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ON FRAMES FOR COUNTABLY GENERATED HILBERT C^*-MODULES

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ABSTRACT. Let V be a countably generated Hilbert C^*-module over a C^*-algebra A. We prove that a sequence $\{f_i : i \in I\} \subseteq V$ is a standard frame for V if and only if the sum $\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ converges in norm for every $x \in V$ and if there are constants $C, D > 0$ such that $C\|x\|^2 \le \|\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle\| \le D\|x\|^2$ for every $x \in V$. We also prove that surjective adjointable operators preserve standard frames. A class of frames for countably generated Hilbert C^*-modules over the C^*-algebra of all compact operators on some Hilbert space is discussed.

1. Introduction and preliminaries

A (*right*) *Hilbert C^*-module V* over a C^*-algebra A (or a *Hilbert A-module*) is a linear space which is a right A-module, together with an A-valued inner product $\langle\cdot,\cdot\rangle$ on $V \times V$ which is linear in the second and conjugate linear in the first variable such that V is a Banach space with the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. We use the symbol $\langle V, V\rangle$ for the closed, two-sided ideal of A spanned by all inner products $\langle x, y\rangle$, $x, y \in V$. V is said to be a *full Hilbert A-module* if $\langle V, V\rangle = A$.

We denote the C^*-algebra of all adjointable operators on a Hilbert C^*-module V by $\mathbf{B}(V)$. We also use $\mathbf{B}(V, W)$ to denote the space of all adjointable operators acting between different Hilbert A-modules. A good reference for Hilbert C^*-modules are the lecture notes of E. C. Lance [12].

The C^*-algebra of all bounded operators and the ideal of all compact operators on a Hilbert space H are denoted by $\mathbf{B}(H)$ and $\mathbf{K}(H)$, respectively.

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6] as part of their research in non-harmonic Fourier series. A *frame* for a separable Hilbert space H is defined to be a finite or countable sequence $\{f_i : i \in I\}$ for which there exists constants $C, D > 0$ such that

$$C\|x\|^2 \le \sum_{i\in I} |(x, f_i)|^2 \le D\|x\|^2, \quad x \in H.$$

M. Frank and D. Larson [7, 8] generalized this definition to the situation of countably generated Hilbert C^*-modules. A *frame* for a countably generated Hilbert

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C^*-module V is a sequence $\{f_i : i \in I\}$ ($I \subseteq \mathbf{N}$ finite or countable) for which there are constants $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{i\in I} \langle x, f_i\rangle\langle f_i, x\rangle \leq D\langle x, x\rangle, \quad x \in V. \tag{1.1}$$

We consider *standard frames* for which the sum in the middle of (1.1) converges in norm for every $x \in V$. (For non-standard frames the sum in (1.1) converges only weakly for at least one element of V.) The numbers C and D are called *frame bounds*. A frame $\{f_i : i \in I\}$ is called a *tight frame* if we can choose $C = D$ and a *Parseval frame* (or a *normalized tight frame*) if $C = D = 1$. A sequence which satisfies only the right-hand inequality in (1.1) is called a *Bessel sequence* with a *Bessel bound* D.

The *frame transform* for a standard frame $\{f_i : i \in I\}$ is the map $\theta : V \to \ell_2(A)$ defined by $\theta(x) = (\langle f_i, x\rangle)_i$, where $\ell_2(A)$ denotes a Hilbert A-module $\{(a_i)_i : a_i \in A, \sum_{i\in I} a_i^* a_i$ converges in norm$\}$ with pointwise operations and the inner product $\langle (a_i)_i, (b_i)_i\rangle = \sum_{i\in I} a_i^* b_i$. The frame transform possesses an adjoint operator and realizes an embedding of V onto an orthogonal summand of $\ell_2(A)$ ([8, Theorem 4.4]). The operator $S = (\theta^*\theta)^{-1} \in \mathbf{B}(V)$ is said to be the *frame operator* for a standard frame $\{f_i : i \in I\}$. The frame operator is positive, invertible, and is the unique operator in $\mathbf{B}(V)$ such that the *reconstruction formula*

$$x = \sum_{i\in I} f_i \langle Sf_i, x\rangle \tag{1.2}$$

holds for all $x \in V$. Let us remark that although M. Frank and D. Larson [7, 8] stated all their results for the unital case, many proofs can be applied to the non-unital situation.

In a countably generated Hilbert C^*-module over a unital C^*-algebra, standard frames always exist [7]. Also, a Hilbert C^*-module over a C^*-algebra of all compact operators $\mathbf{K}(H)$ on some Hilbert space H possesses frames; this follows from [2], where the concept of an orthonormal basis for a Hilbert C^*-module was discussed.

An element v of a Hilbert A-module V is said to be a *basic vector* if $e = \langle v, v\rangle$ is a projection in A such that $eAe = \mathbf{C}e$. The system of basic vectors $\{v_i : i \in I\}$ in V is said to be an *orthonormal basis* for V if $\langle v_i, v_j\rangle = 0$ for all $i \neq j$, and if it generates a dense submodule of V. Every orthonormal basis $\{v_i : i \in I\}$ for a Hilbert C^*-module satisfies $\langle x, x\rangle = \sum_{i\in I} \langle x, v_i\rangle\langle v_i, x\rangle$ for all $x \in V$, with the norm convergence ([2, Theorem 1]).

Recall that, if A is a C^*-subalgebra of $\mathbf{K}(H)$ and $e \in A$ a non-zero projection, the condition $eAe = \mathbf{C}e$ is equivalent to the minimality of e (i.e., the only subprojections of e in A are 0 and e) [1, Lema 1.4.1]. Minimal projections in $\mathbf{K}(H)$ are exactly orthogonal projections of rank 1.

Clearly, an arbitrary Hilbert C^*-module does not possess an orthonormal basis, since there are C^*-algebras without projections. It is known that every Hilbert C^*-module V over $\mathbf{K}(H)$ possesses an orthonormal basis; furthermore, for a fixed orthogonal projection $e \in \mathbf{K}(H)$ of rank 1, there is an orthonormal basis $\{v : i : i \in I\}$ for V such that $\langle v_i, v_i\rangle = e$ for all $i \in I$.

In a Hilbert $\mathbf{K}(H)$-module, the condition of the minimality of supporting projections $e_i = \langle v_i, v_i\rangle, i \in I$, ensures that all orthonormal bases have the same cardinality ([2, Theorem 2]). For a countably generated Hilbert $\mathbf{K}(H)$-module, a set of indices for (all) orthonormal bases is countable. (By choosing an orthonormal basis

$\{v_i : i \in I\}$ such that $\langle v_i, v_i\rangle = e, i \in I$, for some orthogonal projection $e \in \mathbf{K}(H)$ of rank 1, the last statement proves in the same way as in the Hilbert space case.) So, every orthonormal basis for a countably generated Hilbert $\mathbf{K}(H)$-module V is a standard Parseval frame for V.

The paper is organized as follows.

In Section 2 we study standard frames for arbitrary countably generated Hilbert C^*-modules. We first show that an adjointable operator between Hilbert C^*-modules is bounded below with respect to the norm if and only if it is bounded below with respect to the inner product; furthermore, this is equivalent to the surjectivity of its adjoint operator. The first equivalence implies that, in the definition of standard frames, we can replace (1.1) with $C\|x\|^2 \leq \|\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle\| \leq D\|x\|^2$ for all $x \in V$ (Theorem 2.6). From the second equivalence we conclude that surjective adjointable operators preserve standard frames (Theorem 2.5).

In Section 3 we discuss standard frames $\{f_i : i \in I\}$ for which there exists a family of projections $\{e_i : i \in I\}$ such that $e_iAe_i = \mathbf{C}e_i$ and $f_i = f_ie_i$ for every $i \in I$. Surjective images of orthonormal bases are frames of this form. We prove that only a Hilbert C^*-module V for which $\langle V, V\rangle$ is a CCR-algebra admits such frames. Discussion is mainly restricted to countably generated Hilbert $\mathbf{K}(H)$-modules, where such frames always exist; moreover, for every orthogonal projection $e \in \mathbf{K}(H)$ of rank 1, there is a frame $\{f_i : i \in I\}$ such that $f_i = f_ie$ for all $i \in I$. We show that frames $\{f_i : i \in I\}$ for a countably generated Hilbert $\mathbf{K}(H)$-module V such that $\langle f_i, f_i\rangle = e, i \in I$, correspond to frames for a Hilbert space $V_e = \{ve : v \in V\}$ (Theorem 3.4).

2. Some properties of standard modular frames

The results we obtain in this section are the consequences of the statement which generalizes the well known fact: a bounded linear operator between Hilbert spaces is surjective if and only if its adjoint is bounded below.

Proposition 2.1. *Let A be a C^*-algebra, V and W Hilbert A-modules, and $T \in \mathbf{B}(V, W)$. The following statements are mutually equivalent:*

1. *T is surjective.*
2. *T^* is bounded below with respect to the norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in V$.*
3. *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in V$.*

Proof. (1) $\Rightarrow$ (3): Suppose T is surjective. Then $\operatorname{Im} T = W$ is closed. It follows from [12, Theorem 3.2] that $\operatorname{Im} T^*$ is also closed, $\operatorname{Ker} T \oplus \operatorname{Im} T^* = V$ and $\operatorname{Ker} T^* \oplus \operatorname{Im} T = W$. We shall prove that TT^* is bijective.

If $TT^*x = 0$ for some $x \in V$, then $T^*x \in \operatorname{Ker} T \cap \operatorname{Im} T^* = \{0\}$, hence $T^*x = 0$. Now $x \in \operatorname{Ker} T^* = (\operatorname{Im} T)^\perp = W^\perp = \{0\}$ implies $x = 0$. This proves that TT^* is injective.

Let z be an arbitrarily chosen element of W. T is surjective, so $z = Ty$ for some $y \in V$. There are $y_1 \in \operatorname{Ker} T$ and $x \in W$ such that $y = y_1 \oplus T^*x$. Then $z = Ty = T(y_1 \oplus T^*x) = TT^*x$; therefore TT^* is surjective.

Since TT^* is a positive invertible element of the C^*-algebra $\mathbf{B}(V)$, we have

$$0 \leq (TT^*)^{-1} \leq \|(TT^*)^{-1}\| \operatorname{id}_V \Rightarrow TT^* \geq (\|(TT^*)^{-1}\|)^{-1} \operatorname{id}_V,$$

where id_V stands for the identity operator on V. Denoting $m' = \|(TT^*)^{-1}\|^{-1}$ we get $TT^* - m' \mathrm{id}_V \geq 0$. By [12, Lemma 4.1], this is equivalent to

$$\langle (TT^* - m' \mathrm{id}_V)x, x\rangle \geq 0$$

for all $x \in V$, i.e., $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in V$.

The implication (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Suppose that T^* is bounded below with respect to the norm. Then T^* is clearly injective, and it is easy to see that $\mathrm{Im}\, T^*$ is closed. Then T has the closed range, again by [12, Theorem 3.2], and $W = \mathrm{Ker}\, T^* \oplus \mathrm{Im}\, T = \{0\} \oplus \mathrm{Im}\, T = \mathrm{Im}\, T$. Hence T is surjective. □

Corollary 2.2. *Let A be a C^*-algebra, V a Hilbert A-module, and $T \in \mathbf{B}(V)$ such that $T = T^*$. The following statements are mutually equivalent:*

1. *T is surjective.*
2. *There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$ for all $x \in V$.*
3. *There are $m', M' > 0$ such that $m'\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq M'\langle x, x\rangle$ for all $x \in V$.*

Remark 2.3. An operator $T \in \mathbf{B}(V)$ is said to be *coercive* if there is a positive constant m such that $\langle T^*x, T^*x\rangle \geq m\langle x, x\rangle$ holds for all $x \in V$. It follows from Proposition 2.1 that coercive operators in $\mathbf{B}(V)$ are exactly surjections in $\mathbf{B}(V)$.

Theorem 2.4. *Let A be a C^*-algebra, V a countably generated Hilbert A-module, $\{f_i : i \in I\}$ a sequence in V, and $\theta(x) = (\langle f_i, x\rangle)_{i\in I}$ for $x \in V$. The following statements are mutually equivalent:*

1. *$\{f_i : i \in I\}$ is a standard frame for V.*
2. *$\theta \in \mathbf{B}(V, \ell_2(A))$ and θ is bounded below.*
3. *$\theta \in \mathbf{B}(V, \ell_2(A))$ and θ^* is surjective.*

Proof. It follows from [8, Theorem 4.1] and Proposition 2.1 since

$$\langle \theta x, \theta x\rangle = \sum_{i\in I} \langle x, f_i\rangle\langle f_i, x\rangle, \quad x \in V.$$

□

Another direct consequence of Proposition 2.1 is that surjective adjointable operators preserve standard frames.

Theorem 2.5. *Let A be a C^*-algebra, V and W countably generated Hilbert A-modules, and $T \in \mathbf{B}(V, W)$ surjective. If $\{f_i : i \in I\}$ is a standard frame for V with frame bounds C and D, then $\{Tf_i : i \in I\}$ is a standard frame for W with frame bounds $\frac{C}{\|(TT^*)^{-1}\|}$ and $D\|T\|^2$.*

Proof. Since $\{f_i : i \in I\}$ is a standard frame for V, and since $T^*y \in V$ for all $y \in W$, we have

$$C\langle T^*y, T^*y\rangle \leq \sum_{i\in I} \langle T^*y, f_i\rangle\langle f_i, T^*y\rangle \leq D\langle T^*y, T^*y\rangle, \quad y \in W.$$

From the proof of Proposition 2.1 we have $\langle T^*y, T^*y\rangle \geq \|(TT^*)^{-1}\|^{-1}\langle y, y\rangle$ for all $y \in W$, since T is surjective. It follows that

$$\frac{C}{\|(TT^*)^{-1}\|}\langle y, y\rangle \leq \sum_{i\in I} \langle y, Tf_i\rangle\langle Tf_i, y\rangle \leq D\|T\|^2\langle y, y\rangle, \quad y \in W.$$ □

We conclude this section with the result which states that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

Theorem 2.6. *Let A be a C^*-algebra, V a countably generated Hilbert A-module, and $\{f_i : i \in I\}$ a sequence in V such that $\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ converges in norm for every $x \in V$. Then $\{f_i : i \in I\}$ is a standard frame for V if and only if there are constants $C, D > 0$ such that*

$$C\|x\|^2 \leq \|\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle\| \leq D\|x\|^2, \quad x \in V. \tag{2.1}$$

Proof. Evidently, every standard frame for V satisfies (2.1).

For the converse we suppose that a sequence $\{f_i : i \in I\}$ fulfills (2.1). For an arbitrary $x \in V$ and a finite $J \subseteq I$ we define $x_J = \sum_{i\in J} f_i\langle f_i, x\rangle$. Then

$$\|x_J\|^4 = \|\langle x_J, x_J\rangle\|^2 = \|\langle x_J, \sum_{i\in J} f_i\langle f_i, x\rangle\rangle\|^2 = \|\sum_{i\in J}\langle x_J, f_i\rangle\langle f_i, x\rangle\|^2$$
$$\leq \|\sum_{i\in J}\langle x_J, f_i\rangle\langle f_i, x_J\rangle\| \cdot \|\sum_{i\in J}\langle x, f_i\rangle\langle f_i, x\rangle\| \leq D\|x_J\|^2\,\|\sum_{i\in J}\langle x, f_i\rangle\langle f_i, x\rangle\|,$$

therefore

$$\|\sum_{i\in J} f_i\langle f_i, x\rangle\|^2 = \|x_J\|^2 \leq D\|\sum_{i\in J}\langle x, f_i\rangle\langle f_i, x\rangle\|.$$

Since J is arbitrary, the series $\sum_{i\in I} f_i\langle f_i, x\rangle$ converges and

$$\|\sum_{i\in I} f_i\langle f_i, x\rangle\|^2 \leq D\|\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle\| \leq D^2\|x\|^2 \;\Rightarrow\; \|\sum_{i\in I} f_i\langle f_i, x\rangle\| \leq D\|x\|.$$

Since $x \in V$ is arbitrarily chosen, the operator

$$T : V \to V, \quad x \mapsto \sum_{i\in I} f_i\langle f_i, x\rangle$$

is well defined, bounded and A-linear. It is easy to check that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in V$, so $T \in \mathbf{B}(V)$ and $T = T^*$. From $\langle Tx, x\rangle = \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \geq 0$ for all $x \in V$, it follows that $T \geq 0$. Now (2.1) and $\langle T^{1/2}x, T^{1/2}x\rangle = \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ imply $\sqrt{C}\|x\| \leq \|T^{1/2}x\| \leq \sqrt{D}\|x\|$ for all $x \in V$. By Corollary 2.2, there are constants $C', D' > 0$ such that

$$C'\langle x, x\rangle \leq \langle T^{1/2}x, T^{1/2}x\rangle = \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq D'\langle x, x\rangle, \quad x \in V.$$

This proves that $\{f_i : i \in I\}$ is a standard frame for V. □

3. On a class of frames for Hilbert $\mathbf{K}(H)$-modules

The existence of standard frames in countably generated Hilbert $\mathbf{K}(H)$-modules V follows from the existence of orthonormal bases. If $T \in \mathbf{B}(V)$ is a surjection and $\{v_i : i \in I\}$ an orthonormal basis for V, then $\{Tv_i : i \in I\}$ is a standard frame for V which satisfies $Tv_i = T(v_ie_i) = (Tv_i)e_i$, where $e_i := \langle v_i, v_i\rangle$ is an orthogonal projection of rank 1 for every $i \in I$. However, not every standard frame in a Hilbert $\mathbf{K}(H)$-module is of this type, as we show in the next example.

Example 3.1. Let $\{v_i : i \in I\}$ be an orthonormal basis for a countably and not finitely generated Hilbert $\mathbf{K}(H)$-module V with property $e_i e_j = 0, i \neq j$, where $e_i = \langle v_i, v_i\rangle$. (Such a basis can always be constructed by following the procedure described in [2, Remark 4(d)].) Let $I = \bigcup_{j=1}^{\infty} I_j$ be a partition of I such that $|I_j| = j$. Let $f_j = \sum_{i\in I_j} v_i \in V$. Since $\langle x, v_j\rangle\langle v_i, x\rangle = \langle x, v_j e_j\rangle\langle v_i e_i, x\rangle = \langle x, v_j\rangle e_j e_i \langle v_i, x\rangle = \delta_{ij}\langle x, v_j\rangle\langle v_i, x\rangle$ for all $x \in V$, we have

$$\langle x, f_j\rangle\langle f_j, x\rangle = \langle x, \sum_{i\in I_j} v_i\rangle\langle \sum_{i\in I_j} v_i, x\rangle = \sum_{i,j\in I_j}\langle x, v_j\rangle\langle v_i, x\rangle = \sum_{i\in I_j}\langle x, v_i\rangle\langle v_i, x\rangle$$

and then

$$\langle x, x\rangle = \sum_{i\in I}\langle x, v_i\rangle\langle v_i, x\rangle = \sum_{j\in J}\langle x, f_j\rangle\langle f_j, x\rangle.$$

This means that $\{f_j : j \in J\}$ is a standard Parseval frame for V such that $\langle f_j, f_j\rangle = \sum_{i\in I_j} e_i$ is a projection with dim Im $\langle f_i, f_i\rangle = |I_j| = j$ for all $j \in J$.

Proposition 3.2. *Let V be a countably and not finitely generated Hilbert $\mathbf{K}(H)$-module. Let $\{f_i : i \in I\}$ be a standard frame for V such that $f_i = f_i e_i$ for some orthogonal projections $e_i, i \in I$, of rank* 1. *Then there is an orthonormal basis $\{v_i : i \in I\}$ and a surjection $T \in \mathbf{B}(V)$ such that $Tv_i = f_i, i \in I$.*

Proof. Let C and D be frame bounds. Let $\{v_i : i \in I\}$ be an orthonormal basis such that $v_i = v_i e_i, i \in I$. (We may assume that the sets of indices for a standard frame and a basis are the same, since they are both infinite subsets of $\mathbf{N}$.)

We first show that for every $x \in V$ the series $\sum_{i\in I} f_i\langle v_i, x\rangle$ converges. Let J be a finite subset of I and $x_J = \sum_{i\in J} f_i\langle v_i, x\rangle$. Then

$$\begin{aligned}\|x_J\|^4 &= \|\langle x_J, x_J\rangle\|^2 = \|\langle \sum_{i\in J} f_i\langle v_i, x\rangle, x_J\rangle\|^2 = \|\sum_{i\in J}\langle x, v_i\rangle\langle f_i, x_J\rangle\|^2\\ &\leq \|\sum_{i\in J}\langle x, v_i\rangle\langle v_i, x\rangle\|\|\sum_{i\in J}\langle x_J, f_i\rangle\langle f_i, x_J\rangle\| \leq \|\sum_{i\in J}\langle x, v_i\rangle\langle v_i, x\rangle\|\cdot D\|x_J\|^2,\end{aligned}$$

from where we get $\|x_J\|^2 \leq D\|\sum_{i\in J}\langle x, v_i\rangle\langle v_i, x\rangle\|$, that is,

$$\|\sum_{i\in J} f_i\langle v_i, x\rangle\|^2 \leq D\|\sum_{i\in J}\langle x, v_i\rangle\langle v_i, x\rangle\|,\ \text{for every finite } J \subseteq I.$$

Since $\sum_{i\in I}\langle x, v_i\rangle\langle v_i, x\rangle$ converges in norm ([2, Theorem 1]), it follows that the series $\sum_{i\in I} f_i\langle v_i, x\rangle$ converges.

Similarly, we check that the series $\sum_{i\in I} v_i\langle f_i, x\rangle$ converges for every $x \in V$.

Now we can define the operators $T, R : V \to V$ by $Tx = \sum_{i\in I} f_i\langle v_i, x\rangle$ and $Rx = \sum_{i\in I} v_i\langle f_i, x\rangle$. It is straightforward to see that $\langle Tx, y\rangle = \langle x, Ry\rangle$ for all $x, y \in V$. Therefore $T \in \mathbf{B}(V)$ and $R = T^*$. From Proposition 2.1 and

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle = \langle T^*x, T^*x\rangle, \quad x \in V,$$

it follows that T is surjective.

It only remains to note that $Tv_i = f_i\langle v_i, v_i\rangle = f_i e_i = f_i$ for all $i \in I$. □

A Hilbert $\mathbf{K}(H)$-module contains a Hilbert space V_e with respect to the inner product $(x, y) = \mathrm{tr}(\langle y, x\rangle)$, where 'tr' means the trace. More precisely, for a fixed orthogonal projection $e \in \mathbf{K}(H)$ of rank 1, V_e is given as the set of all xe, $x \in V$.

Also, for all $x, y \in V_e$ we obtain that $\langle x, y\rangle = (y, x)e$. V_e is an invariant subspace for each T in $\mathbf{B}(V)$ and the map

$$T \mapsto T|V_e, \quad \mathbf{B}(V) \to \mathbf{B}(V_e) \tag{3.1}$$

establishes an isomorphism of C^*-algebras, where $\mathbf{B}(V_e)$ denotes the C^*-algebra of all bounded operators on V_e. It is known that a family $\{v_i : i \in I\} \subseteq V_e$ is an orthonormal basis for V if and only if it is an orthonormal basis for V_e. (The proofs can be found in [2, Remark 4, Theorem 5]).) We extend the last statement to a standard frame $\{f_i : i \in I\}$ contained in V_e. First we need a lemma which describes some properties of the isomorphism (3.1).

Lemma 3.3. *Let V be a Hilbert $\mathbf{K}(H)$-module, $e \in \mathbf{K}(H)$ an orthogonal projection of rank 1, and $T \in \mathbf{B}(V)$. The following statements hold:*

(1) *T is bounded below if and only if $T|V_e \in \mathbf{B}(V_e)$ is bounded below.*
(2) *T is surjective if and only if $T|V_e \in \mathbf{B}(V_e)$ is surjective.*

Proof. (1) First observe that if $T|V_e \in \mathbf{B}(V_e)$ is a positive operator on the Hilbert space V_e, then $T \in \mathbf{B}(V)$ is a positive element of the C^*-algebra $\mathbf{B}(V)$. This is a consequence of the fact that the map $T \mapsto T|V_e$ is an isomorphism of C^*-algebras.

Suppose $T_e := T|V_e$ is bounded below. Let $m > 0$ be such that $\|T_e(xe)\| \geq m\|xe\|$ for all $x \in V$. In other words, $T_e^*T_e - m^2\mathrm{id}_{V_e}$ is a positive operator on the Hilbert space V_e. By the observation from the beginning of the proof, we get $T^*T - m^2\mathrm{id}_V \geq 0$, i.e., $\langle (T^*T - m^2\mathrm{id}_V)x, x\rangle \geq 0$ for all $x \in V$. Now we have $\langle Tx, Tx\rangle \geq m^2\langle x, x\rangle$, and then $\|Tx\| \geq m\|x\|$ for all $x \in V$.

The opposite statement is obvious.

(2) It follows from (1) and Proposition 2.1. □

Theorem 3.4. *Let V be a countably generated Hilbert $\mathbf{K}(H)$-module, $e \in \mathbf{K}(H)$ an orthogonal projection of rank 1, and $\{f_i : i \in I\}$ a sequence of vectors in V_e. Then $\{f_i : i \in I\}$ is a standard frame for the Hilbert $\mathbf{K}(H)$-module V with frame bounds C and D if and only if $\{f_i : i \in I\}$ is a frame for the Hilbert space V_e with frame bounds C and D.*

Proof. Suppose that $\{f_i : i \in I\}$ is a standard frame for a Hilbert $\mathbf{K}(H)$-module V with frame bounds C and D. It means that

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq D\langle x, x\rangle, \quad x \in V.$$

Since $\langle xe, ye\rangle = (ye, xe)e$ for all $xe, ye \in V_e$, by choosing xe instead of x in the above inequalities, we get

$$C(xe, xe)e \leq \sum_{i\in I}(xe, f_i)(f_i, xe)e \leq D(xe, xe)e, \quad x \in V,$$

which implies $C(x, x) \leq \sum_{i\in I}(x, f_i)(f_i, x) \leq D(x, x)$ for all $x \in V_e$. It proves that $\{f_i : i \in I\}$ is a frame for the Hilbert space V_e with frame bounds C and D.

Now suppose that $\{f_i : i \in I\} \subseteq V_e$ is a frame for the Hilbert space V_e with frame bounds C and D.

First we assume that V is finitely generated. Let $\{v_1, \ldots, v_n\} \subseteq V_e$ be an orthonormal basis for V and $S_e \in \mathbf{B}(V_e)$ the frame operator associated to the (Hilbert space) frame $\{f_i : i \in I\}$. Then

$$xe = \sum_{i\in I} S_e^{\frac{1}{2}} f_i (xe, S_e^{\frac{1}{2}} f_i) = \sum_{i\in I} S_e^{\frac{1}{2}} f_i \langle S_e^{\frac{1}{2}} f_i, xe\rangle, \quad xe \in V_e.$$

Since $x = \sum_{j=1}^{n} v_i \langle v_i, x\rangle$ for all $x \in V$, and since $v_1, \dots, v_n \in V_e$, we immediately get that for all $x \in V$, $x = \sum_{i\in I} S_e^{\frac{1}{2}} f_i \langle S_e^{\frac{1}{2}} f_i, x\rangle$ holds. This proves that $\{S_e^{\frac{1}{2}} f_i : i \in I\}$ is a Parseval standard frame for V. Let $S \in \mathbf{B}(V)$ be the unique extension of $S_e \in \mathbf{B}(V_e)$. Since S_e is invertible and positive, $S \in \mathbf{B}(V)$ is also invertible and positive. Therefore $S^{-\frac{1}{2}}$ preserves standard frames, so the sequence $\{S^{-\frac{1}{2}}(S_e^{\frac{1}{2}} f_i) : i \in I\} = \{S_e^{-\frac{1}{2}} S_e^{\frac{1}{2}} f_i : i \in I\} = \{f_i : i \in I\}$ is a standard frame for V.

Now we assume that V is not finitely generated. Let $\{v_i : i \in I\}$ be an orthonormal basis for V such that $v_i = v_i e$ for all $i \in I$. Then $\{v_i : i \in I\}$ is an orthonormal basis for the Hilbert space V_e. Let $T_e : V_e \to V_e$ be the operator defined as $T_e(xe) = \sum_{i\in I} f_i \langle v_i, xe\rangle$. As in the proof of Proposition 3.2 we show that T_e is well defined, $T_e \in \mathbf{B}(V_e)$ and T_e is surjective. Let $T \in \mathbf{B}(V)$ be the unique extension of $T_e \in \mathbf{B}(V_e)$. By the previous lemma, T is surjective. Now Theorem 2.5 implies that $\{f_i : i \in I\}$ is a standard frame for V, since $T(v_i) = T_e(v_i) = f_i$ for all $i \in I$. □

The concept of an orthonormal basis has been introduced in Hilbert C^*-modules over an arbitrary C^*-algebra. Obviously, there are Hilbert C^*-modules which do not possess an orthonormal basis. Actually, if a Hilbert C^*-module V possesses an orthonormal basis, then $\langle V, V\rangle$ has to be a CCR-algebra. We prove this in the next theorem.

Theorem 3.5. *Let A be a C^*-algebra and V a full countably generated Hilbert A-module. Let $\{e_i : i \in I\}$ be a family of projections in A such that $e_i A e_i = \mathbf{C} e_i, i \in I$, and $\{f_i : i \in I\}$ a standard frame for V such that $f_i = f_i e_i, i \in I$. Then A is a CCR-algebra. In particular, if V possesses an orthonormal basis, then A is a CCR-algebra.*

Proof. By the definition of a CCR-algebra we need to show that for every irreducible representation $\varphi : A \to \mathbf{B}(H)$, $\varphi(A) \subseteq \mathbf{K}(H)$ holds.

Let $0 \neq \varphi : A \to \mathbf{B}(H)$ be an irreducible representation of A. Then $e_i A e_i = \mathbf{C} e_i$ implies $\varphi(e_i)\varphi(A)\varphi(e_i) = \mathbf{C}\varphi(e_i)$ for every $i \in I$.

Let $i \in I$ be such that $\varphi(e_i) \neq 0$. Now $\varphi(e_i)$ is a non-zero projection, so there is a non-zero vector $\xi_0 \in H$ which belongs to $\operatorname{Im}\varphi(e_i)$. Then $\varphi(e_i)\xi_0 = \xi_0$ and

$$\varphi(e_i)\varphi(A)\varphi(e_i)\xi_0 = \mathbf{C}\varphi(e_i)\xi_0 \Rightarrow \varphi(e_i)\varphi(A)\xi_0 = \mathbf{C}\xi_0.$$

φ is irreducible, therefore it is a cyclic representation of A, and every non-zero vector is cyclic for φ. In particular, ξ_0 is a cyclic vector for φ. Therefore

$$\{0\} \neq \varphi(e_i)H = \varphi(e_i)\overline{\varphi(A)\xi_0} \subseteq \overline{\varphi(e_i)\varphi(A)\xi_0} = \mathbf{C}\xi_0 \Rightarrow \operatorname{Im}\,\varphi(e_i) = \mathbf{C}\xi_0.$$

This proves that $\varphi(e_i) \in \mathbf{K}(H)$ for every $i \in I$.

Let $S \in \mathbf{B}(V)$ be the frame operator associated to $\{f_i : i \in I\}$. From the reconstruction formula (1.2) we have $\langle x, y\rangle = \sum_{i\in I} \langle x, Sf_i\rangle \langle f_i, y\rangle$ and then

$$\varphi(\langle x, y\rangle) = \sum_{i\in I} \varphi(\langle x, Sf_i\rangle)\varphi(\langle f_i, y\rangle), \quad x, y \in V. \tag{3.2}$$

From $f_i = f_i e_i$ and compactness of $\varphi(e_i)$ it follows that

$$\varphi(\langle x, Sf_i\rangle)\varphi(\langle f_i, y\rangle) = \varphi(\langle x, Sf_i\rangle)\varphi(e_i)\varphi(\langle f_i, y\rangle) \in \mathbf{K}(H), \quad x, y \in V, i \in I.$$

Finally, we get $\varphi(\langle x, y\rangle) \in \mathbf{K}(H)$ for all $x, y \in V$, as the convergence in (3.2) is in norm. Since V is full, we conclude that $\varphi(A) \subseteq \mathbf{K}(H)$.

This finishes our proof. □

The converse of the previous theorem does not hold. For example, we can take an arbitrary Hilbert C^*-module over the C^*-algebra $A = C([0,1])$ of all continuous complex functions on the unit segment $[0,1]$. A is a CCR-algebra, since it is commutative, and the only projection $e \in A$ which satisfies $eAe = \mathbf{C}e$ is the constant function 0.

Remark 3.6. Frames of subspaces for a separable Hilbert space have been recently introduced and studied in [4]. We can generalize their definition for Hilbert $\mathbf{K}(H)$-modules in the following way.

Let V be a countably generated Hilbert $\mathbf{K}(H)$-module, $\{W_i : i \in I\}$ $(I \subseteq \mathbf{N})$ a family of closed submodules of V, and $\{\lambda_i : i \in I\}$ a family of weights, i.e., a family of positive numbers. We say that $\{W_i : i \in I\}$ is a *standard frame of submodules for* V *with respect to a family of weights* $\{\lambda_i : i \in I\}$, if there are constants $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{i\in I} \lambda_i^2 \langle \pi_i(x), \pi_i(x)\rangle \leq D\langle x, x\rangle, \quad x \in V, \tag{3.3}$$

where $\pi_i \in \mathbf{B}(V)$ denotes the orthogonal projection on W_i for every $i \in I$, and convergence of the sum in the middle of (3.3) is in norm.

Let us fix an orthogonal projection $e \in \mathbf{K}(H)$ of rank 1. It can be proved that a family of closed submodules $\{W_i : i \in I\}$ is a standard frame of submodules for V with respect to the family of weights $\{\lambda_i : i \in I\}$ if and only if $\{(W_i)_e : i \in I\}$ is a frame of subspaces for V_e with respect to the family of weights $\{\lambda_i : i \in I\}$. Therefore many statements from [4] can be extended to countably generated Hilbert $\mathbf{K}(H)$-modules. This will be done in our subsequent paper.

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