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On Framed Link Groups

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Introduction

In this paper we shall study an invariant of framed links in 3-manifolds and apply it to investigate 3- and 4-manifolds.

In recent years relations between framed links and 3- or 4-manifolds have been revealed by Kirby [3], Montesinos [4] and others. In their papers it was proved that closed orientable 3-(resp. 4-)manifolds correspond to some equivalence classes of framed links in S^3 (resp. $\#_n S^1\times S^2$) bijectively (resp. injectively). Thus link-theoretical invariants of the equivalence relation can be regarded as invariants of 3-(or 4-)manifolds.

To find such invariants is an important and interesting problem as is mentioned in [2]. But almost all invariants of links known before, for examples, link groups, signatures and so on, are not efficient since they are not invariant under "band moves". On the contrary, framed link groups which we shall define in the following section are invariant under these moves. Moreover they change in a simple fashion when they are "stabilized".

In § 1 we shall give the definition of framed link groups and prove its invariance under "band moves". In § 2 and § 3 we shall study the relations between framed link groups and 3- and 4-manifold theories.

§ 1. Framed link groups.

We work in the smooth category. Let M^3 be a connected 3-manifold without boundary.

DEFINITION 1. $L\subset M$ is called a framed link in M with k components if L is a disjoint union of circles l_i $(i=1, 2, \cdots, k)$ with tubular neighbourhoods N_i and framings $f_i: l_i\times D^2\to N_i$ such that $N_i\cap N_j=\emptyset$ $(i\neq j)$.

REMARK. For a link in a 3-sphere S^3 or a 3-disk D^3, a framing is indicated by an integer as in [3].

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For a component l_i of L, let $p_i=f_i(l_i\times(1/2, 0))$ where $(1/2, 0)\in D^2=\{(x, y)\in R^2|x^2+y^2\leqq 1\}$ and let $m_i=f_i(q_i\times\partial D^2)$ where $q_i\in l_i$. Here we call p_i a parallel and m_i a meridian of l_i.

Let $G(L)=\pi_1(M-L)$. When we give orientations to loops p_i and m_i, and combine these to the base point by arcs, we obtain elements of $G(L)$, which we also call a parallel and a meridian of l_i. (They are also denoted by p_i and m_i.) Clearly a parallel and a meridian are determined up to conjugation and inversion. Let $Q(L)$ be the normal subgroup of $G(L)$ which is generated by p_i's $(i=1, 2, \cdots, k)$. The subgroup $Q(L)$ does not depend on the choice of parallels.

For subsets A, B of a group G, let $[A, B]=\{\prod a_ib_ia_i^{-1}b_i^{-1}|a_i\in A, b_i\in B\}$ and $N(A)=\{\prod g_ia_ig_i^{-1}|a_i\in A, g_i\in G\}$.

DEFINITION 2. Let $FG(L)=G(L)/[Q(L), G(L)]$. We call it the framed link group of L.

Note that $FG(L)$ is a finitely presented group whenever $G(L)$ is so. More exactly we have the following lemma.

LEMMA. *Let $L\subset M$ be a framed link. Suppose that $G(L)$ is finitely presented as $\langle x_1, \cdots, x_n; r_1, \cdots, r_m\rangle$. Then $FG(L)$ is presented as $\langle x_1, \cdots, x_n; r_1, \cdots, r_m, [p_i(x_1, \cdots, x_n), x_j]$ $(i=1, \cdots, k, j=1, \cdots, n)\rangle$, where $p_i(x_1, \cdots, x_n)$ is a word representing the parallel p_i.*

PROOF. Let $h: \langle x_1, \cdots, x_n; r_1, \cdots, r_m\rangle\to G(L)$ be an isomorphism. Let N be the normal subgroup generated by $[p_i(x_1, \cdots, x_n), x_j]$ $(i=1, \cdots, k, j=1, \cdots, n)$. Note that $N\ni[p_i(x_1, \cdots, x_n), w]$ $(i=1, \cdots, k)$ for any word w in $x_1, \cdots, x_n$. Since h maps $[p_i(x_1, \cdots, x_n), x_j]$ into $[Q(L), G(L)]$ and $[Q(L), G(L)]$ is normal, we have $h(N)\subset[Q(L), G(L)]$. On the other hand, any element of $[Q(L), G(L)]$ is presented by a product of elements $h_i[p_i, g_i]h_i^{-1}$ for some $h_i, g_i\in G(L)$ where $p_i\in G(L)$ is a parallel. This means $[Q(L), G(L)]\subset h(N)$ and thus $[Q(L), G(L)]=h(N)$. From the definitions of factor groups and finitely presented groups, we obtain $\langle x_1, \cdots, x_n; r_1, \cdots, r_m\rangle/N\cong\langle x_1, \cdots, x_n; r_1, \cdots, r_m, [p_i(x_1, \cdots, x_n), x_j]$ $(i=1, \cdots, k, j=1, \cdots, n)\rangle$. Thus $G(L)/[Q(L), G(L)]\cong\langle x_1, \cdots, x_n; r_1, \cdots, r_m, [p_i(x_1, \cdots, x_n), x_j]$ $(i=1, \cdots, k, j=1, \cdots, n)\rangle$. This completes the proof.

Let p denote the canonical surjection $G(L)\to FG(L)$.

DEFINITION 3. Let $P(L)=p(Q(L))$. We call it the parallel subgroup of $FG(L)$.

Clearly $P(L)$ is contained in the center of $FG(L)$.

The rest of this section is offered to prove that $FG(L)$ and $P(L)$ are unchanged by "band moves". Here we recall "band moves".

DEFINITION 4. Let L be a framed link having at least 2 components. Let l_i, l_j be components of L and b be a band which connects l_i and p_j such that $b \cap (L - l_i) = \emptyset$. Let l'_i be the loop obtained as connected sum of l_i and p_j which is indicated by a thick line in Figure 1. The framing of l'_i is determined by the standard way from those of l_i and l_j (For more details, see Montesinos [4].). Then $L' = (L - l_i) \cup l'_i$ is called the framed link obtained from L by a band move.

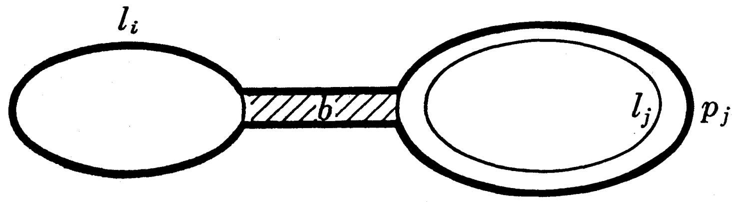

FIGURE 1

Our result on the invariance of framed link groups are stated as follows:

THEOREM 1. *Let L' be a framed link obtained from L by a band move. Then there is an isomorphism*

$$h: FG(L) \longrightarrow FG(L')$$

which preserves the parallel subgroups, i.e., $h(P(L)) = P(L')$.

PROOF. We use the notations in Definition 4. Let $X = N_i \cup N_j \cup N(b)$ where $N(b)$ is a tubular neighbourhood of the band b as in Figure 2.

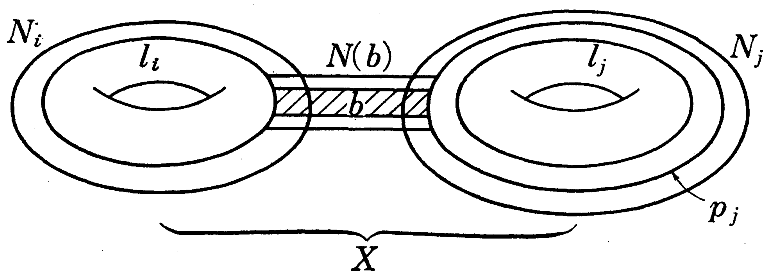

FIGURE 2

Note that ∂X, the boundary of X, is a surface of genus 2. We determine the generaters a, b, x, y of $\pi_1(\partial X)$ as in Figure 3.

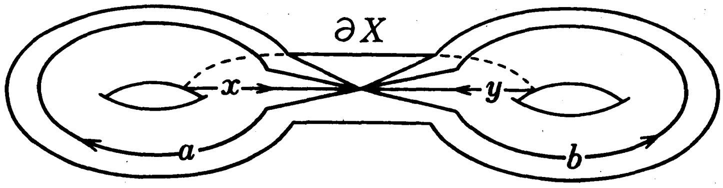

FIGURE 3

Then $\pi_1(\partial X)=\langle a, b, x, y; [a, x]=[b, y]\rangle$. $\pi_1(X-(l_i\cup l_j))$ is also generated by a, b, x, y and we obtain $\pi_1(X-(l_i\cup l_j))=\langle a, b, x, y; [a, x], [b, y]\rangle$. Similarly $\pi_1(X-(l_i'\cup l_j))=\langle a, b, x, y; [a^{-1}b, x], [b, y^{-1}x]\rangle$.

Let $Y=\overline{M-X}$ and $\langle \boldsymbol{x}; \boldsymbol{r}\rangle$, $\boldsymbol{x}=\{x_1, \cdots, x_n\}$, $\boldsymbol{r}=\{r_1, \cdots, r_m\}$ be a finite presentation of $\pi_1(Y-L)$. When we regard a, b, x, y as elements of $\pi_1(Y-L)=\langle \boldsymbol{x}; \boldsymbol{r}\rangle$, they can be represented by words $a(\boldsymbol{x})$, $b(\boldsymbol{x})$, $x(\boldsymbol{x})$, $y(\boldsymbol{x})$ in $\boldsymbol{x}$. Then, by van Kampen's theorem for $M-L=(X-(l_i\cup l_j))\cup(Y-L)$, we obtain $G(L)=\pi_1(M-L)=\langle \boldsymbol{x}, a, b, x, y; \boldsymbol{r}, a=a(\boldsymbol{x}), b=b(\boldsymbol{x}), x=x(\boldsymbol{x}), y=y(\boldsymbol{x}), [a, x], [b, y]\rangle$. Similarly we obtain $G(L')=\pi_1(M-L')=\langle \boldsymbol{x}, a, b, x, y; \boldsymbol{r}, a=a(\boldsymbol{x}), b=b(\boldsymbol{x}), x=x(\boldsymbol{x}), y=y(\boldsymbol{x}), [a^{-1}b, x], [b, y^{-1}x]\rangle$.

Next we shall study presentations of $FG(L)$ and $FG(L')$. Parallels p_l's $(l\neq i, j)$ can be represented by words $p_l(\boldsymbol{x})$ in $\boldsymbol{x}$ both in $G(L)$ and $G(L')$. Furthermore parallels corresponding to l_i and l_i' are presented by a and $a^{-1}b$ in $G(L)$ and $G(L')$ respectively. On the other hand, parallels corresponding to l_j's are presented by b both in $G(L)$ and $G(L')$. Thus we obtain the following presentations of $FG(L)$ and $FG(L')$.

$$\begin{aligned}FG(L)\cong\langle \boldsymbol{x}, a, b, x, y; \boldsymbol{r}, a=a(\boldsymbol{x}), b=b(\boldsymbol{x}), x=x(\boldsymbol{x}), y=y(\boldsymbol{x}), [a, x], [b, y],\\ [p_l(\boldsymbol{x}), \boldsymbol{x}], [p_l(\boldsymbol{x}), a], [p_l(\boldsymbol{x}), b], [p_l(\boldsymbol{x}), x], [p_l(\boldsymbol{x}), y]\ (1\leqq l\leqq k,\\ l\neq i, j)\ [a, \boldsymbol{x}], [a, b], [a, x], [a, y], [b, \boldsymbol{x}], [b, a], [b, x], [b, y]\rangle\end{aligned}$$

$$\begin{aligned}FG(L')\cong\langle \boldsymbol{x}, a, b, x, y; \boldsymbol{r}, a=a(\boldsymbol{x}), b=b(\boldsymbol{x}), x=x(\boldsymbol{x}), y=y(\boldsymbol{x}), [a^{-1}b, x],\\ [b, y^{-1}x], [p_l(\boldsymbol{x}), \boldsymbol{x}], [p_l(\boldsymbol{x}), a], [p_l(\boldsymbol{x}), b], [p_l(\boldsymbol{x}), x], [p_l(\boldsymbol{x}), y]\\ (1\leqq l\leqq k, l\neq i, j)\ [a^{-1}b, \boldsymbol{x}], [a^{-1}b, a], [a^{-1}b, b], [a^{-1}b, x],\\ [a^{-1}b, y], [b, \boldsymbol{x}], [b, a], [b, x], [b, y]\rangle\,.\end{aligned}$$

Note that $[ab, c]=a[b, c]a^{-1}[a, c]$ and $[a, bc]=[a, b]b[a, c]b^{-1}$ hold for elements a, b, c in a group. Thus it is easily ascertained that any relator of $FG(L)$ can be obtained as a product of conjugate elements of relators of $FG(L')$ and vice versa. This shows that $FG(L)$ and $FG(L')$ are isomorphic. Denote this isomorphism by h. Clearly h maps the parallel subgroup $P(L)$ which is generated by $p_l(\boldsymbol{x})$'s, a and b to the parallel subgroup $P(L')$ which is generated by $p_l(\boldsymbol{x})$'s, a and $a^{-1}b$. This completes the proof of Theorem 1.

We now consider the framed link group of a split link.

PROPOSITION 1. *Let L_i be a framed link in M_i $(i=1, 2)$. Let M be a connected sum of M_1 and M_2 which are summed keeping away from L_1 and L_2. When we regard $L=L_1\cup L_2$ as a framed link in M, we obtain the following isomorphisms*:

$$\begin{array}{ll} FG(L)\cong FG(L_1)*FG(L_2)/[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)] & \\ \ \cup \qquad\qquad\quad \cup & \\ \ P(L)\cong N(P(L_1)*P(L_2))/N(P(L_1)*P(L_2))\cap[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]\,. & \end{array}$$

PROOF. Consider the diagram below:

$$\begin{tikzcd} G(L_1)*G(L_2) \arrow[r, "i", "\cong"'] \arrow[d, "p'"] & G(L) \arrow[d, "p"] \\ FG(L_1)*FG(L_2) \arrow[r, dashed, "j"] & FG(L) \end{tikzcd}$$

where i is an isomorphism obtained from van Kampen's theorem, p is the canonical surjection and p' is a homomorphism induced from the canonical surjections $p_i: G(L_i)\to FG(L_i)$. We claim that there is a homomorphism $j: FG(L_1)*FG(L_2)\to FG(L)$ such that $p\circ i=j\circ p'$. We have $\ker p=[Q(L),\ G(L)]$, $\ker p'=N([Q(L_1),\ G(L_1)]*1\cup 1*[Q(L_2),\ G(L_2)])$ and $i([Q(L_1),\ G(L_1)]*1\cup 1*[Q(L_2),\ G(L_2)])\subset[Q(L),\ Q(L)]$, considering $[Q(L),\ G(L)]$ is normal. So we obtain $i(\ker p')\subset\ker p$. This shows the existence of j.

Since $p\circ i=j\circ p'$ is onto, j is also onto. Hence in order to show $\ker j=[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$ it is sufficient to prove $FG(L)\cong FG(L_1)*FG(L_2)/[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$. Note that $\ker j=p'(\ker j\circ p')=p'(\ker p\circ i)=p'(i^{-1}[Q(L),\ G(L)])$. Further, since $i(N(Q(L_1)*Q(L_2)))=Q(L)$, we have $i^{-1}[Q(L),\ G(L)]=[i^{-1}(Q(L)),\ i^{-1}(G(L))]=[N(Q(L_1)*Q(L_2)),\ G(L_1)*G(L_2)]$. Hence $\ker j=p'(i^{-1}[Q(L),\ G(L)])=p'[N(Q(L_1)*Q(L_2)),\ G(L_1)*G(L_2)]=[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$. Thus j induces an isomorphism, say k, $k: FG(L_1)*FG(L_2)/[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]\to FG(L)$.

We denote by H the group $N(P(L_1)*P(L_2))/N(P(L_1)*P(L_2))\cap[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$. Regarding H as a subgroup of $FG(L_1)*FG(L_2)/[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$, we shall show that k maps H to $P(L)$ bijectively. We first note that H is contained in the center of $FG(L_1)*FG(L_2)/[N(P(L_1)*P(L_2)),\ FG(L_1)*FG(L_2)]$. In particular H is a normal subgroup. Since $P(L)$ is normally generated by parallels which come from H by k. This means that k maps the normal subgroup H onto $P(L)$. Thus $k|H: H\to P(L)$ is a bijection. This completes the proof of Proposition 1.

§ 2. 3-manifolds and framed link groups.

Kirby [3] showed that to diffeomorphism classes of connected closed orientable 3-manifolds there correspond equivalence classes of framed link (or more exactly framed link types) in S^3 defined below. The equivalence relation which is called ∂-equivalence are generated by the following two:

(i) (Kirby move or stabilization) Let L be a framed link in S^3 and l_0 be a unknot with framing ± 1 separated from L by a 2-sphere. The insertion of l_0 is called a Kirby move.

(ii) (A band move) This move was described in Definition 4.

We write $L \underset{\partial}{\sim} L'$ when they are ∂-equivalent. For a framed link L, χ_L denotes the 3-manifold obtained by surgeries on L.

THEOREM (Kirby). *For framed links L and L' in S^3, they are ∂-equivalent if and only if χ_L and $\chi_{L'}$ are diffeomorphic.*

The purpose of § 2 is to study the relation between χ_L and $FG(L)$, $P(L)$.

PROPOSITION 2. *There is an exact sequence*

$$1 \longrightarrow P(L) \longrightarrow FG(L) \longrightarrow \pi_1(\chi_L) \longrightarrow 1 .$$

PROOF. From the definition of χ_L, the sequence $1 \to Q(L) \to G(L) \xrightarrow{q} \pi_1(\chi_L) \to 1$ is exact where $q: G(L) = \pi_1(M-L) \to \pi_1(\chi_L)$ is induced from the inclusion map. Since $[Q(L), G(L)] \subset Q(L)$, we have $q([Q(L), G(L)]) = 1$. Thus q induces an epimorphism $q': FG(L) \to \pi_1(\chi_L)$ with $\ker q' = p(\ker q) = p(Q(L)) = P(L)$, where $p: G(L) \to FG(L)$ is the canonical epimorphism. Hence the sequence $1 \to P(L) \to FG(L) \to \pi_1(\chi_L) \to 1$ is exact.

Now we describe how a framed link group changes with a Kirby move. Let Z denote an infinite cyclic group.

PROPOSITION 3. *Let $L' = L \cup l_0$ be obtained from L by a Kirby move. Then the following commutative diagram holds.*

$$\begin{tikzcd}
1 \arrow[r] & P(L') \arrow[r] \arrow[d, "\cong"] & FG(L') \arrow[r] \arrow[d, "\cong"] & \pi_1(\chi_{L'}) \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & P(L) \times Z \arrow[r] & FG(L) \times Z \arrow[r] & \pi_1(\chi_L) \arrow[r] & 1 .
\end{tikzcd}$$

PROOF. We apply Proposition 1 substituting $L_1 = L$ and $L_2 = l_0$. Since l_0 is a unknot with framing ± 1, $FG(l_0) = P(l_0) \cong Z$. From Proposition 1, $FG(L') \cong FG(L) * Z / [N(P(L) * Z), FG(L) * Z]$. Considering that $P(L)$ is in the center of $FG(L)$, we have $[N(P(L) * Z), FG(L) * Z] = [N(1 * Z), FG(L) * Z]$.

This means $FG(L')\cong FG(L)*Z/[N(1*Z), FG(L)*Z]\cong FG(L)\times Z$. Similarly we have $P(L')\cong P(L)\times Z$ and the proposition.

Now we introduce some concept with respect to a pair of groups. By (H, G) we denote the pair of a group G and its normal subgroup H. Let $(H, G)\times K$ stand for $(H\times K, G\times K)$ for any group K.

DEFINITION 5. The pairs of groups (H, G) and (H', G') are called Z-equivalent, denoted by $(H, G)\underset{Z}{\sim}(H', G')$, if $(H, G)\times Z^m$ and $(H', G')\times Z^n$ are isomorphic as pairs for some integers m and n.

Using the definition above we have the following theorem from Theorem 1 and Proposition 3.

THEOREM 2. *Let L and L' be framed links in S^3. If $L\underset{\partial}{\sim}L'$, then $(P(L), FG(L))\underset{Z}{\sim}(P(L'), FG(L'))$.*

This theorem combined with Kirby's theorem says that the Z-equivalence class of $(P(L), FG(L))$ is an invariant of surgery manifold χ_L.

Now we present some examples.

EXAMPLE 1. Let L be a unknot with framing n (Figure 4).

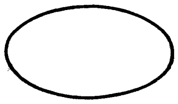

FIGURE 4

In this case we have $\chi_L\cong L(n, 1)$, a lens space of type $(n, 1)$, and clearly $1\to P(L)\to FG(L)\to\pi_1(\chi_L)\to 1$ is isomorphic to $1\to Z\xrightarrow{\times n}Z\to Z/nZ\to 1$.

EXAMPLE 2. Let L be the Borromean rings with framing 0 for any component (Figure 5).

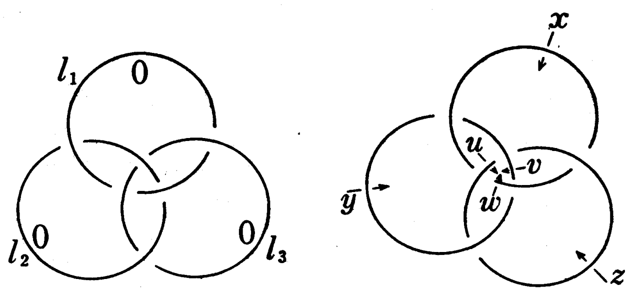

FIGURE 5

Then $\chi_L\cong T^3$, a 3-dimensional torus. Let a^b denote bab^{-1}. Using the Wirtinger presentation, we obtain:

$$G(L)\cong\langle x,\, y,\, z,\, u,\, v,\, w;\, x=w^{z},\, y=v^{x},\, z=u^{y},\, x=w^{u},\, y=v^{w},\, z=u^{v}\rangle\ ,$$

by Tietze's transformation,

$$\cong\langle u,\, v,\, w;\, w^{u}=w^{(u^{v})},\, v^{w}=v^{(w^{u})},\, u^{v}=u^{(v^{w})}\rangle\ .$$

On the other hand parallels are presented as follows:

$$p_1=uz^{-1}=u(u^{v})^{-1}=[u,\, v]\ ,$$

similarly

$$p_2=[w,\, u] \quad \text{and} \quad p_3=[v,\, w]\ .$$

Hence we have:

$$\begin{aligned} FG(L)\cong\langle u,\, v,\, w;\, & w^{u}=w^{(u^{v})},\, v^{w}=v^{(w^{u})},\, u^{v}=u^{(v^{w})}\ , \\ & [u,[v,w]],\, [v,[w,u]],\, [w,[u,v]],\, [v,[v,w]],\, [w,[w,u]]\ , \\ & [u,[u,v]],\, [w,[v,w]],\, [u,[w,u]],\, [v,[u,v]]\rangle\ . \end{aligned}$$

Note that the first 3 relators can be induced from the others. Let G denote the free group of rank 3, then the observation above shows, in this case, $1\to P(L)\to FG(L)\to\pi_1(\chi_L)\to 1$ is isomorphic to $1\to[G,G]/[G,[G,G]]\to G/[G,[G,G]]\to G/[G,G]\to 1$.

§ 3. 4-manifolds and framed link groups.

In this section we deal with framed links in $\#_n S^1\times S^2$. Montesinos [4] showed that the correspondence which assigns connected closed orientable 4-manifolds to equivalence classes described below of framed links in $\#_n S^1\times S^2$ (where n varies within non-negative integers) is injective. The equivalence relation is generated by the following 3 types of moves:

(i′) Let $l_0=S^1\times pt\subset S^1\times S^2$ be a framed link in $S^1\times S^2$ with the standard framing. To a framed link L in $\#_n S^1\times S^2$, add l_0 to obtain a framed link $L'=L\cup l_0$ in $\#_{n+1} S^1\times S^2$.

(ii′) Let L be a framed link in $\#_n S^1\times S^2$ and l_1 be a unknot with framing ± 0 in a 3-ball in $\#_n S^1\times S^2$ which is away from L. Add l_1 to obtain a new framed link $L'=L\cup l_1$ in $\#_n S^1\times S^2$.

(iii′) The band move described in Definition 4.

We shall define a diagram of groups which is associated with a given framed link in $\#_n S^1\times S^2$ and describe how the diagram changes when the link is changed by moves (i′)-(iii′). Let $\tilde{Q}(L)$ be an image of $Q(L)$ under a canonical surjection $G(L)=\pi_1(M-L)\to\pi_1(M)$. Consider an induced homomorphism, say r, $r\colon FG(L)=G(L)/[Q(L), G(L)]\to\pi_1(M)/[\tilde{Q}(L), \pi_1(M)]$. Let $M(L)$ be $\ker r$. It is easy to see that $M(L)$ is a normal subgroup of

$FG(L)$ which is generated by meridians m_i's. So we call it a meridian subgroup of $FG(L)$.

From a given framed link, we obtain the following diagram denoted by $(*)$:

$$
(*) \qquad
\begin{tikzcd}
& & 1 \arrow[d] & & \\
& & M(L) \arrow[d, hook] & & \\
1 \arrow[r] & P(L) \arrow[r, hook] & FG(L) \arrow[r, "q'"] \arrow[d, "r"] & \pi_1(\chi_L) \arrow[r] & 1 \quad \text{(exact)} \\
& & \pi_1(M)/[\tilde{Q}(L), \pi_1(M)] \arrow[d] & & \\
& & 1 \quad \text{(exact)} & &
\end{tikzcd}
$$

where q' denotes the same map as in the proof of Proposition 2.

Let N_L be a closed 4-manifold which corresponds to a framed link $L \subset M$, where $M = \#_n S^1 \times S^2$. More precisely N_L is obtained from $\natural_n S^1 \times D^3$ by adding 2-handles along $L \subset M = \partial(\natural_n S^1 \times D^3)$ and by adding suitably 3- and 4-handles to obtain a closed 4-manifold. From the diagram $(*)$ we can read off the information about $\pi_1(N_L)$.

PROPOSITION 4. *Under the notation above, we have*

$$\pi_1(N_L) \cong \pi_1(\chi_L)/q'(M(L)) \cong (\pi_1(M)/[\tilde{Q}(L), \pi_1(M)])/r(P(L)) .$$

PROOF. The manifold N_L has $(\natural_n S^1 \times D^3) \cup$ (2-handles) as its 2-skeleton, where 2-handles are attached along L. Further, since $\pi_1(\#_n S^1 \times S^2) \cong \pi_1(\natural_n S^1 \times D^3)$, we have

$$\pi_1(N_L) \cong \pi_1(\natural_n S^1 \times D^3 \cup \text{2-handles}) \cong \pi_1(\#_n S^1 \times S^2 \cup \text{2-handles}) .$$

The last group is isomorphic to $\pi_1(M)/\tilde{Q}(L)$. Since $\tilde{Q}(L)$ and $r(P(L))$ are generated by image of parallels, we get

$$\pi_1(M)/\tilde{Q}(L) \cong (\pi_1(M)/[\tilde{Q}(L), \pi_1(M)])/r(P(L)) .$$

The proof of the fact $\pi_1(N_L) \cong \pi_1(\chi_L)/q'(M(L))$ is similar if we consider the dual handle decomposition. This completes the proof.

Suppose that L is changed to L' by a band move. Then L and L' have isomorphic diagrams; in fact, as is described in the proof of Theorem 1, not only $P(L)$ and $FG(L)$ but also $M(L)$ (which is generated by x, yx^{-1} and m_l $(l \neq i, j)$) is invariant. In the remainder of this section we study how the diagram (*) changes under the moves (i′) and (ii′).

Let L' be obtained from L by the move (i′). Then $FG(L')$ is obtained from $FG(L)$ and a generator corresponding to a parallel of l_0. But this element commutes with all elements of $FG(L')$. Thus $FG(L')\cong FG(L)\times \boldsymbol{Z}$. The similar argument for $P(L')$ and $M(L')$ shows the following proposition.

PROPOSITION 5. *The diagram of L' obtained from L by move* (i′) *is isomorphic to*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & M(L) & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & P(L)\times \boldsymbol{Z} & \longrightarrow & FG(L)\times \boldsymbol{Z} & \longrightarrow & \pi_1(\chi_L) & \longrightarrow & 1 \\
 & & & & \downarrow & & & & \\
 & & & & (\pi_1(M)/[\tilde{Q}(M),\ \pi_1(M)])\times \boldsymbol{Z} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

where all maps are naturally constructed from those of the diagram (*) *and* id: $\boldsymbol{Z}\to \boldsymbol{Z}$.

Next suppose that L' is obtained from L by move (ii′). Then $FG(L')$ is obtained from $FG(L)$ and a generater corresponding a meridian of l_1. But, in this case, this element commutes only with parallels, so we get $FG(L')\cong FG(L)*\boldsymbol{Z}/[N(P(L)*1),\ N(1*\boldsymbol{Z})]$. The similar argument for $P(L')$ and $M(L')$ gives the following proposition.

PROPOSITION 6. *The diagram of L' obtained from L by the move* (ii′) *is isomorphic to*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & N(M(L)*\boldsymbol{Z})/[N(P(L)*1),\ N(1*\boldsymbol{Z})]\cap N(M(L)*\boldsymbol{Z}) & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & P(L) & \longrightarrow & FG(L)*\boldsymbol{Z}/[N(P(L)*1),\ N(1*\boldsymbol{Z})] & \longrightarrow & \pi_1(\chi_L)*\boldsymbol{Z} & \longrightarrow & 1 \\
 & & & & \downarrow & & & & \\
 & & & & \pi_1(M)/[\tilde{Q}(L),\ \pi_1(M)] & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1\,. & & & &
\end{array}$$

In relation to the propositions above, we present an example, from which we can read off from the diagram (*) further information about $\pi_1(N_L)$.

Let consider the following link picture:

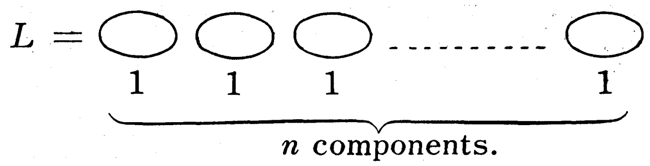

FIGURE 6

Then we get $N_L \cong \#_n CP(2)$. An easy calculation shows that the corresponding diagram is:

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & Z^n \cong M(L) \arrow[d] & & \\
1 \arrow[r] & Z^n \arrow[r] \arrow[d, phantom, "\wr\wr"] & Z^n \arrow[r] \arrow[d] & 1 \arrow[r] & 1 \\
 & P(L) & 1 \cong \pi_1(M)/[\tilde{Q}(L), \pi_1(M)] \arrow[d] & & \\
 & & 1\,. & &
\end{tikzcd}$$

Note that, in the diagram above, we have $M(L) \cap P(L) \cong Z^n$. But this property does not change when L is stabilized by the moves (i′) and (ii′). Since the band move also does not change the diagram, $\#_n CP(2)$ and $\#_m CP(2)$ $(n \neq m)$ are distinguished by our diagram.

More acute observation of the diagram (*) will give us a detailed geometric information about 4-manifolds.

REMARK. The following generalization of framed link groups may be useful in some setting. For a framed link L in M, let $M(L)$ be the kernel of the natural map $G(L) = \pi_1(M-L) \to \pi_1(M)$. Then $M(L)$ is the normal subgroup of $G(L)$ which is normally generated by meridians. As in § 1, let $Q(L)$ be the normal subgroup of $G(L)$ which is normally generated by parallels. Let us define $\widetilde{FG}(L) = G(L)/[Q(L), M(L)]$. Then the invariance of $\widetilde{FG}(L)$ under band moves can be proved by the same way as in Theorem 1. Using $\widetilde{FG}(L)$, one can simplify the statements in § 3.

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