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ON FUBINI THEOREM FOR GENERAL PERRON INTEGRAL

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The approximation by means of integral sums (which is analogous to the usual approach to the Riemann integral) is used to obtain Fubini theorem for the Perron integral in a general form; there are found necessary and sufficient conditions for the existence of the iterated integral.

0 Notations. Let R be the real line, R^+ – the positive (open) real halfline, N – the set of positive integers. It is assumed that the linear space R^n, $n \in N$ is endowed with a norm, $\|x\|$ denoting the norm of x for $x \in R^n$. If $y \in R^n$, $\delta \in R^+$, then $B(y, \delta) = \{x \in R^n \mid \|x - y\| \leqq \delta\}$ is the closed ball in R^n with the center y and radius δ. $d(X)$ is the diameter of X for $X \subset R^n$, $\operatorname{cl} X$ is the closure of X. If Y, Z are sets, $f : Y \to Z$ and $W \subset Y$, then $f|_W$ is the restriction of f to W; if $Z = R$, then f is called a function. $U \times V$ is the cartesian product of the sets U and V. If $f : U \times V \to Z$, $w \in U$ then $f(u, \cdot) : V \to Z$ is defined by $f(u, \cdot)(v) = f(u, v)$ and analogous notations are used in case of three variables.

$\mathfrak{K}(R^n)$ is the set of nondegenerate compact intervals in R^n and if $K \in \mathfrak{K}(R^n)$, then $\mathfrak{K}(K)$ is the set of nondegenerate subintervals of K. $\operatorname{Int} J$ is the interior of J for $J \in \mathfrak{K}(R^n)$ and $|J|$ is the Lebesgue measure of J.

1 Basic concepts. The generalized Perron integral may be introduced in the following way, which is a modification of the usual approach to the Riemann integral (the material of this section is known, for references see Note 1,3).

Let $K \in \mathfrak{K}(R^n)$, $\omega : K \to R^+$. Denote by $\mathscr{A}(\omega)$ the set of such sets $A = \{(J_i, \tau_i) \mid i = 1, 2, \ldots, k\}$ that the following conditions are fulfilled:

(1,1) $$\tau_i \in J_i \in \mathfrak{K}(K) \quad \text{for} \quad i = 1, 2, \ldots, k\,,$$

(1,2) $$\bigcup_{i=1}^{k} J_i = K\,,$$

(1,3) $$\operatorname{Int} J_i \cap \operatorname{Int} J_j = \emptyset \quad \text{for} \quad i \neq j\,, \quad i, j = 1, 2, \ldots, k\,,$$

(1,4) $$J_i \subset B(\tau_i, \omega(\tau_i)) \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

If ω is replaced by $\omega_{[K]}$, which is defined by $\omega_{[K]}(\tau) = d(K)$ for $\tau \in K$, then condition (1,4) may be omitted and $\mathscr{A}(\omega_{[K]})$ is the set of such A that (1,1), (1,2) and (1,3) are fulfilled.

Lemma 1,1. *$\mathscr{A}(\omega) \neq \emptyset$ for any $\omega : K \to R^+$.*

Let the proof be sketched. Fix such $\omega : K \to R^+$ that $\mathscr{A}(\omega) = \emptyset$, put $K_1 = K$ and divide K_1 into a finite number of $L_i \in \mathfrak{K}(K)$ so that $d(L_i) \leqq \frac{1}{2}d(K)$ for every i. Find such a j that $\mathscr{A}(\omega|_{L_j}) = \emptyset$, put $K_2 = L_j$ and repeat this procedure. It follows that $\prod_{s=1}^{\infty} K_s = \{z\}$, $z \in K$. $\omega(z) > 0$ and therefore $\mathscr{A}(\omega|_{K_s}) \neq \emptyset$ for s sufficiently large. This contradiction makes the proof complete.

For $U : \mathfrak{K}(K) \times K \to R$, $A = \{(J_i, \tau_i) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\omega_{[K]})$, $X \subset K$ define

$$S(U, A) = \sum_{i=1}^{k} U(J_i, \tau_i)\,, \tag{1,5}$$

$$S_X(U, A) = \sum_{\tau_i \in X} U(J_i, \tau_i)\,. \tag{1,6}$$

Observe that if $f : K \to R$ and $U(J, \tau) = f(\tau)\,|J|$ for $J \in \mathfrak{K}(K)$, $\tau \in K$, then $S(U, A) = \sum_{i=1}^{k} f(\tau_i)\,|J_i|$, the last sum being of the type that is used in the definition of the Riemann integral of f.

Definition 1,1. U is called (P)-*integrable* (Perron-integrable) in K, if to every $\varepsilon \in R^+$ there exists such an $\omega : K \to R^+$ that

$$|S(U, A_1) - S(U, A_2)| \leqq \varepsilon \quad \text{for} \quad A_1, A_2 \in \mathscr{A}(\omega)\,.$$

The set of functions $U : \mathfrak{K}(K) \times K \to R$ which are (P)-integrable in K is denoted by $\mathfrak{P}(K)$.

Theorem 1,1. *If $U \in \mathfrak{P}(K)$, then there exists such an $I \in R$ that to every $\varepsilon \in R^+$ there is such an $\omega : K \to R^+$ that*

$$|S(U, A) - I| \leqq \varepsilon \quad \textit{for} \quad A \in \mathscr{A}(\omega)\,.$$

Definition 1,2. The number I from Theorem 1,1 is called the *Perron integral* of U and denoted by $(P) \int_K U$.

Note 1,1. Assume that $f : K \to R$ and $U(J, \tau) = f(\tau)\,|J|$. In this special case $U \in \mathfrak{P}(K)$ iff f is Perron-integrable in the classical sense and $(P) \int_K U$ is equal to the classical Perron integral. $\mathfrak{P}(K)$ and $(P) \int_K U$ may be defined equivalently by means of major and minor functions in an analogous manner as in the classical theory of the Perron integral. $(P) \int_K U$ will be called the general Perron integral.

Note 1,2. If $L \in \mathfrak{K}(K)$, we shall write $(P) \int_L U$ instead of $(P) \int_L U|_{\mathfrak{K}(L) \times L}$, provided that the latter integral exists.

A map $V : \mathfrak{K}(K) \to R$ is called additive in case that $V(L) = V(H) + V(J)$ if $H \dotplus J = L \in \mathfrak{K}(K)$ (i.e. if $H, J, L \in \mathfrak{K}(K)$, $H \cup J = L$, $\operatorname{Int} H \cap \operatorname{Int} J = \emptyset$). A map $G : \mathfrak{K}(K) \to R$ is called superadditive provided that $G(L) \geqq \sum_{i=1}^{k} G(J_i)$, if $J_1, \ldots, J_k, L \in \mathfrak{K}(K)$, $L = \bigcup_{i=1}^{k} J_i$ and $\operatorname{Int} J_i \cap \operatorname{Int} J_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, k$.

The set of all superadditive maps $\eta : \mathfrak{K}(K) \to R$ such that $\eta(J) \geqq 0$ for $J \in \mathfrak{K}(K)$ is denoted by $Y(K)$.

Definition 1.3. $U : \mathfrak{K}(K) \times K \to R$ is called *variationally integrable* in K provided that there is such an additive $V : \mathfrak{K}(K) \to R$ that to any $\varepsilon \in R^+$ there exist $\eta \in Y(K)$ and $\omega : K \to R^+$ such that $\eta(K) \leqq \varepsilon$, $|U(J, \tau) - V(J)| \leqq \eta(J)$ for $\tau \in J \in \mathfrak{K}(K)$, $J \subset B(\tau, \omega(\tau))$. The set of functions, which are variationally integrable in K, is denoted by $\mathfrak{B}(K)$.

The following Lemma may be proved easily.

Lemma 1,2. *There is at most one V fulfilling the conditions of Definition* 1,3.

Therefore it may be defined:

Definition 1,4. If V fulfils the conditions of Definition 1,3, then $V(K)$ is called the *variational integral* of U and denoted by $(V) \int_K U$.

The equivalence of the Perron integral and the variational integral is stated in the following

Theorem 1,2. $\mathfrak{B}(K) = \mathfrak{P}(K)$; *if* $U \in \mathfrak{B}(K)$, *then* $(V) \int_K U = (P) \int_K U$.

Therefore Definitions 1,3 and 1,4 may be taken for descriptive definitions of (P)-integrable functions and of the Perron integral. In the sequel there will be needed only the following part of Theorem 1,2:

$$(1,7) \qquad \mathfrak{P}(K) \subset \mathfrak{B}(K)\,; \quad \text{if} \quad U \in \mathfrak{P}(K), \quad \text{then} \quad (V) \int_K U = (P) \int_K U\,,$$

the proof of which is analogous to the proof of Lemma 2,6,

Note 1,3. The proofs of Theorems 1,1, 1,2, Lemmas 1,1, 1,2 and of the assertions from Note 1,1 may be found in [3]; in [3] different notations are used and there is a very slight difference in the concepts of the integral (which is removed, if every $U : \mathfrak{K}(K) \times K \to R$ is assumed to be additive in the following sense: $U(L, \tau) = U(H, \tau) + U(J, \tau)$ holds whenever $L, H, J \in \mathfrak{K}(K)$, $L = H \cup J$, $\operatorname{Int} H \cap \operatorname{Int} J = \emptyset$, $\tau \in H \cap J$).

Definitions 1,1 and 1,2 appeared in [4] (for $n = 1$ and U additive) and there was proved their equivalence to the definitions by means of major and minor functions. The concept of the variational integral is due to R. HENSTOCK, [1].

2 Fubini Theorem. It will be assumed throughout this section that there are given $n, n_1, n_2 \in N$, $n = n_1 + n_2$ and that there is given a representation $R^n = R^{n_1} \times R^{n_2}$; if $x \in R^n$, we shall write $x = (x_1, x_2)$ with $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ and we assume that $\|x\| = \max(\|x_1\|, \|x_2\|)$, $\|x\|$, $\|x_1\|$, $\|x_2\|$ denoting the norms of x, x_1, x_2 respectively. Similarly if $K \in \mathfrak{K}(R^n)$, there exist unique $K_1 \in \mathfrak{K}(R^{n_1})$, $K_2 \in \mathfrak{K}(R^{n_2})$ such that $K = K_1 \times K_2$. If $\omega : K \to R^+$, it will be occasionally written $\omega(\tau_1, \tau_2)$ instead of $\omega(\tau)$ for $\tau = (\tau_1, \tau_2) \in K$.

Definition 2,1. Let $K_1 \in \mathfrak{K}(R^{n_1})$, $U_1 : \mathfrak{K}(K_1) \times K_1 \to R$. Let $T \subset K_1$ have the following property: to every $\varepsilon \in R^+$ there exists such a $\xi : K_1 \to R^+$ that if $(H_1^{(i)}, \sigma_1^{(i)}) \in \mathfrak{K}(K_1) \times T$, $\sigma_1^{(i)} \in H_1^{(i)} \subset B(\sigma_1^{(i)}, \xi(\sigma_1^{(i)}))$ for $i = 1, 2, \ldots, s$, $\operatorname{Int} H_1^{(i)} \cap \operatorname{Int} H_1^{(j)} = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, s$, then $\sum_{i=1}^{s} |U_1(H_1^{(i)}, \sigma_1^{(i)})| \leqq \varepsilon$. Denote the set of such T by $\mathfrak{N}(U_1)$.

Note 2,1. If $U_1(J_1, \tau_1) = |J_1|$ for $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$, then $T \in \mathfrak{N}(U_1)$ iff $|T| = 0$, $|T|$ being the Lebesgue measure of T.

Note 2,2. In the terminology of [3] the corresponding statement to $T \in \mathfrak{N}(U_1)$ is that h is of variation zero in E (cf. [3], § 26).

Theorem 2,1. *Let $U_1 : \mathfrak{K}(K_1) \times K_1 \to R$, $U_2 : \mathfrak{K}(K_2) \times K_1 \times K_2 \to R$, $U = U_1 U_2$ (i.e. $U(J, \tau) = U_1(J_1, \tau_1)\, U_2(J_2, \tau_1, \tau_2)$ for $J = J_1 \times J_2 \in \mathfrak{K}(K)$, $\tau = (\tau_1, \tau_2) \in K$), $U \in \mathfrak{P}(K)$. Let T be the set of of such $\tau_1 \in K_1$ that $U_2(\cdot, \tau_1, \cdot) \in \mathfrak{P}(K_2)$. Then $K_1 - T \in \mathfrak{N}(U_1)$.*

For $\tau_1 \in T$ define $\phi(\tau_1) = (P) \int_{K_2} U_2(\cdot, \tau_1, \cdot)$, for $\tau_1 \in K_1 - T$ choose $\phi(\tau_1) \in R$ arbitrarily and define $W(J_1, \tau_1) = U_1(J_1, \tau_1)\, \phi(\tau_1)$ for $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$. Then $W \in \mathfrak{P}(K_1)$ and

$$(2,1) \qquad (P) \int_K U = (P) \int_{K_1} W$$

((2,1) may be written shortly $(P) \int_K U = (P) \int_{K_1} U_1[(P) \int_{K_2} U_2]$).

Theorem 2,1 is a consequence of Theorems 2,3 and 2,4.

Note 2,3. Theorem 2,1 differs from Theorem 44,1 in [3] that U is not supposed VBG* (and U need not be additive, cf. Note 1,3).

Note 2,4. If $f : K \to R$ is Perron integrable in the classical sense (cf. Note 1,1), put $U_1(J_1, \tau_1) = |J_1|$, $U_2(J_2, \tau_1, \tau_2) = f(\tau_1, \tau_2)\, |J_2|$. Then $(P) \int_{K_2} f(\tau_1, \cdot)$ exists almost everywhere and $(P) \int_K f = (P) \int_{K_1} (P) \int_{K_2} f(\tau_1, \cdot)$. Symmetrically $(P) \int_K f = (P) \int_{K_2} (P) \int_{K_1} f(\cdot, \tau_2)$.

Definition 2,2. Let $\{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\omega_{[K_1]})$. Let $\{(L_2^{(i,j)}, \lambda_2^{(i,j)}) \mid j = 1, 2, \ldots, l^{(i)}\} \in \mathscr{A}(\omega_{[K_2]})$ for $i = 1, 2, \ldots, k$. Put

$$(2,2) \qquad A = \{(J_1^{(i)} \times L_2^{(i,j)}, (\tau_1^{(i)}, \lambda_2^{(i,j)})) \mid i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, l^{(i)}\}.$$

The set of all such A denote by $\mathscr{A}_{1,2}(\omega_{[K]})$ and put

$$\mathscr{A}_{1,2}(\omega) = \mathscr{A}(\omega) \cap \mathscr{A}_{1,2}(\omega_{[K]}) \quad \text{for} \quad \omega : K \to R^+ .$$

Lemma 2,1. *$\mathscr{A}_{1,2}(\omega_{[K]}) \subset \mathscr{A}(\omega_{[K]})$; $\mathscr{A}_{1,2}(\omega) \subset \mathscr{A}(\omega)$ for $\omega : K \to R^+$.*

This is obvious.

Lemma 2,2. *$\mathscr{A}_{1,2}(\omega) \neq \emptyset$ for $\omega : K \to R^+$.*

Proof. For $\sigma_1 \in K_1$ find by Lemma 1,1

$$A(\sigma_1) = \{(H_2^{(j)}(\sigma_1), \sigma_2^{(j)}(\sigma_1)) \in \mathfrak{K}(K_2) \times K_2 \mid j = 1, 2, \ldots, l(\sigma_1)\} \in \mathscr{A}(\omega(\sigma_1, \cdot))$$

and put $\mu(\sigma_1) = \min_{j=1,2,\ldots,l(\sigma_1)} \omega(\sigma_1, \sigma_2^{(j)}(\sigma_1))$. It is $\mu : K_1 \to R^+$ and by Lemma 1,1 there exists $\{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\mu)$. Put $l^{(i)} = l(\tau_1^{(i)})$, $\lambda_2^{(i,j)} = \sigma_2^{(j)}(\tau_1^{(i)})$, $L_2^{(i,j)} = H_2^{(j)}(\tau_1^{(i)})$ for $j = 1, 2, \ldots, l^{(i)}$, $i = 1, 2, \ldots, k$.

Definition 2,3. $U = U_1 U_2$ is called $(\mathrm{P}_{1,2})$-*integrable* in K, if for every $\varepsilon \in R^+$ there exists such an $\omega : K \to R$ that $|S(U, A_1) - S(U, A_2)| \leqq \varepsilon$ for $A_1, A_2 \in \mathscr{A}_{1,2}(\omega)$. The set of functions, which are $(\mathrm{P}_{1,2})$-integrable in K, is denoted by $\mathfrak{P}_{1,2}(K)$.

Theorem 2,2. *If $U \in \mathfrak{P}_{1,2}(K)$, then there exists a unique $I \in R$ such that for every $\varepsilon \in R^+$ there exists such an $\omega : K \to R^+$ that $|S(U, A) - I| \leqq \varepsilon$ for $A \in \mathscr{A}_{1,2}(\omega)$.*

This is obvious.

Definition 2,4. The number I from Theorem 2,2 is called the $(\mathrm{P}_{1,2})$-*integral of U* and is denoted by $(P_{1,2}) \int_K U$.

Theorem 2,3. *$\mathfrak{P}(K) \subset \mathfrak{P}_{1,2}(K)$; if $U \in \mathfrak{P}(K)$ then $(P_{1,2}) \int_K U = (P) \int_K U$.*

This follows immediately from Lemma 2,1.

Lemma 2,3. *Let $X_i \in \mathfrak{N}(U_1)$ for $i \in N$. Then $\bigcup_{i \in N} X_i \in \mathfrak{N}(U_1)$.*

The proof of Lemma 2,3 is quite straightforward.

Lemma 2,4. *Let $X \in \mathfrak{N}(U_1)$, $\phi : K_1 \to R$, $W(J_1, \tau_1) = U_1(J_1, \tau_1)\, \phi(\tau_1)$ for $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$. Then $X \in \mathfrak{N}(W)$.*

The proof follows from the preceding Lemma, as $X = \bigcup_{r \in N} X_r$ with $X_r = \{x \in X \mid |\phi(x)| \leqq r\}$.

Lemma 2,5. *Let* $\xi : K_1 \to R$, $H_1^{(i)} \in \mathfrak{K}(K_1)$, $\sigma_1^{(i)} \in H_1^{(i)} \subset B(\sigma_1^{(i)}, \xi(\sigma_1^{(i)}))$ *for* $i = 1, 2, \ldots, s$, $\operatorname{Int} H_1^{(i)} \cap \operatorname{Int} H_1^{(j)} = \emptyset$ *for* $i \neq j$, $i, j = 1, 2, \ldots, s$. *Then there exists* $A_1 = \{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi)$ *such that* $J_1^{(i)} = H_1^{(i)}$, $\tau_1^{(i)} = \sigma_1^{(i)}$ *for* $i = 1, 2, \ldots, s$.

The proof follows from Lemma 1,1, for either $K_1 = \bigcup_{i=1}^{s} H_1^{(i)}$ holds or $\operatorname{cl}(K_1 - \bigcup_{i=1}^{s} H_1^{(i)})$ is a finite union of intervals from $\mathfrak{K}(K_1)$ whose interiors are mutually disjoint.

If $U \in \mathfrak{P}_{1,2}(K)$, $J_1 \in \mathfrak{K}(K_1)$, put $Q = U|_{\mathfrak{K}(J_1 \times K_2) \times J_1 \times K_2}$. It is easy to deduce from Lemma 2,5 that $Q \in \mathfrak{P}_{1,2}(J_1 \times K_2)$, it will be written $(P_{1,2}) \int_{J_1 \times K_2} U$ instead of $(P_{1,2}) \int_{J_1 \times K_2} Q$.

Lemma 2,6. *Let* $U \in \mathfrak{P}_{1,2}(K)$. *Put* $V(J_1) = (P_{1,2}) \int_{J_1 \times K_2} U$ *for* $J_1 \in \mathfrak{K}(K_1)$. *Then to every* $\varepsilon \in R^+$ *there exist* $\omega : K \to R^+$ *and* $\eta \in Y(K_1)$ *in such a way that* $\eta(K_1) \leqq \varepsilon$ *and*

$$(2,3) \qquad \left|V(J_1) - \sum_{i=1}^{k} U(J_1 \times L_2^{(i)}, (\tau_1, \lambda_2^{(i)}))\right| \leqq \eta(J_1)$$

if $\tau_1 \in J_1 \in \mathfrak{K}(K_1)$, $\{(L_2^{(i)}, \lambda_2^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\omega(\tau_1, \cdot))$, $J_1 \times L_2^{(i)} \subset B((\tau_1, \lambda_2^{(i)}), \omega(\tau_1, \lambda_2^{(i)}))$ *for* $i = 1, 2, \ldots, k$.

Proof. To $\varepsilon \in R^+$ find $\omega : K \to R^+$ according to Definition 2,3 and put $\eta(J_1) = \sup |S(U, C_1) - S(U, C_2)|$, sup being taken for $C_1, C_2 \in \mathscr{A}_{1,2}(\omega_{J_1 \times K_2})$. It is easy to verify that $\eta \in Y(K_1)$, $\eta(K_1) \leqq \varepsilon$ and (2,3) holds, as $S(U, C_1)$ can be made arbitrarily close to $V(J_1)$ while C_2 may be put equal to $\{(J_1 \times L_2^{(i)}, (\tau_1, \lambda_2^{(i)})) \mid i = 1, 2, \ldots, k\}$.

Theorem 2,4. *Let* $U_1 : \mathfrak{K}(K_1) \times K_1 \to R$, $U_2 : \mathfrak{K}(K_2) \times K_1 \times K_2 \to R$, $U = U_1 U_2$, $U \in \mathfrak{P}_{1,2}(K)$. *Let* T *be the set of such* $\tau_1 \in K_1$ *that* $U_2(\cdot, \tau_1, \cdot) \in \mathfrak{P}(K_2)$. *For* $\tau_1 \in T$ *define* $\phi(\tau_1) = (P) \int_{K_2} U_2(\cdot, \tau_1, \cdot)$, *for* $\tau_1 \in K_1 - T$ *choose* $\phi(\tau_1) \in R$ *arbitrarily and define* $W(J_1, \tau_1) = U_1(J_1, \tau_1)\, \phi(\tau_1)$ *for* $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$. *Then*

$$(2,4) \qquad K_1 - T \in \mathfrak{N}(U_1)\,,$$

(2,5) *to every* $\varepsilon \in R^+$ *there exists such a* $v : K \to R^+$ *that*

$$|S_{(K_1 - T) \times K_2}(U, A)| \leqq \varepsilon \quad \textit{for} \quad A \in \mathscr{A}_{1,2}(v)\,,$$

$$(2,6) \qquad W \in \mathfrak{P}(K_1) \quad \textit{and} \quad (P) \int_{K_1} W = (P_{1,2}) \int_{K_1} U\,.$$

Proof. Let us start with the proof of (2,4). Let X_r for $r \in N$ denote the set of such $\tau_1 \in K_1$ that for every $\omega_2 : K_2 \to R^+$ there exist $A_2^{(1)}, A_2^{(2)} \in \mathscr{A}(\omega_2)$ is such a way that

$$|S(U_2(\cdot, \tau_1, \cdot), A_2^{(1)}) - S(U_2(\cdot, \tau_1, \cdot), A_2^{(2)})| \geqq r^{-1}\,.$$

Obviously $K_1 - T = \bigcup_{r \in N} X_r$ and – by Lemma 2,2 – (1,4) will be satisfied, if it will be proved that $X_r \in \mathfrak{N}(U_1)$ for $r \in N$.

Let $r \in N$ be fixed, let $\varepsilon \in R^+$ and let $\omega : K \to R$ correspond to ε according to Definition 1,2. To $\tau_1 \in X_r$ find

$$(2,7) \qquad A_2(\tau_1) = \{(L_2^{(j)}(\tau_1), \lambda_2^{(j)}(\tau_1)) \in \mathfrak{K}(K_2) \times K_2 \mid j = 1, 2, \ldots, l(\tau_1)\},$$
$$\tilde{A}_2(\tau_1) = \{(\tilde{L}_2^{(j)}(\tau_1), \tilde{\lambda}_2^{(j)}(\tau_1)) \in \mathfrak{K}(K_2) \times K_2 \mid j = 1, 2, \ldots, \tilde{l}(\tau_1)\},$$

$A_2(\tau_1)$, $\tilde{A}_2(\tau_1) \in \mathscr{A}(\omega(\tau_1, \cdot))$ in such a way that

$$|S(U_2(\cdot, \tau_1, \cdot), A_2(\tau_1)) - S(U_2(\cdot, \tau_1, \cdot), \tilde{A}_2(\tau_1))| \geqq r^{-1}.$$

Put $\xi(\tau_1) = \min \left(\min_{j=1,2,\ldots,l(\tau_1)} \omega(\tau_1, \lambda_2^{(j)}(\tau_1)), \min_{j=1,2,\ldots,\tilde{l}(\tau_1)} \omega(\tau_1, \tilde{\lambda}_2^{(j)}(\tau_1)) \right)$. To $\tau_1 \in K_1 - X_r$ find

$$A_2(\tau_1) = \{(L_2^{(j)}(\tau_1), \lambda_2^{(j)}(\tau_1)) \in \mathfrak{K}(K_2) \times K_2 \mid j = 1, 2, \ldots, l(\tau_1)\} \in \mathscr{A}(\omega(\tau_1, \cdot))$$

and put $\tilde{A}_2(\tau_1) = A_2(\tau_1)$, $\xi(\tau_1) = \min_{j=1,2,\ldots,l(\tau_1)} \omega(\tau_1, \lambda_2^{(j)}(\tau_1))$. Let $(H_1^{(j)}, \sigma_1^{(j)}) \in \mathfrak{K}(K_1) \times X_r$ for $j = 1, 2, \ldots, s$, $\sigma_1^{(j)} \in H_1^{(j)} \subset B(\sigma_1^{(j)}, \xi(\sigma_1^{(j)}))$ for $j = 1, 2, \ldots, s$, $\operatorname{Int} H_1^{(j)} \cap \operatorname{Int} H_1^{(i)} = \emptyset$ for $j \neq i$, $j, i = 1, 2, \ldots, s$. By Lemma 2,5 there exists $\{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi)$ so that $J_1^{(i)} = H_1^{(i)}$, $\tau_1^{(i)} = \sigma_1^{(i)}$ for $i = 1, 2, \ldots, s$. Without loss on generality we may assume that

$$\operatorname{sgn} (S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1))) = \operatorname{sgn} U_1(J_1^{(i)}, \tau_1^{(i)})$$

if $\tau_1^{(i)} \in X_r$ and $U_1(J_1^{(i)}, \tau_1^{(i)}) \neq 0$. Put

$$(2,8) \quad A = \{(J_1^{(i)} \times L_2^{(j)}(\tau_1^{(i)}), (\tau_1^{(i)}, \lambda_2^{(j)}(\tau_1^{(i)}))) \mid i = 1, 2, \ldots, k, j = 1, 2, \ldots, l(\tau_1^{(i)})\},$$
$$\tilde{A} = \{(J_1^{(i)} \times L_2^{(j)}(\tau_1^{(i)}), (\tau_1^{(i)}, \tilde{\lambda}_2^{(j)}(\tau_1^{(i)}))) \mid i = 1, 2, \ldots, k, j = 1, 2, \ldots, \tilde{l}(\tau_1^{(i)})\}.$$

It may be verified easily that

$$S(U, A) - S(U, \tilde{A}) =$$
$$= \sum_{\tau_1^{(i)} \in X_r} U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1^{(i)}))] \geqq$$
$$\geqq r^{-1} \sum_{\tau_1^{(i)} \in X_r} |U_1(J_1^{(i)}, \tau_1^{(i)})| \geqq r^{-1} \sum_{j=1}^{s} |U_1(H_1^{(j)}, \sigma_1^{(j)})|.$$

On the other hand $A, \tilde{A} \in \mathscr{A}_{1,2}(\omega)$, hence $|S(U, A) - S(U, \tilde{A})| \leqq \varepsilon$, so that $\sum_{j=1}^{s} |U(H_1^{(j)}, \sigma_1^{(j)})| \leqq r\varepsilon$ and (2,4) holds, as $\varepsilon \in R^+$ may be chosen arbitrarily to a fixed r.

In order to prove (2,6) it is to be proved that to any $\varepsilon \in R^+$ there exists such an $\omega_1 : K_1 \to R^+$ that

$$(2,9) \qquad \left| S(W, A_1) - (P_{1,2}) \int_K U \right| \leqq \varepsilon \quad \text{for} \quad A_1 \in \mathscr{A}(\omega_1) .$$

Let $\varepsilon \in R^+$ be fixed. Find such an $\omega : K \to R^+$ that

$$(2,10) \qquad |S(U, A) - S(U, \tilde{A})| \leqq \tfrac{1}{4}\varepsilon \quad \text{for} \quad A, \tilde{A} \in \mathscr{A}_{1,2}(\omega) .$$

Let $\tau_1 \in K_1 - T$; to every such τ_1 find $A_2(\tau_1) \in \mathscr{A}(\omega(\tau_1, \cdot))$ and such a $\delta(\tau_1) \in R^+$ that $\tau_1 \in J_1 \in \mathfrak{K}(K_1)$, $J_1 \subset B(\tau_1, \delta(\tau_1))$, $(M, \sigma) \in A_2(\tau_1)$ implies $J_1 \times M \subset B((\tau_1, \sigma), \omega(\tau_1, \sigma))$. (Using notations of (2,2) we may put $\delta(\tau_1) = \min_{j=1,2,\ldots,l(\sigma)_1} \omega(\tau_1, \lambda_2^{(j)}(\tau_1))$.)
Put

$$Q_1 = \{\tau_1 \in K_1 - T \mid |\phi(\tau_1)| + |S(U_2(\cdot, \tau_1, \cdot), A_2(\tau_1))| \leqq 1\} ,$$

$$Q_r = \{\tau_1 \in K_1 - T \mid r - 1 < |\phi(\tau_1)| + |S(U_2(\cdot, \tau_1, \cdot), A_2(\tau_1))| \leqq r\}$$

for $r = 2, 3, \ldots$ By (2,4) $Q_r \in \mathfrak{N}(U_1)$ for $r \in N$. By Definition 2,1 there exists such a $\xi_r : K_1 \to R^+$ for $r \in N$ that $\sum_{i=1}^{s} |U_1(H_1^{(i)}, \sigma_1^{(i)})| \leqq \varepsilon/(r \,.\, 2^{r+2})$ provided that $(H_1^{(i)}, \sigma_1^{(i)}) \in \mathfrak{K}(K_1) \times Q_r$ $\sigma_1^{(i)} \in H_1^{(i)} \subset B(\sigma_1^{(i)}, \xi_r(\sigma_1^{(i)}))$, Int $H_1^{(i)} \cap$ Int $H_1^{(j)} = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, s$. Finally put $\tilde{A}_2(\tau_1) = A_2(\tau_1)$ for $\tau_1 \in K_1 - T$ and $\omega_1(\tau_1) = \min(\delta(\tau_1), \xi_r(\tau_1))$ provided that $\tau_1 \in Q_r$, $r \in N$; ω_1 is defined for $\tau_1 \in K_1 - T$, as $K_1 - T = \bigcup_{r \in N} Q_r$ and $Q_r \cap Q_s = \emptyset$ for $r \neq s$, $r, s \in N$.

Let $\tau_1 \in T$; to every such τ_1 find $A_2^{(1)}(\tau_1) \in \mathscr{A}(\omega(\tau_1, \cdot))$. and then find such a $A_2^{(2)}(\tau_1) \in \mathscr{A}(\omega(\tau_1, \cdot))$ that

$$|S(U_2(\cdot, \tau_1, \cdot), A_2^{(2)}(\tau_1)) - \phi(\tau_1)| \leqq \tfrac{1}{2}(S(U_2(\cdot, \tau_1, \cdot), A_2^{(1)}(\tau_1)) - \phi(\tau_1)| .$$

Find such an $\omega_1(\tau_1) \in R^+$ that $\tau_1 \in J_1 \subset B(\tau_1, \omega_1(\tau_1))$, $J_1 \in \mathfrak{K}(K_1)$, $(M, \sigma) \in A_2^{(1)}(\tau_1) \cup \cup A_2^{(2)}(\tau_1)$ implies that $J_1 \times M \subset B((\tau_1, \sigma), \omega(\tau_1, \sigma))$. Choose $A_1 = \{J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\omega_1)$. If $\tau_1^{(i)} \in T$ and $U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2^{(1)}(\tau_1^{(i)})) - \phi(\tau_1^{(i)})] > 0$, put $A_2(\tau_1^{(i)}) = A_2^{(1)}(\tau_1^{(i)})$, $\tilde{A}_2(\tau_1^{(i)}) = A_2^{(2)}(\tau_1^{(i)})$; otherwise (for $\tau_1^{(i)} \in T$) put $A_2(\tau_1^{(i)}) = A_2^{(2)}(\tau_1^{(i)})$, $\tilde{A}_2(\tau_1^{(i)}) = A_2^{(1)}(\tau_1^{(i)})$. Using the notations of (2,7) and (2,3) define A and $\tilde{A}$ by (2,8). It is not difficult to verify that $A, \tilde{A} \in \mathscr{A}(\omega)$, so that

$$(2,11) \qquad |S(U, A) - S(U, \tilde{A})| \leqq \tfrac{1}{4}\varepsilon .$$

Obviously

$$(2,12) \qquad |S(U, A) - S(W, A_1)| \leqq$$
$$\leqq |S_{(K_1 - T) \times K_2}(U, A) - S_{K_1 - T}(W, A_1)| + |S_{T \times K_2}(U, A) - S_T(W, A_1)| .$$

It follows from the choice of ω_1 that

$$(2,13) \qquad |S_{(K_1-T)\times K_2}(U, A) - S_{K_1-T}(W, A_1)| =$$
$$= |\sum_{\tau_1^{(i)}\in K_1-T} U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - \phi(\tau_1^{(i)})] \leqq$$
$$\leqq \sum_{r\in N} \sum_{\tau_1^{(i)}\in Q_r} |U_1(J_1^{(i)}, \tau_1^{(i)})| \,.\, r \leqq \sum_{r\in N} \varepsilon/2^{r+2} = \varepsilon/4\,.$$

Let $\tau_1^{(i)} \in T$. If

$$(2,14) \qquad U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2^{(1)}(\tau_1^{(i)})) - \phi(\tau_1^{(i)})] > 0\,,$$

then

$$0 < U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - \phi(\tau_1^{(i)})] \leqq$$
$$\leqq 2U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1^{(i)}))]$$

if (2,14) does not hold, then

$$|U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - \phi(\tau_1^{(i)})]| \leqq$$
$$\leqq U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1^{(i)}))]$$

so that

$$(2,15) \qquad |U_1(J_1^{(i)}, \tau_1^{(i)}) S[(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - \phi(\tau_1^{(i)})]| \leqq$$
$$\leqq 2U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1^{(i)}))]$$

holds, if $\tau_1^{(i)} \in T$.

It may be seen that $S(U, A) - S(U, \tilde{A}) = S_{T\times K_2}(U, A) - S_{T\times K_2}(U, \tilde{A}) = = \sum_{\tau_1^{(i)}\in T} U_1(J_1^{(i)}, \tau_1^{(i)}) [S(U_2(\cdot, \tau_1^{(i)}, \cdot), A_2(\tau_1^{(i)})) - S(U_2(\cdot, \tau_1^{(i)}, \cdot), \tilde{A}_2(\tau_1^{(i)}))]$. Hence it follows by (2,11) and (2,15) that

$$(2,16) \qquad |S_{T\times K_1}(U, A) - S_T(W, A_1)| \leqq \tfrac{1}{2}\varepsilon\,.$$

This together with (2,12) and (2,13) gives

$$(2,17) \qquad |S(U, A) - S(W, A_1)| \leqq \tfrac{3}{4}\varepsilon$$

and (2,9) holds by (2,17) and (2,10), as $\hat{A} \in \mathscr{A}_{1,2}(\omega)$ may be chosen in such a way that $S(U, \hat{A})$ is arbitrarily close to $(P_{1,2}) \int_K U$. The proof of (2,6) is complete.

It remains to prove that (2,5) holds. By (2,4) and Lemma 2,4 there exists such a $\xi_1 : K_1 \to R^+$ that

$$(2,18) \qquad |S_{K_1-T}(W, C_1)| \leqq \tfrac{1}{3}\varepsilon \quad \text{for} \quad C_1 \in \mathscr{A}(\xi_1)\,.$$

By Lemma 2,6 there exists such a $\xi : K \to R^+$ that

$$\left| S_{(K_1 - T) \times K_2}(U, A) - \sum_{\tau_1^{(i)} \in K_1 - T} (P_{1,2}) \int_{J_1^{(i)} \times K_2} U \right| \leqq \tfrac{1}{3}\varepsilon \tag{2,19}$$

for $A \in \mathscr{A}_{1,2}(\xi)$, A being described in (2,2). Finaly by (1,7) and by Definition 1,3 there exists such a $\vartheta_1 : K_1 \to R^+$ that

$$\left| S_{K_1 - T}(W, C_1) - \sum_{\sigma_1^{(i)} \in K_1 - T} (P) \int_{M_1^{(i)}} W \right| \leqq \tfrac{1}{3}\varepsilon \tag{2,20}$$

for $C_1 = \{(M_1^{(i)}, \sigma_1^{(i)}) \mid i = 1, 2, \ldots, m\} \in \mathscr{A}(\vartheta_1)$. Put $\nu(\tau) = \min(\xi(\tau), \xi_1(\tau_1), \vartheta_1(\tau_1))$ for $\tau = (\tau_1, \tau_2) \in K$. Obviously $\nu : K \to R^+$ and if $A \in \mathscr{A}_{1,2}(\nu)$ (cf. (2,2)), then $A_1 = \{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi_1) \cap \mathscr{A}(\vartheta_1)$, so that we may put $C_1 = A_1$ in (2,18) and (2,20). Moreover, by (2,6) $(P_{1,2}) \int_{J_1^{(i)} \times K_2} U = (P) \int_{J^{(i)}} W$ for $i = 1, 2, \ldots, k$. Hence (2,5) holds by (2,18), (2,19) and (2,20). The proof of Theorem 2,4 is complete.

Definition 2,4. Let $U_1 : \mathfrak{K}(K_1) \times K_1 \to R$, $X \subset K_1$. U_1 is said to be *of bounded variation in X* (BV in X), if there are $\varkappa \in R^+$ and $\xi : K_1 \to R$ in such a way that $\sum_{\tau_1^{(i)} \in X} |U_1(J_1^{(i)}, \tau_1^{(i)})| \leqq \varkappa$ for any $\{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi)$.

U_1 is said to be *of generalized bounded variation in X* (BVG in X), if there are $X_r \subset X$ for $r \in N$ in such a way that $\bigcup_{r \in N} X_r = X$ and U_1 is BV in each X_r, $r \in N$.

Note 2,5. If $W : \mathfrak{K}(K_1) \times K_1 \to R$, $W \in \mathfrak{P}(K_1)$, $T \subset K_1$, $K_1 - T \in \mathfrak{N}(W)$, $\Psi : K_1 \to R$, $\Psi(\tau_1) = 1$ for $\tau_1 \in T$, $\hat{W}(J_1, \tau_1) = W(J_1, \tau_1) \Psi(\tau_1)$ for $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$, then $\hat{W} \in \mathfrak{P}(K_1)$ and $(P) \int_{K_1} \hat{W} = (P) \int_K W$.

The following theorem is the converse to Theorem 2,4.

Theorem 2,5. *Let* $U_1 : \mathfrak{K}(K_1) \times K_1 \to R$, $U_2 : \mathfrak{K}(K_2) \times K_1 \times K_2 \to R$, $U = U_1 U_2$. *Let T be the set of such* $\tau_1 \in K_1$ *that* $U_2(\cdot, \tau_1, \cdot) \in \mathfrak{P}(K_2)$. *For* $\tau_1 \in T$ *define* $\phi(\tau_1) = (P) \int_{K_2} U_2(\cdot, \tau_1, \cdot)$, *for* $\tau_1 \in K_1 - T$ *put* $\phi(\tau_1) = 0$ *and define* $W(J_1, \tau_1) = U_1(J_1, \tau_1) \phi(\tau_1)$ *for* $(J_1, \tau_1) \in \mathfrak{K}(K_1) \times K_1$. *Assume that*

$$K_1 - T \in \mathfrak{N}(U_1), \tag{2,21}$$

(2,22) *to every* $\varepsilon \in R^+$ *there exists such a* $\nu : K \to R^+$ *that*

$$|S_{(K_1 - T) \times K_2}(U, A)| \leqq \varepsilon \quad \text{for} \quad A \in \mathscr{A}_{1,2}(\nu),$$

$$W \in \mathfrak{P}(K_1), \tag{2,23}$$

$$U_1 \quad \text{is BVG in} \quad K_1. \tag{2,24}$$

Then $U \in \mathfrak{P}_{1,2}(K)$ *and* $(P_{1,2}) \int_K U = (P) \int_{K_1} W$.

Proof. It is sufficient to prove that to any $\varepsilon \in R^+$ there exists such an $\omega : K \to R^+$ that

$$(2,25) \qquad \left|S(U, A) - (P)\int_{K_1} W\right| \leqq \varepsilon \quad \text{for} \quad A \in \mathscr{A}_{1,2}(\omega).$$

Fix $\varepsilon \in R^+$. By (2,24) there are such $X_r \subset K_1$, $\xi_r : K_1 \to R^+$ and $\varkappa_r \in R^+$ for $r \in N$ that $\bigcup_{r \in N} X_r = K_1$ and

$$(2,26) \qquad \sum_{\tau_1^{(i)} \in X_r} |U_1(J_1^{(i)}, \tau_1^{(i)})| \leqq \varkappa_r$$

holds for $A = \{(J_1^{(i)}, \sigma_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi_r)$. Without loss on generality it may be assumed that the sets X_r are mutually disjoint and it is easy to show that (2,26) holds for any $A \in \mathscr{A}(\xi)$, ξ being defined by $\xi(\tau_1) = \xi_r(\tau_1)$ for $\tau_1 \in X_r$, $r \in N$.

For $\tau_1 \in T$ find $r \in N$ such that $\tau_1 \in X_r$. By the definition of T there exists such a $\vartheta_{\tau_1} : K_2 \to R^+$ that

$$(2,27) \qquad |S(U_2(\cdot, \tau_1, \cdot), A_2) - \phi(\tau_1)| \leqq \varepsilon/(\varkappa_r \,.\, 2^{r+2}) \quad \text{for} \quad A_2 \in \mathscr{A}(\vartheta_{\tau_1}).$$

Find v by (2,22), ε being replaced by $\frac{1}{4}\varepsilon$. By (2,21) and Lemma 2,4 there exists such a $\varrho : K_1 \to R^+$ that

$$(2,28) \qquad \sum_{\tau_1^{(i)} \in K_1 - T} |W(J_1^{(i)}, \tau_1^{(i)})| \leqq \tfrac{1}{4}\varepsilon$$

for $A_1 = \{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k) \in \mathscr{A}(\varrho)$. By (2,23) there exists such a $\eta : K_1 \to$ $\to R^+$ that

$$(2,29) \qquad \left|S(W, A_1) - (P)\int_{K_1} W\right| \leqq \tfrac{1}{4}\varepsilon \quad \text{for} \quad A_1 \in \mathscr{A}(\eta).$$

Put $\omega(\tau) = \min(\vartheta_{\tau_1}(\tau_2), v(\tau), \xi(\tau_1), \varrho(\tau_1), \eta(\tau_1))$ for $\tau = (\tau_1, \tau_2) \in K$. Let $A \in \mathscr{A}_{1,2}(\omega)$. Then – using the same notations as in (2,2) –

$$A_1 = \{(J_1^{(i)}, \tau_1^{(i)}) \mid i = 1, 2, \ldots, k\} \in \mathscr{A}(\xi) \cap \mathscr{A}(\varrho) \cap \mathscr{A}(\eta),$$

$\{(L_2^{(i,j)}, \lambda_2^{(i,j)})|, j = 1, 2, \ldots, l^{(i)}\} \in \mathscr{A}(\vartheta_{\tau_1^{(i)}})$, so that by (2,27), (2,28) and (2,29)

$$\left|S(U, A) - (P)\int_{K_1} W\right| \leqq$$

$$\leqq \left|\sum_{i=1}^{k} U_1(J_1^{(i)}, \tau_1^{(i)}) \sum_{j=1}^{l^{(i)}} U_2(L_2^{(i,j)}, \tau_1^{(i)}, \lambda_2^{(i,j)}) - \sum_{i=1}^{k} U_1(J_1^{(i)}, \tau_1^{(i)})\, \phi(\tau_1)\right| +$$

$$+ \left|\sum_{i=1}^{k} U(J_1^{(i)}, \tau_1^{(i)})\, \phi(\tau_1^{(i)}) - (P)\int_{K_1} W\right| \leqq$$

$$\leqq \sum_{r \in N} \sum_{\tau_1^{(i)} \in T \cap X_r} |U_1(J_1^{(i)}, \tau_1^{(i)})| \, |\sum_{j=1}^{l(i)} U(L_2^{(i,j)}. \tau_1^{(i)}, \lambda_2^{(i,j)}) - \phi(\tau_1^{(i)})| +$$

$$+ |\sum_{\tau_1^{(i)} \in K_1 - T} U_1(J_1^{(i)}, \tau_1^{(i)}) \sum_{j=1}^{l(i)} U_2(L_2^{(i,j)}, \tau_1^{(i)}, \lambda_2^{(i,j)})| +$$

$$+ |\sum_{\tau_1^{(i)} \in K_1 - T} U_1(J_1^{(i)}, \tau_1^{(i)}) \, \phi(\tau_1^{(i)})| + \tfrac{1}{4}\varepsilon \leqq$$

$$\leqq \sum_{r \in N} \sum_{\tau_1^{(i)} \in T \cap X_r} |U_1(J_1^{(i)}, \tau_1^{(i)})| \, . \, \varepsilon/(\varkappa_r \, . \, 2^{r+2}) + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon \leqq \sum_{r \in N} \varepsilon/2^{r+2} + \tfrac{3}{4}\varepsilon = \varepsilon$$

and (2,25) holds, which makes the proof complete.

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