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On fractional order multiple integral transforms technique to handle three dimensional heat equation

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Abstract

In this article, we extend the notion of double Laplace transformation to triple and fourth order. We first develop theory for the extended Laplace transformations and then exploit it for analytical solution of fractional order partial differential equations (FOPDEs) in three dimensions. The fractional derivatives have been taken in the Caputo sense. As a particular example, we consider a fractional order three dimensional homogeneous heat equation and apply the extended notion for its analytical solution. We then perform numerical simulations to support and verify our analytical calculations. We use Fox-function theory to present the derived solution in compact form.

Keywords: Fractional integral; Caputo fractional derivative; FOPDEs; Multiple Laplace transform

1 Introduction

Most of the real world problems in engineering, physics, biology, and other applied sciences involve differential equations. This emphasizes the importance of understanding and investigation of differential equations. Unfortunately, very few types of differential equations can be solved analytically. One of the effective techniques for analytical solution of differential equations is the Laplace transform method, which has been used by a number of researchers, see for instance [1, 2]. The idea of Laplace transformation was extended to double Laplace transform and Sumudu by Kiliçman *et al.* for solution of wave and Poisson equations [3]. The Sumudu transform technique was used to solve differential equations in control engineering problems [4, 5].

Because of variety of applications in applied sciences, fractional calculus has attracted the attention of numerous researchers in the past decades, see, e.g., [6–9]. Fractional order models are more general compared with integer order [9] and are helpful in understanding the dynamics of real world problems in a better way. Plenty of open problems can be found in the field of fractional calculus that need investigation both from theoretical and experimental sides [10, 11]. Yang *et al.* [12] introduced an updated fractional operator with variable order to describe spontaneous behavior of the process of diffusion.

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Integral transform procedure due to Laplace was used to investigate solution of FOPDEs [13]. Oldham and Spanier exploited Laplace transform approach to compute solution of homogeneous FOPDEs [14]. Using nonsingular kernel operator of derivative, Yang *et al.* [15] proposed solutions of problems involving steady heat flow. Recently, fixed and variable order derivatives have been applied to investigate the anomalous relaxation models in heat-transfer problems [16]. The Laplace transform technique has been used by various authors to analytically solve FOPDEs. A third order Laplace transform method was used by Tahir *et al.* to solve a fractional order heat equation in two dimensions [17]. Sarwar *et al.* computed series type solution to fractional heat equation in three dimensions [18]. To the best of our knowledge, the homogeneous three dimensional heat equation with non-integer order has not been investigated through Laplace transform yet. In this paper, we extend the notion of triple Laplace transform to fourth order Laplace transform and use the developed theory to solve a fractional order heat equation in three dimensions.

One of the important applications of the heat equation is the measurement of thermal diffusivity in polymers [19]. Heat equation can be utilized to describe the diffusion of pressure in a porous medium. The generalization of heat equation into a fractional order is very important in the nonlocal phenomenon.

As mentioned earlier, fractional calculus generalizes the concept of integrals and derivatives from integer to any positive real order. It means that fractional derivatives, which are in fact definite integrals, provide geometric accumulation of functions. The corresponding accumulation contains the integer order counterpart as a special case. This feature of fractional calculus leads to global dynamics of real world problems, whereas classical calculus describes the local dynamics of the corresponding problem. Further, in many real world phenomena, due to hereditary axioms as well as description of memory, the fractional order models are more beneficial than the classical ones. These interesting and useful features of fractional calculus motivated us to study heat equations under fractional order concept for its global and comprehensive structure analysis. For some recent and useful studies, the reader is advised to see work presented in [20–31].

We consider our problem as

$${}^c_0D_t^\alpha u(x, y, z, t) = \frac{1}{\pi^2} \nabla^2 u(x, y, z, t), \quad x > 0, y > 0, z > 0, t > 0, \quad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. We solve (1) subject to the following initial/ boundary conditions:

$$\begin{aligned} u(x, y, z, 0) &= (\sin \pi x)(\sin \pi y)(\sin \pi z), \\ u(0, 0, 0, t) &= u(0, y, z, t) = u(x, 0, z, t) = u_t(x, y, 0, t) = 0, \\ u_x(0, y, z, t) &= u_y(x, 0, z, t) = u_z(x, y, 0, t) = \pi E_\alpha(-t^\alpha), \end{aligned} \quad (2)$$

where subscripts denote partial derivatives and $0 < \alpha \leq 1$. Here we remark that the considered heat equation is formulated in the Caputo sense.

The fractional order heat equation in three dimensions is obtained by replacing the first order time derivatives of integer order with fractional order time derivative such that $0 < \alpha \leq 1$. In many, situations we need first order time–space derivatives, e.g., in heat transfer etc. In many scientific problems, during their modeling, we need their exact solution which

is quite difficult for most of the nonlinear problems. Therefore, some sophisticated tools are required to deal with such problems. Researchers have used numerical and analytical techniques to handle the problems for corresponding numerical and analytical solutions. Here we use updated tools of a multiple integral transform method based on the Laplace transform to handle the considered problem for exact analytical solution. The result is presented in compact form using the concept of fox function.

The report is structured as follows. In Sect. 2 we give basic definitions regarding fractional derivatives and Laplace transform. Section 3 is devoted to the derivation of Laplace transform of partial derivatives and integrals. We then consider in Sect. 4 the fractional heat equation in three dimensions and apply the fourth order Laplace transform for its solution. Finally, we conclude our work in Sect. 5. References are given at the end of the manuscript.

2 Preliminaries

In the following we summarize basic definitions regarding the terms involved in the problem under consideration. These include the definition of fractional derivatives, the Laplace transforms of first, second, third, and fourth orders.

Definition 2.1 Let $f(t)$ be a function defined on the interval $(0, \infty)$. For $\alpha > 0$, the Riemann–Liouville fractional integral of $f(t)$ of order α is defined as (see, for instance, [7, 8, 32])

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \tag{3}$$

provided the integral on the right converges.

Definition 2.2 Let $f(t)$ be a function defined on the interval $(0, \infty)$. For $\alpha > 0$, the Riemann–Liouville fractional derivative of order α of the function $f(t)$ is defined by

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, \quad \alpha \in (n - 1, n], \tag{4}$$

where the right-hand integral is pointwise defined on $(0, \infty)$ [7, 8].

Definition 2.3 For the function $f(t)$ defined in the interval $(0, \infty)$, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \quad \alpha \in (n - 1, n], \tag{5}$$

where the right-hand integral is pointwise defined on \mathcal{R}^+ , see [8].

Definition 2.4 Let $f(t)$ be a function defined for all $t \geq 0$, where $t \in \mathcal{R}$. The Laplace transform of the function $f(t)$, denoted by $F(s)$, is the function defined by the integral [2]

$$F(s) = \int_0^\infty f(t)e^{-st} dt, \tag{6}$$

where $s > 0$.

Definition 2.4 can be extended into the double Laplace transform, the triple Laplace transform, and the Laplace transform of fourth order as follows.

Definition 2.5 Let $f(x, t)$ be a function of two variables x and t defined for all $x, t \geq 0$, where $x, t \in \mathcal{R}$. “The double Laplace transform” of the function $f(x, t)$ is defined as follows [2, 7]:

$$\mathcal{L}_t \mathcal{L}_x \{f(x, t)(s_1, s_2)\} = F(s_1, s_2) = \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x} f(x, t) dx dt, \tag{7}$$

where $s_1, s_2 > 0$.

Definition 2.6 Let $f(x, y, t)$ be a function of three variables x, y , and t defined for all $x, y, t \geq 0$, and $x, y, t \in \mathcal{R}$. For $s_1, s_2, s_3 > 0$, the triple “Laplace transform” of the function $f(x, y, t)$ is given by

$$\begin{aligned} \mathcal{L}_t \mathcal{L}_x \mathcal{L}_y \{f(x, y, t)(s_1, s_2, s_3)\} &= F(s_1, s_2, s_3) \\ &= \int_0^\infty e^{-s_3 t} \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 t} f(x, y, t) dx dy dt. \end{aligned} \tag{8}$$

Definition 2.7 Let $f(x, y, z, t)$ be a function of four variables x, y, z , and t defined for all $x, y, z, t \geq 0$, $x, y, z, t \in \mathcal{R}$. The Laplace transform of the fourth order for the function $f(x, y, z, t)$ is given by

$$\begin{aligned} \mathcal{L}_t \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \{f(x, y, z, t)(s_1, s_2, s_3, s_4)\} \\ &= F(s_1, s_2, s_3, s_4) \\ &= \int_0^\infty e^{-s_4 t} \int_0^\infty e^{-s_3 t} \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 t} f(x, y, z, t) dx dy dz dt, \end{aligned} \tag{9}$$

such that $s_1, s_2, s_3, s_4 > 0$.

Definition 2.8 Let $\alpha, \beta, t \in \mathbb{C}$ such that $\Re(\alpha) > 0$. The “Mittag-Leffler” function is defined as [11]

$$E_{\alpha, \beta}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\alpha + \beta)}.$$

Definition 2.9 The Laplace transforms of the functions $t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)$ and $t^{\beta-1} E_{\alpha, \beta}(-\lambda t^\alpha)$ can be respectively defined as follows [33]:

$$\mathcal{L} [t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda} \quad \text{for } |\lambda| < |s^\alpha|, \tag{10}$$

$$\mathcal{L} [t^{\beta-1} E_{\alpha, \beta}(-\lambda t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda} \quad \text{for } |\lambda| < |s^\alpha|. \tag{11}$$

Definition 2.10 The Fox function, also referred to as the Fox’s H-function, generalizes the Mellin–Barnes function. The importance of the Fox function lies in the fact that it

includes nearly all special functions occurring in applied mathematics and statistics as special cases. In 1961, Fox defined the H-function as the Mellin–Barnes type path integral:

$$H_{p,q}^{m,n} \left[-\sigma \left| \begin{matrix} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_l \frac{\prod_{k=1}^m (b_k - B_k s) \prod_{j=1}^n \Gamma(1 - a_j + sA_j) \sigma^s ds}{\prod_{k=m+1}^q \Gamma(1 - b_k + B_k s) \prod_{j=n+1}^p (a_j - sA_j)}, \tag{12}$$

where l is a suitable contour, the orders (m, n, p, q) are integers $0 \leq m \leq q, 0 \leq n \leq p$, and the parameters $a_j \in \mathbb{R}, A_j > 0, j = 1, 2, \dots, p, b_k \in \mathbb{R}, B_k > 0, k = 1, 2, \dots, q$, are such that $A_j(b_k + i) \neq B_k(a_j - i - 1), i = 0, 1, 2, \dots$

$$H_{s,t+1}^{1,s} \left[-\sigma \left| \begin{matrix} (1 - a_1, A_1), \dots, (1 - a_s, A_s) \\ (1, 0), (1 - b_1, B_1), \dots, (1 - b_t, B_t) \end{matrix} \right. \right] = \sum_{r=0}^{\infty} \frac{\Gamma(a_1 + A_1 r) \dots \Gamma(a_s + A_s r) \sigma^r}{r! \Gamma(b_1 + B_1 r) \dots \Gamma(b_t + B_t r)}. \tag{13}$$

3 Preliminaries regarding Laplace transforms of first, second, third, and fourth order

In this section, we recall some basic results and notions which we consider helpful for readers to understand the present work. Proof of theorems are omitted as they can be proved by the following steps similar to the case of classical derivatives.

Theorem 3.1 *If $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$ and $l = \max\{m_1, m_2\}$, where $m_1, m_2 \in \mathbb{Z}$. For $i = 1, 2, \dots, m_1$ and $j = 1, 2, \dots, m_2$, there exist $k, \tau_1, \tau_2 > 0$ such that $|\frac{\partial^{i+j} f(x,t)}{\partial x^i \partial t^j}| < k e^{x\tau_1 + t\tau_2}$, then the “double Laplace transform” satisfies the following formulae [34]:*

$$\begin{aligned} \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{m_1} f(x,t)}{\partial x^{m_1}} \right\} &= s_1^{m_1} \mathcal{L}_x \mathcal{L}_t \{f(x,t)\} - \sum_{i=0}^{m_1-1} s_1^{m_1-1-i} \mathcal{L}_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\}, \\ \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{m_2} f(x,t)}{\partial t^{m_2}} \right\} &= s_2^{m_2} \mathcal{L}_x \mathcal{L}_t \{f(x,t)\} - \sum_{j=0}^{m_2-1} s_1^{m_2-1-j} \mathcal{L}_x \left\{ \frac{\partial^j f(0,t)}{\partial x^j} \right\}, \\ \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{m_1+m_2} f(x,t)}{\partial t^{m_1} \partial x^{m_2}} \right\} &= s_1^{m_1} s_2^{m_2} \left[\mathcal{L}_t \mathcal{L}_x \{f(x,t)\} - \sum_{j=0}^{m_2-1} s_2^{-j-1} \mathcal{L}_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\} \right. \\ &\quad \left. - \sum_{i=0}^{m_1-1} s_1^{-i-1} \mathcal{L}_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\} + \sum_{i=0}^{m_2-1} \sum_{j=0}^{m_1-1} s_1^{-i-1} s_2^{-j-1} \left\{ \frac{\partial^{i+j} f(0,0)}{\partial x^i \partial t^j} \right\} \right], \end{aligned} \tag{14}$$

where $\frac{\partial^{m_1+m_2}}{\partial t^{m_1} \partial x^{m_2}} f(x,t)$ denotes a mixed partial derivative at the point (x,t) .

Proof The proof is similar to that of the Laplace transforms of the ordinary derivatives of functions of a single variable, see for more detail [35]. □

Theorem 3.2 *Let $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$, and $l = \max\{m_1, m_2, m_3\}$, where $m_1, m_2, m_3 \in \mathbb{Z}$. For $i_1 = 1, 2, \dots, m_1, i_2 = 1, 2, \dots, m_2$, and $i_3 = 1, 2, \dots, m_3$, there exist $k, \tau_1, \tau_2, \tau_3 > 0$ such*

that $\left| \frac{\partial^{i_1+i_2+i_3} f(x,y,t)}{\partial x^{i_1} \partial y^{i_2} \partial t^{i_3}} \right| < ke^{x\tau_1+y\tau_2+t\tau_3}$, then the triple Laplace transform satisfies the following formulae [34]:

$$\begin{aligned}
 & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_1} f(x,y,t)}{\partial x^{m_1}} \right\} \\
 &= s_1^{m_1} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_t \{f(x,y,t)\} - \sum_{i_1=0}^{m_1-1} s_1^{m_1-1-i_1} \mathcal{L}_t \mathcal{L}_y \left\{ \frac{\partial^{i_1} f(0,y,t)}{\partial x^{i_1}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_2} f(x,y,t)}{\partial y^{m_2}} \right\} \\
 &= s_2^{m_2} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_t \{f(x,y,t)\} - \sum_{i_2=0}^{m_2-1} s_2^{m_2-1-i_2} \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x,0,t)}{\partial y^{i_2}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_3} f(x,y,t)}{\partial t^{m_3}} \right\} \\
 &= s_3^{m_3} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_t \{f(x,y,t)\} - \sum_{i_3=0}^{m_3-1} s_3^{m_3-1-i_3} \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x,y,0)}{\partial t^{i_3}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_1+m_2+m_3} f(x,y,t)}{\partial t^{m_1} \partial y^{m_2} \partial x^{m_3}} \right\} \\
 &= s_1^{m_1} s_2^{m_2} s_3^{m_3} [\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{f(x,y,t)\}] \\
 &\quad - \sum_{i_3=0}^{m_3-1} \sum_{i_2=0}^{m_2-1} s_2^{m_2-i_2-1} s_3^{m_3-i_3-1} \mathcal{L}_x \left\{ \frac{\partial^{i_2+i_3} f(x,0,0)}{\partial t^{i_3} \partial y^{i_2}} \right\} \\
 &\quad - \sum_{i_1=0}^{m_1-1} \sum_{i_3=0}^{m_3-1} s_1^{m_1-i_1-1} s_3^{m_3-i_3-1} \mathcal{L}_y \left\{ \frac{\partial^{i_1+i_3} f(0,y,0)}{\partial t^{i_3} \partial x^{i_1}} \right\} \\
 &\quad - \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} s_1^{m_1-i_1-1} s_2^{m_2-i_2-1} \mathcal{L}_t \left\{ \frac{\partial^{i_1+i_2} f(0,0,t)}{\partial x^{i_1} \partial y^{i_2}} \right\} \\
 &\quad + \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \sum_{i_3=0}^{m_3-1} s_1^{-m_1-1} s_2^{-m_2-1} s_3^{-m_3-1} \left\{ \frac{\partial^{i_1+i_2+i_3} f(0,0,0)}{\partial x^{i_1} \partial y^{i_2} \partial t^{i_3}} \right\}
 \end{aligned} \tag{15}$$

where $\frac{\partial^{i_1+i_2+i_3}}{\partial x^{i_1} \partial y^{i_2} \partial t^{i_3}} f(x,y,t)$ denotes a mixed partial derivative at the point (x,t) .

Proof In a similar fashion it can be proved as the Laplace transforms of the ordinary derivatives of functions of a single variable [35]. □

Theorem 3.3 Let $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$ and $l = \max\{m_1, m_2, m_3, m_4\}$, where $m_1, m_2, m_3, m_4 \in \mathbb{Z}$. For $i_1 = 1, 2, \dots, m_1, i_2 = 1, 2, \dots, m_2, i_3 = 1, 2, \dots, m_3$, and $i_4 = 1, 2, \dots, m_4$, there exist $k, \tau_1, \tau_2, \tau_3, \tau_4 > 0$ such that $\left| \frac{\partial^{i_1+i_2+i_3+i_4} f(x,y,z,t)}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3} \partial t^{i_4}} \right| < ke^{x\tau_1+y\tau_2+z\tau_3+t\tau_4}$, then the

fourth order Laplace transform satisfies the following formulae:

$$\begin{aligned}
 & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_1} f(x, y, z, t)}{\partial x^{m_1}} \right\} \\
 &= s_1^{m_1} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \mathcal{L}_t \{f(x, y, z, t)\} - \sum_{i_1=0}^{m_1-1} s_1^{m_1-1-i_1} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \left\{ \frac{\partial^{i_1} f(0, y, z, t)}{\partial x^{i_1}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_2} f(x, y, z, t)}{\partial y^{m_2}} \right\} \\
 &= s_2^{m_2} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \mathcal{L}_t \{f(x, y, z, t)\} - \sum_{i_2=0}^{m_2-1} s_2^{m_2-1-i_2} \mathcal{L}_t \mathcal{L}_x \mathcal{L}_z \left\{ \frac{\partial^{i_2} f(x, 0, z, t)}{\partial y^{i_2}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_3} f(x, y, z, t)}{\partial t^{m_3}} \right\} \\
 &= s_3^{m_3} \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \mathcal{L}_t \{f(x, y, z, t)\} - \sum_{i_3=0}^{m_3-1} s_3^{m_3-1-i_3} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x, y, z, 0)}{\partial t^{i_3}} \right\}, \\
 & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{m_1+m_2+m_3} f(x, y, z, t)}{\partial t^{m_1} \partial y^{m_2} \partial x^{m_3}} \right\} \tag{16} \\
 &= s_1^{m_1} s_2^{m_2} s_3^{m_3} s_4^{m_4} [\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{f(x, y, z, t)\}] \\
 &\quad - \sum_{i_4=0}^{m_4-1} \sum_{i_3=0}^{m_3-1} \sum_{i_2=0}^{m_2-1} s_2^{m_2-i_2-1} s_3^{m_3-i_3-1} s_4^{m_4-i_4-1} \mathcal{L}_x \left\{ \frac{\partial^{i_2+i_3+i_4} f(x, 0, 0, 0)}{\partial t^{i_4} \partial z^{i_3} \partial y^{i_2}} \right\} \\
 &\quad - \sum_{i_1=0}^{m_1-1} \sum_{i_3=0}^{m_3-1} \sum_{i_4=0}^{m_4-1} s_1^{m_1-i_1-1} s_3^{m_3-i_3-1} s_4^{m_4-i_4-1} \mathcal{L}_y \left\{ \frac{\partial^{i_1+i_3+i_4} f(0, y, 0, 0)}{\partial t^{i_4} \partial z^{i_3} \partial x^{i_1}} \right\} \\
 &\quad - \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \sum_{i_4=0}^{m_4-1} s_1^{m_1-i_1-1} s_2^{m_2-i_2-1} s_4^{m_4-i_4-1} \mathcal{L}_z \left\{ \frac{\partial^{i_1+i_2+i_4} f(0, 0, z, 0)}{\partial t^{i_4} \partial y^{i_2} \partial x^{i_1}} \right\} \\
 &\quad - \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \sum_{i_3=0}^{m_3-1} s_1^{m_1-i_1-1} s_2^{m_2-i_2-1} s_3^{m_3-i_3-1} \mathcal{L}_t \left\{ \frac{\partial^{i_1+i_2+i_3} f(0, 0, 0, t)}{\partial z^{i_3} \partial y^{i_2} \partial x^{i_1}} \right\} \\
 &\quad + \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \sum_{i_3=0}^{m_3-1} \sum_{i_4=0}^{m_4-1} s_1^{-m_1-1} s_2^{-m_2-1} s_3^{-m_3-1} s_4^{-m_4-1} \left\{ \frac{\partial^{i_1+i_2+i_3+i_4} f(0, 0, 0, 0)}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3} \partial t^{i_4}} \right\},
 \end{aligned}$$

where $\frac{\partial^{i_1+i_2+i_3+i_4}}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3} \partial t^{i_4}}$ denotes a mixed partial derivative at the point (x, y, z, t) .

Proof The proof is similar to that of the Laplace transforms of the ordinary derivatives of functions of a single variable, and therefore the readers are suggested to see [35] and the references therein. □

In the following theorems we define the double, triple, and fourth order Laplace transform of fractional integrals.

Theorem 3.4 *Let $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) \geq 0$. Let $a, b \in \mathbb{R}$ with $a, b > 0$ and $f \in \mathcal{L}_1[(0, a) \times (0, b)]$. Further assume that for $x > a, t > b$ and constants $k, \tau_1, \tau_2 > 0$ the inequality $|f(x, t)| \leq ke^{x\tau_1+t\tau_2}$ holds. Then the double Laplace transform of fractional integral*

is given by [34]

$$\mathcal{L}_t \mathcal{L}_x \{ {}_0 I_x^\alpha f(x, t) \} (s_1, s_2) = \frac{1}{s_1^\alpha} \mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} (s_1, s_2), \tag{17}$$

$$\mathcal{L}_t \mathcal{L}_x \{ {}_0 I_t^\beta f(x, t) \} (s_1, s_2) = \frac{1}{s_2^\beta} \mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} (s_1, s_2), \tag{18}$$

and

$$\mathcal{L}_t \mathcal{L}_x \{ {}_0 I_t^\beta {}_0 I_x^\alpha f(x, t) \} (s_1, s_2) = \frac{1}{s_1^\alpha s_2^\beta} \mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} (s_1, s_2), \tag{19}$$

where s_1 and s_2 are parameters of Laplace transforms of x and t respectively.

Proof Formula (17) can be derived by taking the double Laplace transform of the convolution with respect to x . By taking the double Laplace transform of the convolution with respect to t , one can easily prove formula (18). For the proof of formula (19), one may consider the double Laplace transform of the double convolution. For further details on the double Laplace transforms, see [3, 36] and the references therein. \square

Theorem 3.5 *Let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta), \Re(\gamma) \geq 0$. Let $a, b, c \in \mathbb{R}$ with $a, b, c > 0$ and $f \in \mathcal{L}_1[(0, a) \times (0, b) \times (0, c)]$. Further assume that for $x > a, y > b, t > c$ and constants $k, \tau_1, \tau_2, \tau_3 > 0$ the inequality $|f(x, y, t)| \leq ke^{x\tau_1 + y\tau_2 + t\tau_3}$ holds. Then the triple Laplace transform of fractional integrals is given by [34]*

$$\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_x^\alpha f(x, y, t) \} (s_1, s_2, s_3) = \frac{1}{s_1^\alpha} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} (s_1, s_2, s_3), \tag{20}$$

$$\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_y^\beta f(x, y, t) \} (s_1, s_2, s_3) = \frac{1}{s_2^\beta} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} (s_1, s_2, s_3), \tag{21}$$

$$\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_t^\gamma f(x, y, t) \} (s_1, s_2, s_3) = \frac{1}{s_3^\gamma} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} (s_1, s_2, s_3), \tag{22}$$

and

$$\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_t^\gamma {}_0 I_y^\beta {}_0 I_x^\alpha f(x, y, t) \} (s_1, s_2, s_3) = \frac{1}{s_1^\alpha s_2^\beta s_3^\gamma} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} (s_1, s_2, s_3), \tag{23}$$

where $s_1, s_2,$ and s_3 are the parameters of Laplace transforms of $x, y,$ and t respectively.

Theorem 3.6 *Let $\alpha, \beta, \gamma, \sigma \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\sigma) \geq 0$. Let $a, b, c, d \in \mathbb{R}$ with $a, b, c, d > 0$ and $f \in L^1[(0, a) \times (0, b) \times (0, c) \times (0, d)]$. Further assume that for $x > a, y > b, z > c, t > d$ and constants $k, \tau_1, \tau_2, \tau_3, \tau_4 > 0$ the inequality $|f(x, y, z, t)| \leq ke^{x\tau_1 + y\tau_2 + z\tau_3 + t\tau_4}$ holds. Then the triple Laplace transform of fractional integrals is given by*

$$\begin{aligned} &\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_x^\alpha f(x, y, z, t) \} (s_1, s_2, s_3, s_4) \\ &= \frac{1}{s_1^\alpha} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} (s_1, s_2, s_3, s_4), \end{aligned} \tag{24}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_y^\beta f(x, y, z, t) \} (s_1, s_2, s_3, s_4) \\ &= \frac{1}{s_2^\beta} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} (s_1, s_2, s_3, s_4), \end{aligned} \tag{25}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_z^\gamma f(x, y, z, t) \} (s_1, s_2, s_3, s_4) \\ &= \frac{1}{s_3^\gamma} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} (s_1, s_2, s_3, s_4), \end{aligned} \tag{26}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_t^\alpha f(x, y, z, t) \} (s_1, s_2, s_3, s_4) \\ &= \frac{1}{s_2^\alpha} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} (s_1, s_2, s_3, s_4), \end{aligned} \tag{27}$$

and

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}_0 I_t^\alpha {}_0 I_z^\gamma {}_0 I_y^\beta f(x, y, z, t) \} (s_1, s_2, s_3, s_4) \\ &= \frac{1}{s_1^\alpha s_2^\beta s_3^\gamma s_4^\sigma} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} (s_1, s_2, s_3, s_4), \end{aligned} \tag{28}$$

where $s_1, s_2, s_3,$ and s_4 are the parameters of Laplace transforms of $x, y, z,$ and t respectively.

In the theorems given below, we give the double, triple, and fourth order Laplace transforms of the fractional Caputo derivatives.

Theorem 3.7 *Let $m_1, m_2 \in \mathbb{N}$ and $\alpha, \beta > 0$ such that $m_2 - 1 < \alpha \leq m_2, m_1 - 1 < \beta \leq m_1$. Let us choose $l = \max\{m_1, m_2\}$ and let $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$. Assume further that for $a, b > 0$ we have $f^{(l)} \in \mathcal{L}_1[(0, a) \times (0, b)]$. Further assume that for $x > a, t > b$ and constants $k, \tau_1, \tau_2 > 0$ the inequality $|f(x, t)| \leq ke^{x\tau_1 + t\tau_2}$ holds. The double Laplace transforms of the partial fractional Caputo derivatives can be defined as*

$$\mathcal{L}_t \mathcal{L}_x \{ {}_0^c D_x^\alpha f(x, t) \} = s_1^\alpha \left[\mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} - \sum_{i_1=0}^{m_1-1} s_1^{-1-i_1} \mathcal{L}_t \left\{ \frac{\partial^{i_1} f(0, t)}{\partial x^{i_1}} \right\} \right], \tag{29}$$

$$\mathcal{L}_t \mathcal{L}_x \{ {}_0^c D_t^\beta f(x, t) \} = s_2^\beta \left[\mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} - \sum_{i_2=0}^{m_2-1} s_2^{-1-i_2} \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(0, t)}{\partial t^{i_2}} \right\} \right], \tag{30}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_x \{ {}_0^c D_x^\alpha {}_0^c D_t^\beta f(x, t) \} \\ &= s_1^\alpha s_2^\beta \left[\mathcal{L}_t \mathcal{L}_x \{ f(x, t) \} - \sum_{i_1=0}^{m_2-1} s_1^{-1-i_1} \mathcal{L}_t \left\{ \frac{\partial^{i_1} f(0, t)}{\partial x^{i_1}} \right\} \right. \\ & \quad \left. - \sum_{i_2=0}^{m_1-1} s_2^{-1-i_2} \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x, 0)}{\partial t^{i_2}} \right\} + \sum_{i_2=0}^{m_1-1} \sum_{i_1=0}^{m_2-1} s_1^{-i_1-1} s_2^{-i_2-1} \frac{\partial^{i_1+i_2} f(0, 0)}{\partial x^{i_1} \partial t^{i_2}} \right]. \end{aligned} \tag{31}$$

Theorem 3.8 *Let $m_1, m_2, m_3 \in \mathbb{N}$ and $\alpha, \beta, \gamma > 0$ such that $m_3 - 1 < \gamma \leq m_3, m_2 - 1 < \beta \leq m_2, m_1 - 1 < \alpha \leq m_1$. Let us choose $l = \max\{m_1, m_2, m_3\}$ and let $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$. Assume further that for $a, b, c > 0$ we have $f^{(l)} \in \mathcal{L}_1[(0, a) \times (0, b) \times (0, c)]$. Further assume that for $x > a, y > b, t > c$ and constants $k, \tau_1, \tau_2, \tau_3 > 0$ the inequality $|f(x, y, t)| \leq ke^{x\tau_1 + y\tau_2 + t\tau_3}$ holds. The triple Laplace transforms of the partial fractional Caputo derivatives can be*

defined as [17]

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}^c D_x^\alpha f(x, y, t) \} \\ &= s_1^\alpha \left[\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} - \sum_{i_1=0}^{m_1-1} s_1^{-1-i_1} \mathcal{L}_y \mathcal{L}_t \left\{ \frac{\partial^{i_1} f(0, y, t)}{\partial x^{i_1}} \right\} \right], \end{aligned} \tag{32}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}^c D_y^\beta f(x, y, t) \} \\ &= s_2^\beta \left[\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} - \sum_{i_2=0}^{m_2-1} s_2^{-1-i_2} \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x, 0, t)}{\partial y^{i_2}} \right\} \right], \end{aligned} \tag{33}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}^c D_t^\gamma f(x, y, t) \} \\ &= s_3^\gamma \left[\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} - \sum_{i_3=0}^{m_3-1} s_3^{-1-i_3} \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x, y, 0)}{\partial t^{i_3}} \right\} \right], \end{aligned} \tag{34}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ {}^c D_x^\alpha {}^c D_y^\beta {}^c D_t^\gamma f(x, y, t) \} \\ &= s_1^\alpha s_2^\beta s_3^\gamma \left[\mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \{ f(x, y, t) \} - \sum_{i_1=0}^{m_1-1} s_1^{-1-i_1} \mathcal{L}_t \mathcal{L}_y \left\{ \frac{\partial^{i_1} f(0, y, t)}{\partial x^{i_1}} \right\} \right. \\ & \quad - \sum_{i_2=0}^{m_2-1} s_2^{-1-i_2} \mathcal{L}_t \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x, 0, t)}{\partial y^{i_2}} \right\} - \sum_{i_3=0}^{m_3-1} s_3^{-1-i_3} \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x, y, 0)}{\partial t^{i_3}} \right\} \\ & \quad \left. + \sum_{i_3=0}^{m_3-1} \sum_{i_2=0}^{m_2-1} \sum_{i_1=0}^{m_1-1} s_1^{-i_1-1} s_2^{-i_2-1} s_3^{-i_3-1} \frac{\partial^{i_1+i_2+i_3} f(0, 0, 0)}{\partial x^{i_1} \partial y^{i_2} \partial t^{i_3}} \right]. \end{aligned} \tag{35}$$

Theorem 3.9 *Let $m_1, m_2, m_3, m_4 \in \mathbb{N}$ and $\alpha, \beta, \gamma, \sigma > 0$ such that $m_4 - 1 < \sigma \leq m_4$, $m_3 - 1 < \gamma \leq m_3$, $m_2 - 1 < \beta \leq m_2$, $m_1 - 1 < \alpha \leq m_1$. Let us choose $l = \max\{m_1, m_2, m_3, m_4\}$ and let $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$. Assume further that for $a, b, c, d > 0$ we have $f^{(l)} \in \mathcal{L}_1[(0, a) \times (0, b) \times (0, c) \times (0, d)]$. Further assume that for $x > a, y > b, z > c, t > d$ and constants $k, \tau_1, \tau_2, \tau_3, \tau_4 > 0$ the inequality $|f(x, y, z, t)| \leq ke^{x\tau_1 + y\tau_2 + z\tau_3 + t\tau_4}$ holds. The fourth order Laplace transforms of the partial fractional Caputo derivatives can be defined as*

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}^c D_x^\alpha f(x, y, z, t) \} \\ &= s_1^\alpha \left[\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} - \sum_{i_1=0}^{m_1-1} s_1^{-1-i_1} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_t \left\{ \frac{\partial^{i_1} f(0, y, z, t)}{\partial x^{i_1}} \right\} \right], \end{aligned} \tag{36}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}^c D_y^\beta f(x, y, z, t) \} \\ &= s_2^\beta \left[\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} - \sum_{i_2=0}^{m_2-1} s_2^{-1-i_2} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x, 0, z, t)}{\partial y^{i_2}} \right\} \right], \end{aligned} \tag{37}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}^c D_z^\gamma f(x, y, z, t) \} \\ &= s_3^\gamma \left[\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} - \sum_{i_3=0}^{m_3-1} s_3^{-1-i_3} \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x, y, 0, t)}{\partial z^{i_3}} \right\} \right], \end{aligned} \tag{38}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}^c D_t^\alpha f(x, y, z, t) \} \\ &= s_4^\alpha \left[\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} - \sum_{i_4=0}^{m_4-1} s_4^{-1-i_4} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_4} f(x, y, z, 0)}{\partial t^{i_4}} \right\} \right], \end{aligned} \tag{39}$$

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ {}^c D_x^\alpha {}^c D_y^\beta {}^c D_z^\gamma {}^c D_t^\sigma f(x, y, z, t) \} \\ &= s_1^\alpha s_2^\beta s_3^\gamma \left[\mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{ f(x, y, z, t) \} \right. \\ & \quad - \sum_{i_1=0}^{m_2-1} s_1^{-1-i_1} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \left\{ \frac{\partial^{i_1} f(0, y, z, t)}{\partial x^{i_1}} \right\} - \sum_{i_2=0}^{m_2-1} s_2^{-1-i_2} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x \left\{ \frac{\partial^{i_2} f(x, 0, z, t)}{\partial y^{i_2}} \right\} \\ & \quad - \sum_{i_3=0}^{m_3-1} s_3^{-1-i_3} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_3} f(x, y, 0, t)}{\partial z^{i_3}} \right\} - \sum_{i_3=0}^{m_3-1} s_4^{-1-i_4} \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_4} f(x, y, z, 0)}{\partial t^{i_4}} \right\} \\ & \quad - \sum_{i_4=0}^{m_4-1} \sum_{i_3=0}^{m_3-1} s_4^{-1-i_4} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \left\{ \frac{\partial^{i_4} f(x, y, z, 0)}{\partial t^{i_4}} \right\} \\ & \quad \left. + \sum_{i_4=0}^{m_4-1} \sum_{i_3=0}^{m_3-1} \sum_{i_2=0}^{m_2-1} \sum_{i_1=0}^{m_1-1} s_1^{-i_1-1} s_2^{-i_2-1} s_3^{-i_3-1} s_4^{-i_4-1} \frac{\partial^{i_1+i_2+i_3+i_4} f(0, 0, 0, 0)}{\partial x^{i_1} \partial y^{i_2} \partial z^{i_3} \partial t^{i_4}} \right]. \end{aligned} \tag{40}$$

4 Solution of third order fractional heat equation

In this section of the manuscript we use the ideas discussed in the previous sections to solve a third order fractional heat equation (1) subject to initial/ boundary conditions (2). Applying fourth order Laplace transformation to both sides of (1) and exploiting the linearity of the fourth order Laplace transform, one can write

$$\mathcal{L}_t \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \{ {}^c D_0^\alpha u(x, y, z, t) \} = \frac{1}{\pi^2} \mathcal{L}_t \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z \{ \nabla^2 u(x, y, z, t) \}. \tag{41}$$

In the light of Theorem 3.7, Eq. (41) takes the form

$$\begin{aligned} & s_4^\alpha \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) - s_4^{\alpha-1} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) - s_4^{\alpha-2} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) \\ &= \frac{1}{\pi^2} \left\{ s_1^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) - s_1 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y u(0, y, z, t) - \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \frac{\partial}{\partial x} u(0, y, z, t) \right. \\ & \quad + s_2^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) - s_2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x u(x, 0, z, t) - \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x \frac{\partial}{\partial y} u(x, 0, z, t) \\ & \quad \left. + s_3^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) - s_3 \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x u(x, y, 0, t) - \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \frac{\partial}{\partial z} u(x, y, 0, t) \right\}, \end{aligned}$$

which implies that

$$\begin{aligned} & s_4^\alpha \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) - \frac{1}{\pi^2} \left\{ s_1^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) + s_2^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) \right. \\ & \quad \left. + s_3^2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) \right\} \\ &= s_4^{\alpha-1} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) + s_4^{\alpha-2} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) \\ & \quad + \frac{1}{\pi^2} \left\{ s_1 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y u(0, y, z, t) + \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \frac{\partial}{\partial x} u(0, y, z, t) + s_2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x u(x, 0, z, t) \right. \end{aligned}$$

$$+ \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x \frac{\partial}{\partial y} u(x, 0, z, t) + s_3 \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x u(x, y, 0, t) + \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \frac{\partial}{\partial z} u(x, y, 0, t) \Big\}.$$

Using some algebraic manipulation, the above equation looks like

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) \left\{ s_4^\alpha - \frac{1}{\pi^2} (s_1^2 + s_2^2 + s_3^2) \right\} \\ &= s_4^{\alpha-1} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) \\ & \quad - s_4^{\alpha-2} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) - \frac{1}{\pi^2} \left\{ s_1 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y u(0, y, z, t) + \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \frac{\partial}{\partial x} u(0, y, z, t) \right. \\ & \quad + s_2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x u(x, 0, z, t) + \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x \frac{\partial}{\partial y} u(x, 0, z, t) + s_3 \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x u(x, y, 0, t) \\ & \quad \left. + \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x \frac{\partial}{\partial z} u(x, y, 0, t) \right\}. \end{aligned}$$

The simplification and little re-arrangement of the above equation leads to the following assertion:

$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, t) \\ &= \frac{\pi^2}{\pi^2 s_4^\alpha - (s_1^2 + s_2^2 + s_3^2)} \left\{ s_4^{\alpha-1} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) - s_4^{\alpha-2} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x u(x, y, z, 0) \right. \\ & \quad - \frac{1}{\pi^2} \left[s_1 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y u(0, y, z, t) + \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y u_x(0, y, z, t) + s_2 \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x u(x, 0, z, t) \right. \\ & \quad \left. \left. + \mathcal{L}_t \mathcal{L}_z \mathcal{L}_x u_y(x, 0, z, t) + s_3 \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x u(x, y, 0, t) + \mathcal{L}_t \mathcal{L}_y \mathcal{L}_x u_z(x, y, 0, t) \right] \right\}. \tag{42} \end{aligned}$$

Now, applying the definition of triple Laplace transformation given in Theorem 3.8 to the initial/boundary conditions (2), we obtain

$$\begin{aligned} \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{u(x, y, z, 0)\} &= \frac{\pi^3}{(s_1^2 + \pi^2)(s_2^2 + \pi^2)(s_3^2 + \pi^2)}, \\ \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{u_t(x, y, z, 0)\} &= \mathcal{L}_z \mathcal{L}_x \mathcal{L}_t u(x, 0, z, t) = \mathcal{L}_z \mathcal{L}_y \mathcal{L}_t u(0, y, z, t) \\ &= \mathcal{L}_y \mathcal{L}_x \mathcal{L}_t u(x, y, 0, t) = 0, \\ \mathcal{L}_z \mathcal{L}_y \mathcal{L}_t \{u_x(x, y, z, t)\} &= \frac{\pi s_4^{\alpha-1}}{s_3 s_2 (1 + s_4^\alpha)}, \mathcal{L}_z \mathcal{L}_x \mathcal{L}_t \{u_y(x, y, z, t)\} \\ &= \frac{\pi s_4^{\alpha-1}}{s_3 s_1 (1 + s_4^\alpha)}, \\ \mathcal{L}_y \mathcal{L}_x \mathcal{L}_t \{u_z(x, y, z, t)\} &= \frac{\pi s_4^{\alpha-1}}{s_1 s_2 (1 + s_4^\alpha)}. \end{aligned} \tag{43}$$

The introduction of Eq. (43) to Eq. (42) leads to

$$\begin{aligned} \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{u(x, y, z, t)\} &= \frac{\pi^2 s_4^{\alpha-1}}{\pi^2 s_4^\alpha - (s_1^2 + s_2^2 + s_3^2)} \left[\frac{\pi^3}{(s_1 + \pi^2)(s_2 + \pi^2)(s_3 + \pi^2)} \right. \\ & \quad \left. + \frac{1}{\pi (1 + s_4^\alpha)} \left\{ \frac{1}{s_1 s_2} + \frac{1}{s_2 s_3} + \frac{1}{s_1 s_3} \right\} \right]. \end{aligned} \tag{44}$$

To find the inverse Laplace of Eq. (44), we shall exploit the following formula:

$$(1+x)^\alpha = \sum_{p=0}^\infty \frac{1}{p!} \frac{\Gamma(p-\alpha)}{\Gamma(\alpha)} (-x)^p. \tag{45}$$

As a result, Eq. (44), after some algebraic manipulation, can be written as follows:

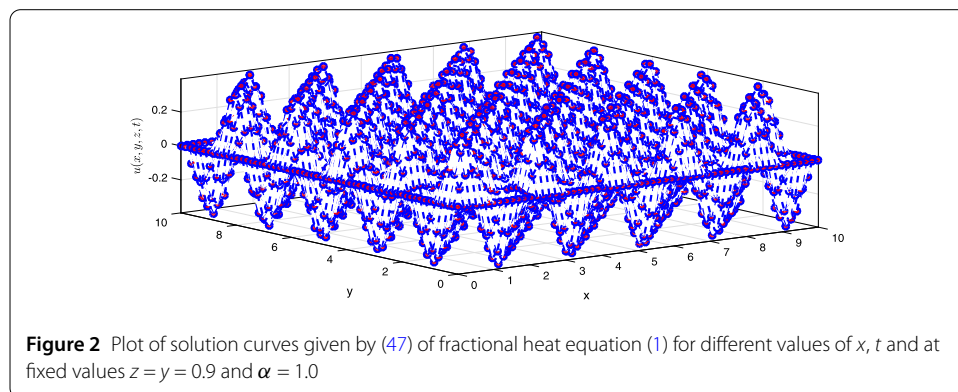
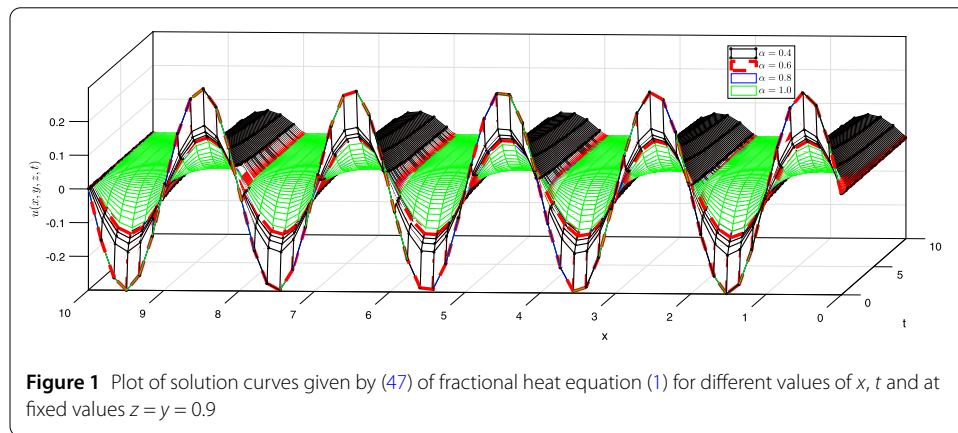
$$\begin{aligned} & \mathcal{L}_t \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x \{u(x, y, z, t)\} \\ &= \sum_{p=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{\Gamma(v-p)\Gamma(v-w)}{v!w!\Gamma(p)\Gamma(v)} \\ & \times \left[\sum_{q=0}^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(-1)^{q+v+w+r}}{\Gamma(w)\pi^{2(p+q+r+s+w)} s_1^{2(v-p-q-r)} s_2^{-2(v+r+w)} s_3^{-2(p+q+r)} s_4^{2\alpha-\alpha p+\alpha w+1}} \right. \\ & - \frac{(-1)^{v+w+m}}{\pi^{2(p+w+v+1)}} \left\{ \sum_{m=0}^\infty \frac{1}{s_1^{2(m-v-p)} s_2^{(v-r+w)+1} s_3^{2(m-r+w)+1} s_4^{2(\alpha m+p+w)}} \right. \\ & - \sum_{n=0}^\infty \frac{1}{s_1^{2(m-v-p)+1} s_2^{2(v-r+w)} s_3^{2(n-r+w)+1} s_4^{2(\alpha n+p+w)}} \\ & \left. \left. - \sum_{c=0}^\infty \frac{1}{s_1^{2(m-v-p)+1} s_2^{2(v-r+w)+1} s_3^{2(c-r+w)} s_4^{2(\alpha c+p+w)}} \right\} \right]. \tag{46} \end{aligned}$$

Finally, by the application of inverse Laplace transform to Eq. (46) and using the Fox H -function, the solution of the fractional three dimensional heat equation can be expressed as follows:

$$\begin{aligned} & u(x, y, z, t) \\ &= \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty \sum_{v=0}^\infty \frac{(-1)^{q+v+r} x^{2v-2p-2q-2r-1} y^{-2v-2r-1} z^{-2p-2q-1} t^{2\alpha-\alpha p}}{v! \pi^{2p+2q+2r+2s}} \\ & \times \mathbf{H}_{1,12}^{2,1} \left[-\left(\frac{t^\alpha \pi^2}{y^2 z^2}\right) \left| \begin{matrix} (1-v+p, 0), (1-v, 1), \\ (0, 1), (2, 0)^4, (1-p, 0), (1-v, 0), (1, 1), \\ (1+2v+2p+2q+2r, 0), (1+2v+2r, 2), \\ (1+2p+2q, 2), (-2\alpha+\alpha p, \alpha). \end{matrix} \right. \right] \\ & - \sum_{p=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{(-1)^{v+w} x^{-2v-2p-1} y^{2v-2r+2w} z^{-2r+2w} t^{2p+2w-1}}{v!w!\pi^{2+2p+2w+2v}} \\ & \times \mathbf{H}_{1,8}^{2,1} \left[-(xzt^\alpha)^2 \left| \begin{matrix} (1-v+p, 0), (1-v-w, 0), \\ (2, 0)^2, (1-p, 0), (1-v, 0), (1+2v+2p, 2), \\ (-2v+2r-2w, 0), (2r-2w, 0), (1-2p-2w, 2\alpha). \end{matrix} \right. \right] \\ & - \sum_{p=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{(-1)^{v+w} x^{-2v-2p} y^{2v-2r+2w-1} z^{-2r+2w} t^{2p+2w-1}}{v!w!\pi^{2+2p+2w+2v}} \\ & \times \mathbf{H}_{1,8}^{2,1} \left[-(xzt^\alpha)^2 \left| \begin{matrix} (1-v+p, 0), (1-v-w, 0), \\ (2, 0)^2, (1-p, 0), (1-v, 0), (2v+2p, 2), \\ (1-2v+2r-2w, 0), (2r-2w, 0), (1-2p-2w, 2\alpha). \end{matrix} \right. \right] \end{aligned}$$

$$\begin{aligned}
 & - \sum_{p=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^{v+w} x^{-2v-2p} y^{2v-2r+2w} z^{-2r+2w-1} t^{2p+2w-1}}{v!w! \pi^{2+2p+2w+2v}} \\
 & \times \mathbf{H}_{1,8}^{2,1} \left[-(xzt^\alpha)^2 \begin{matrix} (1-v+p, 0), (1-v-w, 0), \\ (2, 0)^2, (1-p, 0), (1-v, 0), (2v+2p, 2), \\ (-2v+2r-2w, 0), (1-2r-2w, 0), (1-2p-2w, 2\alpha). \end{matrix} \right]. \quad (47)
 \end{aligned}$$

We present some graphical visualization of the analytical results carried out to study the effect of temperature profile of the proposed problem by varying x, y, z and fractional order parameter α . We study the effect of the variables x, y, z and the fractional order α upon the temperature profile $u(x, y, z, t)$ as a function of the time variable t in the first two figures, i.e., Figs. 1 and 2. To study the effect of t and x on the temperature profile $u(x, y, z, t)$ along different value of fractional order α , we fixed the value of z and y , while varying x and t as shown in Fig. 1. Likewise the effect of x and y with $\alpha = 1$ is given in Fig. 2. Moreover, to describe the relative impact of each component x, y, z , and t against various values of fractional order α , we plotted the contour plots as demonstrated in Figs. 3, 4, 5, and 6 in a different plane against various values of fractional order. Here the graphical results are presented to show the effect of fractional order on the temperature profile $u(x, y, z, t)$. Clearly we observed that the fractional order parameter α has a significant impact on the temperature profile as seen in the graphical results. Likewise, to compare our results with other methods which have been utilized to the proposed fractional order heat equation, a natural transform decomposition technique has been reported by Hassan Khan *et al.*



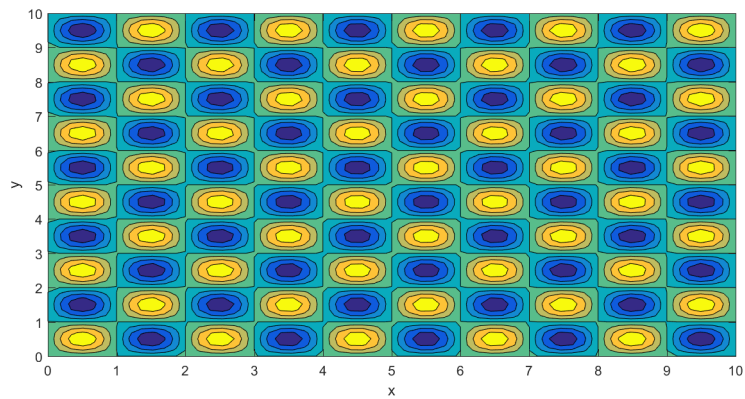


Figure 3 The contour plot of the solution curves given by (47) of fractional heat equation (1) for constant values $t = 0.5, z = 0.5$ at different values of $\alpha = 0.4, 0.6, 0.8, 1.0$ in the xy -plane

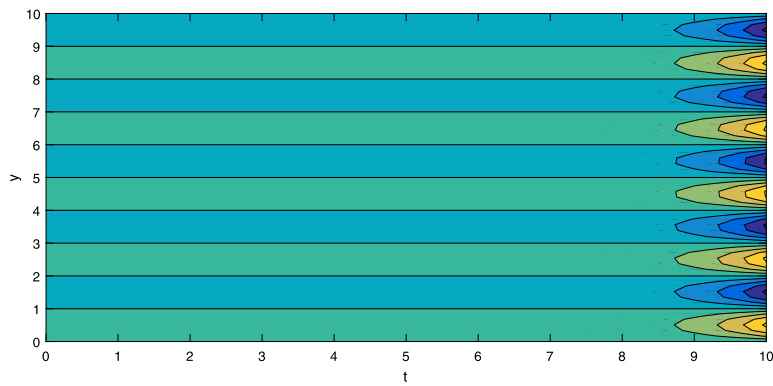


Figure 4 The contour plot of the solution curves given by (47) of fractional heat equation (1) for constant values of $x = 0.5, z = 0.5$ at different values of $\alpha = 0.4, 0.6, 0.8, 1.0$ in the ty -plane

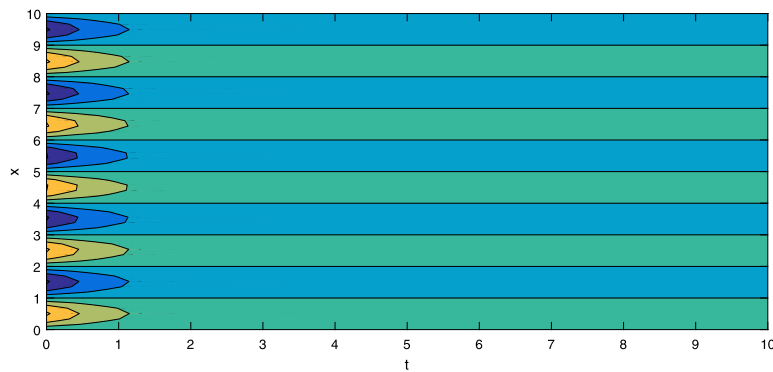
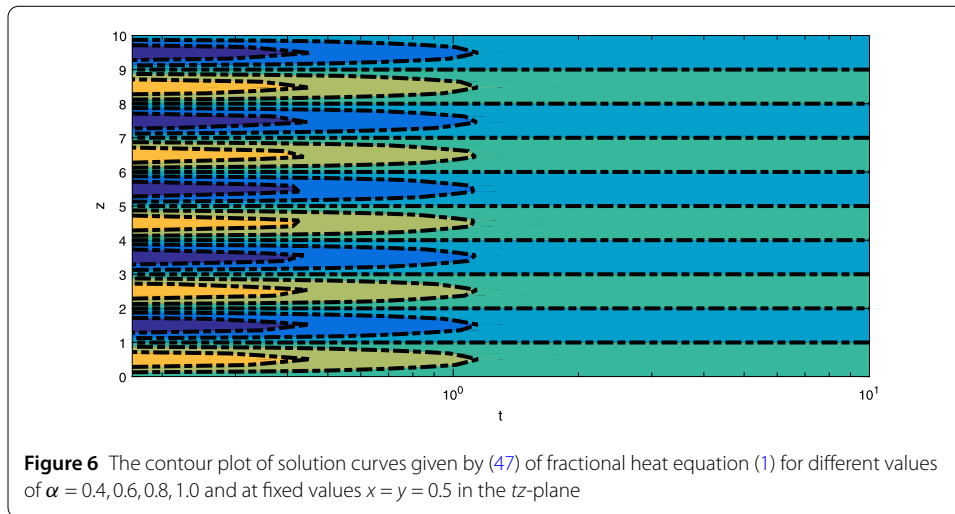


Figure 5 The contour solution plot given by (47) of fractional heat equation (1) for different values of $\alpha = 0.4, 0.6, 0.8, 1.0$ and fixed values of the variables $z = y = 0.5$ in the tx -plane

in [25]. It could be noted that the solution obtained by the fourth order Laplace transform revealed the highest degree of accuracy. It is also analyzed that the fractional order



solutions are more feasible while comparing with the integer order. Thus, the method of generalized Laplace transform is one of the best methods in order to find the solution of partial differential equations having fractional order.

5 Conclusion

Fractional calculus is a developing area in the field of mathematics, science, and technology. It is observed that the anomalous behavior of complex systems in various fields of science and technology is studied via fractional-order systems, e.g., the anomalous behavior of dynamical systems in electrochemistry, physics, viscoelasticity, biology, and chaotic systems. In this work, we have extended a three dimensional heat equation from integer order to fractional order and investigated its associated analytical solution via the generalized Laplace transformation. For this purpose, first we generalized the notion of double Laplace transformation to the triple one and then to fourth order. We have developed theory related to the Laplace transform of the third and fourth orders. The established theory has been used to analytically solve a fractional order partial differential equation. We have solved a three dimensional heat problem with noninteger order using the multiple transform method of Laplace. Numerical simulations have been presented to verify our analytical calculations.

In the future, the concerned technique of multiple transform method may be used to study more general and complex problems of higher dimension with noninteger order derivatives.

Acknowledgements

J. Alzabut is thankful to Prince Sultan University and OSTIM Technical University for their endless support for writing this paper.

Funding

Not applicable.

Availability of data and materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate

This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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Received: 12 October 2021 Accepted: 9 March 2022 Published online: 01 April 2022

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