I$. Let $0 \le \lambda \le 1$. Then we have

$$\lambda g(t) + (1-\lambda)g'(t) = a_\zeta\big(\lambda f(t) + (1-\lambda)f'(t)\big) + b_\zeta \int_0^t \big(\lambda f(s) + (1-\lambda)f'(s)\big)\,ds.$$

Since $S_{\mathcal{F},x}$ is convex, $\lambda g + (1-\lambda)g' \in \Phi x$. This completes the proof. □

Note that the fixed point set of Φ is equal to the set of solutions for the inclusion problem (2). Now by using some different conditions, we show that the set of solutions for the fractional integro-differential inclusion problem could be infinite dimensional.

Theorem 16 *Suppose that* $\eta \in L^1(I, \mathbb{R}^+)$, $\mathcal{F} : I \times \mathbb{R}^{m+n+3} \to \mathcal{P}_{cv,cp}(\mathbb{R})$ *is a multivalued map such that the function* $t \vdash \mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3})$ *is measurable,*

$$H\big(\mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3}), \mathcal{F}(t, y_1, y_2, \ldots, y_{m+n+3})\big) \le \eta(t) \sum_{i=1}^{m+n+3} |x_i - y_i|$$

and $\|\mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3})\| = \sup\{|f| : f \in \mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3})\} \le \eta(t)$ *for almost all* $t \in I$ *and* $\in x_1, x_2, \ldots, x_{m+n+3}, y_1, y_2, y_{m+n+3} \in \mathbb{R}$. *If Lebesgue measure of the set*

$$\big\{t : \dim \mathcal{F}(t, x_1, x_2, \ldots, x_{m+n+3}) < 1 \textit{ for some } x_1, x_2, \ldots, x_{m+n+3} \in \mathbb{R}\big\}$$

is zero and $\Delta < 1$, *then the set of all solutions for problem* (2) *is infinite dimensional, where* $\Delta = \eta^*(1 + n + \zeta_0 + \iota_0 + \sum_{i=1}^m \frac{B(\beta_i)}{(1-\beta_i)^2})(1 + n + \sum_{i=1}^m \frac{B(\beta_i)}{(1-\beta_i)^2})$.

Proof Similar to Lemma 15, define the multivalued map $\Phi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\Phi(x) = \left\{g \in \mathcal{X} : \text{ there is } f \in S_{\mathcal{F},x} \text{ such that } g(t) = a_\zeta f(t) + b_\zeta \int_0^t f(s)\,ds \text{ for all } t \in I\right\}.$$

By using Lemma 15, $\Phi x \in \mathcal{P}_{cp,cv}(\mathcal{X})$ for all $x \in \mathcal{X}$. By using a similar proof in Theorem 12, we can prove that Φ is a contractive multivalued map. Now, we show that $\dim \Phi x > k$ for all $x \in \mathcal{X}$ and $k \ge 1$. Let $k \ge 1$, $x \in \mathcal{X}$, and

$$\begin{aligned}\mathcal{G}(t) = \mathcal{F}\big(&t, x(t), (\phi x)(t), (\psi x)(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_1} x(t), {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_2} x(t), \ldots, {}^{\mathrm{CF}}_{N}\mathcal{D}^{\beta_m} x(t),\\ &{}^{cF}\mathcal{I}^{\gamma_1} x(t), {}^{cF}\mathcal{I}^{\gamma_2} x(t), \ldots, {}^{cF}\mathcal{I}^{\gamma_n} x(t)\big)\end{aligned}$$

for all $t \in I$. By using Lemma 9, there are linearly independent measurable selections $g_1, \dots, g_k$ for $\mathcal{G}$. Consider the maps $h_i(t) = a_\zeta g_i(t) + b_\zeta(t) \int_0^t g_i(s)\,ds$ for $i = 1, \dots, k$. Assume that $\sum_{i=1}^k a_i h_i(t) = 0$ for almost $t \in I$. Since $a_\zeta, b_\zeta \neq 0$, by using the Caputo–Fabrizio derivatives, we get $\sum_{i=1}^k a_i g_i(t) = 0$ for almost $t \in I$. Hence, $a_1 = \cdots = a_k = 0$. This implies that $h_1, \dots, h_k$ are linearly independent, and so $\dim \Phi x \geq k$. Hence, we conclude that the set of fixed points of Φ is infinite dimensional by using Theorem 10. Thus, the set of all solutions for problem (2) is infinite dimensional. □

3 Conclusion

We guess that researchers will review different more fractional integro-differential inclusions in the near future. In this manuscript, we first investigate the existence of solutions for four fractional integro-differential inclusions including the new Caputo–Fabrizio derivation which has been introduced recently. Also, we show that dimension of the set of solutions for the second fractional integro-differential inclusion problem is infinite dimensional under some different conditions.

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