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On Full Uniform Simplification of Even Order Linear Differential Equations with a Parameter

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1. Introduction

We shall be concerned with the system of linear ordinary differential equations with a parameter

$$\varepsilon \frac{dX}{dt} = A(t, \varepsilon)X, \tag{1.1}$$

where the n by n matrix function $A(t, \varepsilon)$ is holomorphic in the domain

$$D(t_0, \varepsilon_0, \theta_0) = \{(t, \varepsilon) \mid |t| \leqq t_0, \quad 0 < |\varepsilon| \leqq \varepsilon_0, \ |\arg \varepsilon| \leqq \theta_0\}$$

and admits the uniform asymptotic expansion

$$A(t, \varepsilon) \sim \sum_{i=0}^{\infty} A_i(t)\varepsilon^i \qquad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{in} \quad |\arg \varepsilon| \leqq \theta_0. \tag{1.2}$$

The coefficients $A_i(t)$ $(i=0, 1,\dots)$ of (1.2) are holomorphic in the closed disk $|t| \leqq t_0$. We here assume that

$$A_0(t) = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ t^q & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

q being a positive integer, which implies that the origin $t=0$ is a turning point of (1.1).

In order to investigate asymptotic behaviors of solutions of (1.1) in a full neighborhood of the turning point $t=0$, we usually try to find a matrix $Q(t, \varepsilon)$, which is holomorphic in $D(t_1, \varepsilon_1, \theta_1,)$ $(0 < t_1 \leqq t_0,\ 0 < \varepsilon_1 \leqq \varepsilon_0,\ 0 < \theta_1 \leqq \theta_0)$ and admits an asymptotic expansion of the form

$$Q(t, \varepsilon) \sim \sum_{i=0}^{\infty} P_i(t)\varepsilon^i \qquad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{in} \quad |\arg \varepsilon| \leqq \theta_1, \tag{1.3}$$

where the coefficients $P_i(t)$ $(i=0, 1,\dots)$ are holomorphic in $|t| \leqq t_1$, such that the transformation $Y=Q(t,\varepsilon)X$ reduces (1.1) to a system of linear differential equations for Y, whose asymptotic behaviors can then be easily analyzed in $|t|\leqq t_1$. In this analysis, however, we encounter a troublesome fact that (1.3) does not hold uniformly in the full neighborhood of the turning point $|t|\leqq t_1$.

To our knowledge, such a problem to seek a simplifying transformation $Q(t,\varepsilon)$ admitting the uniform asymptotic expansion in a full neighborhood of a turning point has been completely solved only for second order linear differential equations (see [7], [2], [4]). W. Wasow [7] first succeeded in solving the problem in the case when $n=2$ and $q=1$ by an elegant method. The purpose of this paper is to show that an extended Airy function of the first kind defined by M. Kohno [1] plays an important role in solving clearly the full uniform simplification problem for (1.1), where n is even, i.e., $n=2N$ $(N\geqq 1)$ and $q=1$, by only following W. Wasow's method.

2. Formal reduction

It is well known that when $q=1$, (1.1) can be reduced to the system of linear differential equations

$$\varepsilon\frac{dY}{dt} = A_0(t)Y \tag{2.1}$$

by a formal transformation

$$Y = P(t,\varepsilon)X, \quad P(t,\varepsilon) = \sum_{i=0}^{\infty} P_i(t)\varepsilon^i,$$

where the coefficient matrices $P_i(t)\,(i=0, 1,\ldots)$ are holomorphic for $|t|\leqq t_0$ and $P_0(0)=I$, I denoting the identity matrix. This result was proved by K. Okubo [3] and independently by W. Wasow [5].

We here restate the method of reduction in [3] to obtain a slightly modified result in the case when q is an arbitrary positive integer.

Let $\mu(t,\varepsilon)$ be a row vector and put

$$\left.\begin{aligned} p_1(t,\varepsilon) &= \mu(t,\varepsilon), \\ p_k(t,\varepsilon) &= \varepsilon\frac{d}{dt}p_{k-1}(t,\varepsilon) + p_{k-1}(t,\varepsilon)A(t,\varepsilon) \\ &\qquad\qquad (k = 2, 3,\ldots, n+1). \end{aligned}\right\} \tag{2.2}$$

Then it is easily seen that the transformation

$$Y = \begin{pmatrix} p_1(t,\varepsilon) \\ p_2(t,\varepsilon) \\ \vdots \\ p_n(t,\varepsilon) \end{pmatrix} X$$

reduces (1.1) to the system of linear differential equations

$$\varepsilon \frac{dY}{dt} = B(t, \varepsilon)Y,$$

where $B(t, \varepsilon)$ is a companion matrix with the form

$$B(t, \varepsilon) = \begin{pmatrix} 0 & 1 & & & 0 \\ \vdots & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & 0 & 1 \\ b_1(t, \varepsilon) & b_2(t, \varepsilon) & \cdots & & b_n(t, \varepsilon) \end{pmatrix}$$

under the condition that

$$p_{n+1}(t, \varepsilon) = \sum_{k=1}^{n} b_k(t, \varepsilon) p_k(t, \varepsilon). \tag{2.3}$$

We shall now attempt to determine the row vector $\mu(t, \varepsilon)$ so that the functions $b_k(t, \varepsilon)$ $(k=1, 2,\ldots, n)$ are as simple as possible. The substitution of the power series of (1.2) into (2.2) leads to

$$p_{k+1} = \mu A_0^k + \varepsilon(k\mu' A_0^{k-1} + \mu\psi_k(A_0; A_1)) + \varepsilon^2 f_k(t, \varepsilon, \mu)$$
$$(k = 0, 1,\ldots, n),$$

where

$$\psi_k(A_0; A_1) = \sum_{j=1}^{k} A_0^{k-j} A_1 A_0^{j-1} + \sum_{j=1}^{k-1} (A_0^j)' A_0^{k-1-j}$$

and $f_k(t, \varepsilon, \mu)$ is a linear form in $\mu, \mu',\ldots, \mu^{(k)}$. Considering this and putting

$$b_1(t, \varepsilon) = t^q + \varepsilon^2 \hat{b}_1(t, \varepsilon),$$
$$b_k(t, \varepsilon) = \varepsilon^2 \hat{b}_k(t, \varepsilon) \qquad (k = 2, 3,\ldots, n-1),$$
$$b_n(t, \varepsilon) \equiv 0,$$

we can write the condition (2.3) as follows:

$$nt^q\mu' + \mu\psi_n(A_0; A_1)A_0 + \varepsilon f_n(t, \varepsilon, \mu)A_0$$
$$= \varepsilon \sum_{k=1}^{n-1} \hat{b}_k(t, \varepsilon)\{\mu A_0^{k-1} + \varepsilon[(k-1)\mu' A_0^{k-2} + \mu\psi_{k-1}(A_0; A_1)]$$
$$+ \varepsilon^2 f_{k-1}(t, \varepsilon, \mu)\}A_0$$
$$= \varepsilon \sum_{k=1}^{n-1} \hat{b}_k(t, \varepsilon)\mu A_0^k + \varepsilon^2 h_n(t, \varepsilon, \mu, \hat{b})A_0, \tag{2.4}$$

where $h_n(t, \varepsilon, \mu, \hat{b})$ is a bilinear form in $\mu^{(k-1)}$ and $\hat{b}_k$ $(k=1, 2,\dots, n-1)$. We here substitute the formal series

$$\mu(t, \varepsilon) = \sum_{i=0}^{\infty} \mu_i(t)\varepsilon^i,$$

$$\hat{b}_k(t, \varepsilon) = \sum_{i=0}^{\infty} b_{k,i}(t)\varepsilon^i \qquad (k = 1, 2,\dots, n-1)$$

into (2.4) and equate the coefficients of like powers of ε in both sides, thereby obtaining the following systems of linear differential equations

$$nt^q\mu_0' + \mu_0\psi_n(A_0; A_1)A_0 = 0, \tag{2.5}$$

$$nt^q\mu_i' + \mu_i\psi_n(A_0; A_1)A_0$$
$$= \sum_{k=1}^{n-1}\{b_{k,i-1}\mu_0 + b_{k,i-2}\mu_1 + \cdots + b_{k,0}\mu_{i-1}\}A_0^k + G_{i-1}(t, \mu, \hat{b})A_0$$
$$(i = 1, 2,\dots), \tag{2.6}$$

where $G_{i-1}(t, \mu, \hat{b})$ includes $\mu_0(t), \mu_1(t),\dots, \mu_{i-1}(t)$ and their derivatives linearly, and $b_{k,j}(t)$ $(j=0, 1,\dots, i-2)$ linearly. Since

$$A_0^j = \begin{pmatrix} 0 & I_{n-j} \\ t^qI_j & 0 \end{pmatrix}, \quad A_0^{n-j}A_1A_0^j = \begin{pmatrix} t^qA_{22}^{(j)} & A_{21}^{(j)} \\ t^{2q}A_{12}^{(j)} & t^qA_{11}^{(j)} \end{pmatrix},$$

where I_j denotes the j by j identity matrix and we have put

$$A_1(t) = (a_{ik}(t);\ i, k = 1, 2,\dots, n) = \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{pmatrix} \qquad (j = 1, 2,\dots, n),$$

$A_{11}^{(j)}$, $A_{12}^{(j)}$, $A_{21}^{(j)}$ and $A_{22}^{(j)}$ being matrices of type $(n-j, n-j)$, $(n-j, j)$, $(j, n-j)$ and (j, j), respectively, we see that

$$\psi_n(A_0; A_1)A_0 = \sum_{j=1}^{n} A_0^{n-j}A_1A_0^{j-1} + \sum_{j=1}^{n-1}(A_0^j)'A_0^{n-j}$$
$$= \sum_{j=1}^{n}\begin{pmatrix} t^qA_{22}^{(j)} & A_{21}^{(j)} \\ t^{2q}A_{12}^{(j)} & t^qA_{11}^{(j)} \end{pmatrix} + \sum_{j=1}^{n-1}\begin{pmatrix} 0 & 0 \\ 0 & qt^{q-1}I_j \end{pmatrix}$$

and hence (2.5) can be rewritten in the form

$$t\mu_0' = -\frac{1}{n}\mu_0\left\{\sum_{j=1}^{n}\begin{pmatrix} tA_{22}^{(j)}(t) & t^{-q+1}A_{21}^{(j)}(t) \\ t^{q+1}A_{12}^{(j)}(t) & tA_{11}^{(j)}(t) \end{pmatrix} + \sum_{j=1}^{n-1}\begin{pmatrix} 0 & 0 \\ 0 & qI_j \end{pmatrix}\right\}$$
$$= \mu_0\Phi(t).$$

Hence, if it is assumed that

$$a_{ik}(t) = O(t^{q-1}) \qquad (k < i;\ i, k = 1, 2, \dots, n), \tag{2.7}$$

then the matrix $\Phi(t)$ is holomorphic at $t=0$ and $\Phi(0)$ is of the form

$$\Phi(0) = \begin{pmatrix} 0 & & & & * \\ & -\frac{q}{n} & & & \\ & & -\frac{2q}{n} & & \\ & & & \ddots & \\ 0 & & & & -\frac{(n-1)q}{n} \end{pmatrix}.$$

This fact implies that the origin $t=0$ is a regular singular point of (2.5). We can then find a holomorphic solution, which corresponds to the characteristic exponent 0, of the form

$$\mu_0(t) = \sum_{m=0}^{\infty} \alpha_0(m) t^m,$$

where

$$\alpha_0(0) = (1, 0, \dots, 0). \tag{2.8}$$

Next we shall seek a holomorphic solution $\mu_i(t)$ of (2.6). For that purpose, we have to investigate the nonhomogeneous term of (2.6). We put

$$\left.\begin{aligned} &\mu_j(t) = (\mu_j^{(1)}(t), \mu_j^{(2)}(t), \dots, \mu_j^{(n)}(t)), \\ &\gamma_{k,j}^{(\nu)}(t) = b_{k,j}(t)\mu_0^{(\nu)}(t) + b_{k,j-1}(t)\mu_1^{(\nu)}(t) + \cdots + b_{k,0}(t)\mu_j^{(\nu)}(t), \\ &\Gamma_j^{(\nu)}(t) = \gamma_{\nu-1,j}^{(1)}(t) + \gamma_{\nu-2,j}^{(2)}(t) + \cdots + \gamma_{1,j}^{(\nu-1)}(t) + t^q \sum_{k=\nu}^{n-1} \gamma_{\nu,j}^{(n-k+\nu)}(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\nu = 1, 2, \dots, n), \end{aligned}\right\} \tag{2.9}$$

and

$$G_j(t, \mu, \hat{b}) = (g_j^{(1)}(t, \mu, \hat{b}), g_j^{(2)}(t, \mu, \hat{b}), \dots, g_j^{(n)}(t, \mu, \hat{b})).$$

Then the nonhomogeneous term of (2.6) can be written as

$$\begin{aligned} F_{i-1}(t) &= (\Gamma_{i-1}^{(1)} + t^q g_{i-1}^{(n)}, \Gamma_{i-1}^{(2)} + g_{i-1}^{(1)}, \dots, \Gamma_{i-1}^{(n)} + g_{i-1}^{(n-1)}) \\ &= (t^q f_{i-1}^{(1)}(t), f_{i-1}^{(2)}(t), \dots, f_{i-1}^{(n)}(t)). \end{aligned}$$

According to the theory of a regular singularity, if

$$f_{i-1}^{(\nu)}(t) = O(t^{q-1}) \qquad (\nu = 2, 3,\dots, n) \tag{2.10}$$

are satisfied, together with the assumption (2.7), then we can find a solution $\mu_i(t)$ of (2.6) which is holomorphic at $t=0$. In order that the conditions (2.10) are satisfied, we may only choose $b_{k,i-1}(t)$ $(k=1, 2,\dots, n-1)$ as polynomials of degree at most $q-2$, i.e.,

$$b_{k,i-1}(t) = \sum_{m=0}^{q-2} \beta_{k,i-1}(m)t^m \qquad (k = 1, 2,\dots, n-1). \tag{2.11}$$

In fact, we assume that we have already obtained holomorphic solutions $\mu_1(t)$, $\mu_2(t),\dots, \mu_{i-1}(t)$, determining $b_{k,0}(t)$, $b_{k,1}(t),\dots, b_{k,i-2}(t)$ $(k=1, 2,\dots, n-1)$ as such polynomials as (2.11). Then $g_{i-1}^{(\nu)}(t, \mu, \hat{b})$ $(\nu=1, 2,\dots, n)$ are known functions which are holomorphic at $t=0$. On the other hand, from (2.9) we have

$$\begin{aligned}
\Gamma_{i-1}^{(2)}(t) &= \gamma_{1,i-1}^{(1)} + O(t^q)\\
&= b_{1,i-1}(t)\mu_0^{(1)}(t) + b_{1,i-2}(t)\mu_1^{(1)}(t) + \cdots + b_{1,0}(t)\mu_{i-1}^{(1)}(t) + O(t^q)\\
&= \left(\sum_{m=0}^{q-2} \beta_{1,i-1}(m)t^m\right)\left(1 + \sum_{m=1}^{\infty} \alpha_0^{(1)}(m)t^m\right)\\
&\quad + b_{1,i-2}(t)\mu_1^{(1)}(t) + \cdots + b_{1,0}(t)\mu_{i-1}^{(1)}(t) + O(t^q)\\
&= \beta_{1,i-1}(0) + \{\beta_{1,i-1}(1) + \alpha_0^{(1)}(1)\beta_{1,i-1}(0)\}t + \cdots\\
&\quad + \{\beta_{1,i-1}(q-2) + \alpha_0^{(1)}(1)\beta_{1,i-1}(q-3) + \cdots\\
&\qquad\qquad + \alpha_0^{(1)}(q-2)\beta_{1,i-1}(0)\}t^{q-2}\\
&\quad + b_{1,i-2}(t)\mu_1^{(1)}(t) + \cdots + b_{1,0}(t)\mu_{i-1}^{(1)}(t) + O(t^{q-1}).
\end{aligned}$$

From this it is easily seen that we can determine the constants $\beta_{1,i-1}(m)$ $(m=0, 1,\dots, q-2)$ successively so that $f_{i-1}^{(2)}(t)=O(t^{q-1})$. Since $b_{1,i-1}(t)$ has been determined, $\gamma_{1,i-1}^{(2)}$ becomes a known holomorphic function. We can then determine $b_{2,i-1}(t)$ as a polynomial of the form (2.11) so that $f_{i-1}^{(3)}(t)=O(t^{q-1})$. Continuing this process, we at last obtain (2.10) by means of determining $b_{k,i-1}(t)$ $(k=1, 2,\dots, n-1)$ as appropriate polynomials of the form (2.11). We have thus derived a desired formal reduction. Lastly, we remark that from (2.2) and (2.8) there holds

$$\begin{pmatrix} p_1(0, 0)\\ p_2(0, 0)\\ \vdots\\ p_n(0, 0) \end{pmatrix} = I.$$

We summarize the above results in the following

THEOREM 1. *Let us denote the matrix* $A_1(t)=(a_{ik}(t);\ i,\ k=1, 2,\dots, n)$. *Under the assumption that* $a_{ik}(t)=O(t^{q-1})\ (k<i;\ i,\ k=1, 2,\dots, n)$, *we can find a formal transformation*

$$Y = P(t, \varepsilon)X = \left(\sum_{i=0}^{\infty} P_i(t)\varepsilon^i\right)X,$$

where the coefficient matrices $P_i(t)\,(i=0, 1,\dots)$ *are holomorphic for* $|t|\leqq t_0$ *and in particular,* $P_0(0)=1$, *which reduces* $\varepsilon\frac{dX}{dt}=A(t,\varepsilon)X$ *to* $\varepsilon\frac{dY}{dt}=B(t,\varepsilon)Y$ *with*

$$B(t, \varepsilon) = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ b_1(t, \varepsilon) & b_2(t, \varepsilon) & \cdots & b_{n-1}(t, \varepsilon) & 0 \end{pmatrix},$$

where

$$b_1(t, \varepsilon) = t^q + \varepsilon^2 \sum_{m=0}^{q-2} \beta_1^m(\varepsilon)t^m,$$

$$b_k(t, \varepsilon) = \varepsilon^2 \sum_{m=0}^{q-2} \beta_k^m(\varepsilon)t^m \qquad (k = 2, 3,\dots, n-1),$$

$$\beta_k^m(\varepsilon) = \sum_{i=0}^{\infty} \beta_k^m(i)\varepsilon^i \qquad (k = 1, 2,\dots, n-1;\ m = 0, 1,\dots, q-2).$$

As a matter of course, if $q=1$, then $B(t,\varepsilon)=A_0(t)$. Considering the first component of the column vector Y and putting $z=t\varepsilon^{-\frac{n}{n+q}}$, we easily see that the reduced system of linear differential equations is closely related to a single linear differential equation of the form

$$\frac{d^n y}{dz^n} = \left(\sum_{m=0}^{q-2} a_{n-2}(m)z^m\right)\frac{d^{n-2}y}{dz^{n-2}} + \cdots + \left(\sum_{m=0}^{q-2} a_1(m)z^m\right)\frac{dy}{dz}$$
$$+ \left(z^q + \sum_{m=0}^{q-2} a_0(m)z^m\right)y\,.$$

3. Analytic reduction

Hereafter we shall consider the system of linear differential equations (1.1), where $n=2N\ (N\geqq 1)$ and $q=1$. Although an analytic reduction theorem for such a system of linear differential equations has been established in the paper [6], in this section we give a simple proof quite similar to that in the case $n=2$ [7] by using an extended Airy function of the first kind.

To begin with, we shall explain the extended Airy function of the first kind $Ai(z)$ (see [1]), which is a particular solution of the linear differential equation

$$\frac{d^n y}{dz^n} - zy = 0 \qquad (n = 2N) \tag{3.1}$$

and is defined by

$$Ai(z) = \gamma \sum_{j=1}^{n} (-1)^j (n+1)^{\frac{n}{n+1}(j-1)} \eta_{n-j} y_j(z),$$

where

$$y_j(z) = z^{n-j} \sum_{m=0}^{\infty} \left(\prod_{k=1}^{n} \Gamma\left(m + 1 + \frac{k-j}{n+1}\right)\right)^{-1} (z^{n+1}(n+1)^{-n})^m$$

$$(j = 1, 2,\dots, n)$$

and the real constants γ and η_j $(j=1, 2,\dots, n)$ are given by

$$\gamma = \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} n^{\frac{1}{2}} (n+1)^{\frac{(2n+1)(1-n)}{2(n+1)}} \left(\prod_{k=1}^{n-1} \sin\left(\frac{\pi}{n+1}k\right)\right)^{-1},$$

$$\eta_0 x^{n-1} + \eta_1 x^{n-2} + \cdots + \eta_{n-2} x + \eta_{n-1} = (x+1) \prod_{k=1}^{N-1} \left(x^2 + 2x \cos\left(\frac{\pi}{n+1}k\right) + 1\right).$$

$Ai(z)$ has several properties entirely analogous to the classical Airy function of the first kind. We state its properties needed later in the following

LEMMA 1. (i) *Let k_i $(i=1, 2,\dots, n)$ be mutually distinct integers modulo $(n+1)$. Then $Ai(\omega^{k_i} z)$ $(i=1, 2,\dots, n)$ form a fundamental set of solutions of* (3.1) *and the Wronskian of them is calculated as follows*:

$$W[Ai(\omega^{k_1} z), Ai(\omega^{k_2} z),\dots, Ai(\omega^{k_n} z) : z]$$
$$= c_1 c_2 \cdots c_n \prod_{1 \leq i < j \leq n} \left[\sin\left(\frac{k_j - k_i}{n+1}\pi\right) \exp\left\{\left(\frac{k_j + k_i}{n+1} + \frac{1}{2}\right)\pi i\right\}\right],$$

where $\omega = \exp\left(\frac{2\pi}{n+1} i\right)$ *and* $c_j = \gamma(-1)^j (n+1)^{\frac{n}{n+1}(n-j)} \eta_{n-j} = Ai^{(n-j)}(0)$.

(ii) *There holds the connection formula*

$$Ai(z) + \sum_{k=1}^{n} \beta_k Ai(\omega^{-k} z) = 0,$$

where

$$\beta_k = -\omega^{-k}\prod_{j=1}^{k-1}\left|\frac{\sin\left(\frac{j-k}{n+1}\pi\right)}{\sin\left(\frac{j}{n+1}\pi\right)}\right| \qquad (k = 1, 2, \ldots, n).$$

(iii) *$Ai(z)$ has the following asymptotic behaviors:*

$$Ai(z) \sim \bar{d}_1 \exp\left(-\frac{n}{n+1}z^{\frac{n+1}{n}}\omega_n\right)z^{-\frac{n-1}{2n}}\left\{1 + O(z^{-\frac{n-1}{n}})\right\} \qquad (-2\pi \leqq \arg z < -\pi),$$

$$\sim \exp\left(-\frac{n}{n+1}z^{\frac{n+1}{n}}\right)z^{-\frac{n-1}{2n}}\left\{1 + O(z^{-\frac{n+1}{n}})\right\} \qquad (-\pi \leqq \arg z < \pi),$$

$$\sim d_1 \exp\left(-\frac{n}{n+1}z^{\frac{n+1}{n}}\omega_n^{-1}\right)z^{-\frac{n-1}{2n}}\left\{1 + O(z^{-\frac{n+1}{n}})\right\} \qquad (\pi \leqq \arg z < 2\pi),$$

where $\omega_n = \exp\left(\frac{2\pi}{n}i\right)$, $d_1 = \exp\left(\frac{n-1}{n}\pi i\right)$ *and* $\bar{d}_1$ *is the complex conjugate of* d_1.

We here put

$$\mathscr{A}_\nu(z) = \begin{pmatrix} Ai(\omega^\nu z) \\ \omega^\nu Ai(\omega^\nu z) \\ \vdots \\ \omega^{\nu(n-1)}Ai^{(n-1)}(\omega^\nu z) \end{pmatrix} \qquad (\nu = 0, \pm 1, \ldots) \tag{3.2}$$

and

$$U_k(z) = [\mathscr{A}_{-(k-1)}(z), \mathscr{A}_{-(k-2)}(z), \ldots, \mathscr{A}_0(z), \ldots, \mathscr{A}_{n-k-1}(z), \mathscr{A}_{n-k}(z)] \qquad (k = 1, 2, \ldots, n+1). \tag{3.3}$$

From Lemma 1 it follows that for all ν ($\nu = -k+1, -k+2, \ldots, n-k$)

$$\omega^{\nu j}Ai^{(j)}(\omega^\nu z) \sim \exp\left(-\frac{n}{n+1}z^{\frac{n+1}{n}}\omega_n^\nu\right)z^{\frac{j}{n}-\frac{n-1}{2n}}\left[(-1)^j\omega^{\nu\{(\frac{n+1}{n})j-\frac{n-1}{2n}\}} + O(z^{-\frac{1}{n}})\right] \qquad (j = 0, 1, \ldots, n-1)$$

as $z \to \infty$ in the sector

$$-\pi + \frac{2\pi}{n+1}(k-1) + \delta \leqq \arg z < -\pi + \frac{2\pi}{n+1}(k+1) - \delta, \tag{3.4}$$

δ being an arbitrarily small positive number. Combining this with the definition (3.2–3), we immediately obtain

LEMMA 2. *If we put*

$$U_k(z) = (z)^{\sigma(z)\Omega_0}\hat{U}_k(z)\exp(p(z)\Omega_1^k) \qquad (k = 1, 2,\dots, n+1),$$

where $p(z)=\frac{n}{n+1}z^{\frac{n+1}{n}}$ *such that* $p(z)>0$ *for* $z>0$,

$$\sigma(z) = \begin{cases} 0 & \text{for } |z| < z_0, \\ 1 & \text{for } |z| \geqq z_0, \end{cases}$$

z_0 *being a sufficiently large positive number,*

$$\Omega_0 = \begin{pmatrix} -\frac{n-1}{2n} & & & 0 \\ & \frac{1}{n}-\frac{n-1}{2n} & & \\ & & \ddots & \\ 0 & & & \frac{n-1}{n}-\frac{n-1}{2n} \end{pmatrix} \quad \text{and} \quad \Omega_1^k = \begin{pmatrix} -\omega_n^{-k+1} & & & 0 \\ & -\omega_n^{-k+2} & & \\ & & \ddots & \\ 0 & & & -\omega_n^{n-k} \end{pmatrix}$$

then $\hat{U}_k(z)$ *as well as its inverse are uniformly bounded in the sector* (3.4).

The uniform boundedness of $\hat{U}_k(z)^{-1}$ is seen form Lemma 1 (i) and the fact that $\hat{U}_k(\infty)=((-1)^j\omega^{\nu\{(\frac{n+1}{n})j-\frac{n-1}{2n}\}};\ j=0, 1,\dots, n-1,\ \nu=-k+1, -k+2,\dots, n-k)$ is nonsingular.

All preparations having done, we now return to the problem of an analytic reduction. In the preceding section we have proved that there exists a power series $P(t, \varepsilon)$ which formally satisfies the system of linear differential equations

$$\varepsilon\frac{dP}{dt} = A_0(t)P - PA(t, \varepsilon). \tag{3.5}$$

We shall now show that we can find an actual solution of (3.5) which is holomorphic in a certain sectorial domain and there admits the formal power series $P(t, \varepsilon)$ as its asymptotic expansion as $\varepsilon\to 0$. For that purpose, taking account of Borel-Ritt's theorem which guarantees the existence of a holomorphic function $P^*(t, \varepsilon)$ such that $P^*(t, \varepsilon)\sim P(t, \varepsilon)$ as $\varepsilon\to 0$, we first put $P=W+P^*(t, \varepsilon)$ and change (3.5) into

$$\varepsilon\frac{dW}{dt} = A_0(t)W - WA_0(t) + G(t, \varepsilon, W), \tag{3.6}$$

where

$$G(t, \varepsilon, W) = W(A_0(t) - A(t, \varepsilon)) + F(t, \varepsilon),$$

$$F(t, \varepsilon) = -\varepsilon \frac{dP^*}{dt} + A_0(t)P^* - P^*A(t, \varepsilon).$$

We then have only to seek a holomorphic solution $W(t, \varepsilon)$ of (3.6) such that $W(t, \varepsilon) \sim 0$ as $\varepsilon \to 0$.

Putting $z = t\varepsilon^{-\frac{n+1}{n}}$ in (3.3), we can easily verify that each

$$U_k(t, \varepsilon) = \begin{pmatrix} 1 & & & & 0 \\ & \varepsilon^{\frac{1}{n+1}} & & & \\ & & \varepsilon^{\frac{2}{n+2}} & & \\ & & & \ddots & \\ 0 & & & & \varepsilon^{\frac{n-1}{n+1}} \end{pmatrix} U_k(t\varepsilon^{-\frac{n}{n+1}}) = \mathscr{E} U_k(t\varepsilon^{-\frac{n}{n+1}}) \tag{3.7}$$

$$(k = 1, 2, \dots, n+1)$$

forms a fundamental matrix of solutions of (2.1), i.e.,

$$\varepsilon \frac{dU}{dt} = A_0(t)U.$$

Let

$$\Pi = \begin{pmatrix} 0 & & & & & -1 \\ & & & & 1 & \\ & & & \cdot^{\cdot^{\cdot}} & & \\ & & -1 & & & \\ & 1 & & & & \\ -1 & & & & & \\ 1 & & & & 0 & \end{pmatrix}$$

with $\det \Pi = (-1)^N$, and then $\Pi^{-1} = \Pi_* = -\Pi$, where here and below the subscript $*$ denotes the transposition of a matrix. A direct calculation shows that each $V_k(t, \varepsilon) = \Pi U_k(t, \varepsilon)$ $(k = 1, 2, \dots, n+1)$ forms a fundamental matrix of solutions of the system of linear differential equations

$$\varepsilon \frac{dV}{dt} = -A_0(t)_* V.$$

Then from [7; Lemma 30.1], we can immediately convert (3.6) into an integral equation of the Volterra type as follows:

$$W(t, \varepsilon)$$
$$= U_k(t, \varepsilon) \left[\int_{I(t)} U_k(\tau, \varepsilon)^{-1} \varepsilon^{-1} G(\tau, \varepsilon, W(\tau, \varepsilon)) (V_k(\tau, \varepsilon)_*)^{-1} d\tau \right] V_k(t, \varepsilon)_*$$
$$(k = 1, 2, \dots, n+1), \tag{3.8}$$

where $I(t)$ denotes a set of n^2 paths of integration ending at t, which correspond to n^2 entries in the integrand matrix. Moreover, putting

$$\tilde{U}_k(t,\varepsilon) = U_k(t,\varepsilon)\exp(-p(z)\Omega_1^k),$$

$$\tilde{V}_k(t,\varepsilon) = V_k(t,\varepsilon)\exp(-p(z)\Omega_1^k) \qquad (z = t\varepsilon^{-\frac{n}{n+1}})$$

and

$$\mathscr{W}_k(t,\varepsilon) = \tilde{U}_k(t,\varepsilon)W(t,\varepsilon)\tilde{V}_k(t,\varepsilon)_*^{-1} \qquad (k = 1, 2,\dots, n+1), \tag{3.9}$$

we rewrite (3.8) in the form

$$\begin{aligned}\mathscr{W}(t,\varepsilon) &= \int_{I(t)} \exp\{(p(z)-p(\xi))\Omega_1^k\}\mathscr{W}(\tau,\varepsilon)M_k(\tau,\varepsilon)\\ &\qquad \times \exp\{(p(z)-p(\xi))\Omega_1^k\}d\tau + H_k(t,\varepsilon)\\ &= L_k[\mathscr{W}] + H_k(t,\varepsilon) \qquad (\xi = \tau\varepsilon^{-\frac{n}{n+1}}),\end{aligned} \tag{3.10}$$

where

$$M_k(t,\varepsilon) = \tilde{V}_k(t,\varepsilon)_*\varepsilon^{-1}(A_0(t) - A(t,\varepsilon))\tilde{V}_k(t,\varepsilon)_*^{-1},$$

$$\begin{aligned}H_k(t,\varepsilon) &= \int_{I(t)} \exp\{(p(z)-p(\xi))\Omega_1^k\}\tilde{U}_k(\tau,\varepsilon)^{-1}\varepsilon^{-1}F(\tau,\varepsilon)\tilde{V}_k(\tau,\varepsilon)_*^{-1}\\ &\qquad \times \exp\{(p(z)-p(\xi))\Omega_1^k\}d\tau.\end{aligned}$$

Since in the domain $D(t_0, \varepsilon_0, \theta_0)$ the norms of the matrices $\mathscr{E}z^{\Omega_0}$ and $\mathscr{E}^{-1}z^{-\Omega_0}$ are estimated as follows:

$$\|\mathscr{E}z^{\Omega_0}\| \leqq \max\{\max_{0\leqq j\leqq N-1}(\varepsilon^{\frac{j}{n+1}}z_0^{\frac{2j-(n-1)}{2n}}),\ \max_{N\leqq j\leqq n-1}(\varepsilon^{\frac{n-1}{2(n+1)}}t_0^{\frac{2j-(n-1)}{2n}})\},$$

$$\|\mathscr{E}^{-1}z^{-\Omega_0}\| \leqq \max\{\max_{0\leqq j\leqq N-1}(\varepsilon^{-\frac{n-1}{n+1}}t_0^{\frac{-2j+(n-1)}{2n}}),\ \max_{N\leqq j\leqq n-1}(\varepsilon^{-\frac{j}{n+1}}z_0^{\frac{-2j+(n-1)}{2n}})\},$$

we observe, taking account of Lemma 2, that each pair $\tilde{U}_k(t,\varepsilon)$ and $\tilde{V}_k(t,\varepsilon)$ $(k=1, 2,\dots, n+1)$ as well as their inverses are $O(\varepsilon^{-\frac{n-1}{n+1}})$ uniformly for $|t|\leqq t_0$ in the sector

$$-\pi + \frac{2\pi}{n+1}(k-1) + \delta \leqq \arg(t\varepsilon^{-\frac{n}{n+1}}) < -\pi + \frac{2\pi}{n+1}(k+1) - \delta$$

$$(k = 1, 2,\dots, n+1). \tag{3.11}$$

This implies from (3.9) that $W(t,\varepsilon)\sim 0$ whenever $\mathscr{W}(t,\varepsilon)\sim 0$. We therefore have only to seek a solution of (3.10) which is holomorphic in a sectorial domain defined by (3.11) and $|t|\leqq t_0$ and there $\mathscr{W}(t,\varepsilon)\sim 0$ as $\varepsilon\to 0$ in $|\arg\varepsilon|\leqq\theta_0$.

For each k ($k=1, 2,\dots, n+1$), in the sectorial domain stated above we have

$$\|M_k(t,\varepsilon)\| \leqq C\varepsilon^{-\frac{n-1}{n+1}(1-\sigma(z))}|t|^{-\left(\frac{n-1}{n}\right)\sigma(z)},$$

where C is a constant independent of ε. We now choose the set of paths of integration $I(t)$ lying in that sectorial domain so that along $I(t)$ the exponential factors in the integral (3.10) are bounded as $\varepsilon\to 0$. Then we can prove the following

LEMMA 3. *Let* $S_k(t_1,\varepsilon_1,\theta_1)$ ($k=1, 2,\dots, n+1$) *be the intersection of* (3.11) *and* $D(t_1,\varepsilon_1,\theta_1)$, *where* $0<t_1\leqq t_0$, $0<\varepsilon_1\leqq\varepsilon_0$ *and* $0<\theta_1\leqq\theta_0$, t_1 *and* ε_1 *being appropriately small. If* $(t,\varepsilon)\in S_k(t_1,\varepsilon_1,\theta_1)$ ($k=1, 2,\dots, n+1$), *then we have*

$$H_k(t,\varepsilon)\sim 0 \qquad \text{as} \quad \varepsilon\longrightarrow 0$$

and

$$\|L_k[\mathscr{W}]\| \leqq K\sup\|\mathscr{W}\| \qquad (0<K<1)$$
$$(k=1, 2,\dots, n+1),$$

where K is a constant and supremum is taken in the sectorial domain defined by (3.11) *and* $|t|\leqq t_1$.

The proof of Lemma 3 is also quite similar to that in the case $n=2$ (see [7; Lemma 30.6]). From Lemma 3 the usual method of successive approximation leads to the required result.

We have thus obtained the following theorem of analytic reductions.

THEOREM 2. *In each* $S_k(t_1,\varepsilon_1,\theta_1)$ ($k=1, 2,\dots, n+1$) *there exists a holomorphic matrix* $P_k(t,\varepsilon)$ ($k=1, 2,\dots, n+1$), *which admits the uniform asymptotic expansion* $P_k(t,\varepsilon)\sim P(t,\varepsilon)$ *as* $\varepsilon\to 0$, *such that the linear transformation* $Y=P_k(t,\varepsilon)X$ *changes* $\varepsilon\frac{dX}{dt}=A(t,\varepsilon)X$ *into* $\varepsilon\frac{dX}{dt}=A_0(t)Y$.

Lastly we remark that $n+1$ sectorial domains $S_1(t_1,\varepsilon_1,\theta_1)$, $S_2(t_1,\varepsilon_1,\theta_1),\dots$, $S_{n+1}(t_1,\varepsilon_1,\theta_1)$ cover the full neighborhood of the turning point $|t|\leqq t_1$.

4. Uniform simplification in a full neighborhood of the turning point

This section deals with the construction of a matrix $Q(t,\varepsilon)$ such that $Y=Q(t,\varepsilon)X$ reduces (1.1) to (2.1) and the uniform asymptotic expansion $Q(t,\varepsilon)\sim P(t,\varepsilon)$ as $\varepsilon\to 0$ holds in a full neighborhood $|t|\leqq t_1$. For such a construction we only follow W. Wasow's idea.

In this section, for simplicity, we assume that $\arg\varepsilon=0$, which is not an essential restriction. From Lemma 1 it is easily seen that

$$y_j(t,\varepsilon)=\beta_j\begin{pmatrix}1 & & & 0\\ & \varepsilon^{\frac{1}{n+1}} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{\frac{n-1}{n+1}}\end{pmatrix}\begin{pmatrix}Ai(\omega^{-j}z)\\ \omega^{-j}A_i'(\omega^{-j}z)\\ \vdots\\ \omega^{-j(n-1)}Ai^{(n-1)}(\omega^{-j}z)\end{pmatrix} \tag{4.1}$$

$$(z=t\varepsilon^{-\frac{n}{n+1}},\ \beta_0=1;\ j=0,1,\dots,n)$$

are solutions of (2.1) and they satisfy the connection formula

$$y_0(t,\varepsilon)+y_1(t,\varepsilon)+\cdots+y_n(t,\varepsilon)=0. \tag{4.2}$$

Since $P_k(0,\varepsilon)\to I$ as $\varepsilon\to 0$, $P_k(t,\varepsilon)^{-1}$ $(k=1,2,\dots,n+1)$ are well-defined and holomorphic in $S_k(t_1,\varepsilon_1,0)$ $(k=1,2,\dots,n+1)$ for sufficiently small t_1 and ε_1. Theorem 2 implies that each $P_k(t,\varepsilon)^{-1}y_j(t,\varepsilon)$ $(j=0,1,\dots,n;\ k=1,2,\dots,n+1)$ is a holomorphic and asymptotically known solution of (1.1) in $S_k(t_1,\varepsilon_1,0)$. Those solutions can be analytically continued in the full domain $D(t_1,\varepsilon_1,0)$ and will be denoted by the same notation in the below.

We shall now prove two lemmas.

LEMMA 4. *There exist scalar functions $\alpha_j^\nu(\varepsilon)$ $(\nu=1,2,\dots,n;j=0,1,\dots,n)$ of ε, which admit asymptotic behaviors*

$$\alpha_j^\nu(\varepsilon)\sim 1 \qquad as \quad \varepsilon\longrightarrow 0,$$

such that there hold

$$\sum_{j=0}^{n}\alpha_j^\nu(\varepsilon)P_{j+\nu}(t,\varepsilon)^{-1}y_j(t,\varepsilon)=0 \qquad (\nu=1,2,\dots,n).$$

Here and below all subscripts and superscripts are to be interpreted modulo $n+1$.

PROOF. For each ν $(\nu=1,2,\dots,n)$, $n+1$ solutions $P_{j+\nu}(t,\varepsilon)^{-1}y_j(t,\varepsilon)$ $(j=0,1,\dots,n)$ must satisfy some linear relation

$$\sum_{j=0}^{n}c_j^\nu(\varepsilon)P_{j+\nu}(t,\varepsilon)^{-1}y_j(t,\varepsilon)=0.$$

Setting $t=0$, we solve $c_j^\nu(\varepsilon)$ by the Cramer rule and then obtain

$$c_j^\nu(\varepsilon) = \det [P_{j+\nu+1}(0, \varepsilon)^{-1} y_{j+1}(0, \varepsilon),\ P_{j+\nu+2}(0, \varepsilon)^{-1} y_{j+2}(0, \varepsilon), \dots,$$
$$P_{j+\nu+n}(0, \varepsilon)^{-1} y_{j+n}(0, \varepsilon)] \qquad (j = 0, 1, \dots, n).$$

Since for all ν, $P_{j+\nu}(0,\varepsilon)^{-1} \sim P(0,\varepsilon)^{-1}$ as $\varepsilon \to 0$ and $\det P(0,\varepsilon)^{-1} \neq 0$ for sufficiently small ε, it follows that

$$c_j^\nu(\varepsilon) \sim \det [P(0, \varepsilon)^{-1}] \det [y_{j+1}(0, \varepsilon),\ y_{j+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon)]$$
$$(j = 0, 1, \dots, n)$$

as $\varepsilon \to 0$. From Lemma 1 and (4.1) we easily see that

$$\det [y_{j+1}(0, \varepsilon),\ y_{j+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon)] \neq 0 \qquad (j = 0, 1, \dots, n)$$

for $\varepsilon \neq 0$ and moreover, we can prove that for any j and j' $(0 \leqq j, j' \leqq n)$

$$\det [y_{j+1}(0, \varepsilon),\ y_{j+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon)]$$
$$= \det [y_{j'+1}(0, \varepsilon),\ y_{j'+2}(0, \varepsilon), \dots,\ y_{j'+n}(0, \varepsilon)]$$

holds. In fact, let j and j' be two distinct integers between 0 and n such that $j' = j + k$ $(1 \leqq k \leqq n)$. Then, using the connection formula $y_j(0, \varepsilon) = -\sum_{l=1}^{n} y_{j+l}(0, \varepsilon)$, we have

$$\det [y_{j'+1}(0, \varepsilon),\ y_{j'+2}(0, \varepsilon), \dots,\ y_{j'+n}(0, \varepsilon)]$$
$$= \det [y_{j+k+1}(0, \varepsilon),\ y_{j+k+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon),\ y_j(0, \varepsilon), \dots,\ y_{j+k-1}(0, \varepsilon)]$$
$$= \det [y_{j+k+1}(0, \varepsilon),\ y_{j+k+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon),$$
$$-y_{j+k}(0, \varepsilon),\ y_{j+1}(0, \varepsilon), \dots,\ y_{j+k-1}(0, \varepsilon)]$$
$$= (-1)^{(n-k+1)k} \det [y_{j+1}(0, \varepsilon),\ y_{j+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon)],$$

which is the required result since the integer $(n-k+1)k$ is always even. We here put

$$\alpha_j^\nu(\varepsilon) = c_j^\nu(\varepsilon) / \det [P(0, \varepsilon)^{-1}] \det [y_{j+1}(0, \varepsilon),\ y_{j+2}(0, \varepsilon), \dots,\ y_{j+n}(0, \varepsilon)]$$

and thus the proof of Lemma 4 is completed.

Now we put

$$x_j^\nu(t, \varepsilon) = \alpha_j^\nu(\varepsilon) P_{j+\nu}(t, \varepsilon)^{-1} y_j(t, \varepsilon) \qquad (j = 0, 1, \dots, n;\ \nu = 0, 1, \dots, n),$$

where $\alpha_j^0(\varepsilon) = 1$. As is easily seen, for each ν $(\nu = 1, 2, \dots, n)$, any n functions of $x_j^\nu(t, \varepsilon)$ $(j = 0, 1, \dots, n)$ are linearly independent solutions of (1.1) and Lemma 4 implies that just the same connection formula as (4.2)

$$x_0^\nu(t, \varepsilon) + x_1^\nu(t, \varepsilon) + \cdots + x_n^\nu(t, \varepsilon) = 0 \qquad (\nu = 1, 2, \dots, n)$$

is satisfied.

Before we state the next lemma, we make some preparations. Hereafter, for abbreviation, denote $S_k(t_1, \varepsilon_1, 0)$ only by S_k and also consider S_k as a sectorial domain in the complex t-plane.

Put

$$q_j(t, \varepsilon) = \frac{n}{n+1} \frac{1}{\varepsilon} t^{\frac{n+1}{n}} \omega_n^{-j} \qquad (j = 0, 1, \dots, n)$$

and define

$$\tilde{q}_0(t, \varepsilon) = \begin{cases} q_1(t, \varepsilon), & \pi \leqq \arg t < \pi + \dfrac{2\pi}{n+1}, \\ q_0(t, \varepsilon), & -\pi \leqq \arg t < \pi, \end{cases} \tag{4.3}$$

$$\tilde{q}_j(t, \varepsilon) = \begin{cases} q_j(t, \varepsilon), & -\pi + \dfrac{2\pi}{n+1} j \leqq \arg t < \pi + \dfrac{2\pi}{n+1}, \\ q_{j-1}(t, \varepsilon), & -\pi \leqq \arg t < -\pi + \dfrac{2\pi}{n+1} j \end{cases} \tag{4.4}$$

$$(j = 1, 2, \dots, n).$$

Moreover, let us define the functions $\hat{y}_j(t, \varepsilon)$ by

$$\hat{y}_j(t, \varepsilon) = y_j(t, \varepsilon) \exp(\tilde{q}_j(t, \varepsilon)) \qquad (j = 0, 1, \dots, n)$$

for $-\pi \leqq \arg t < \pi + \frac{2\pi}{n+1}$. Then from Lemma 1 and the definition (4.1) we observe that the functions $\hat{y}_j(t, \varepsilon)$ $(j = 0, 1, \dots, n)$ are uniformly bounded for $0 < \varepsilon \leqq \varepsilon_1$ and $t \in S_1 \cup S_2 \cup \cdots \cup S_{n+1}$.

We are now in a position to state the following important lemma.

LEMMA 5. *Let us define the functions* $\hat{x}_j^\nu(t, \varepsilon)$ $(\nu, j = 0, 1, \dots, n)$ *by*

$$\hat{x}_j^\nu(t, \varepsilon) = x_j^\nu(t, \varepsilon) \exp(\tilde{q}_j(t, \varepsilon)) \tag{4.5}$$

for $t \in S_1 \cup S_2 \cup \cdots \cup S_{n+1}$. *Then we have*

$$\hat{x}_j^{\nu+1}(t, \varepsilon) - \hat{x}_j^\nu(t, \varepsilon) \sim 0 \quad as \quad \varepsilon \longrightarrow 0$$

uniformly in $t \in S_1 \cup S_2 \cup \cdots \cup S_{n+1}$, $|t| \leqq t_2 < t_1$, *and hence for any* ν *and* μ

$$\hat{x}_j^\nu(t, \varepsilon) - \hat{x}_j^\mu(t, \varepsilon) \sim 0 \quad as \quad \varepsilon \longrightarrow 0$$

uniformly in $t \in S_1 \cup S_2 \cup \cdots \cup S_{n+1}$, $|t| \leqq t_2 < t_1$.

PROOF. Since $P_{j+\nu+1}(t, \varepsilon)^{-1} \sim P(t, \varepsilon)^{-1}$ in $S_{j+\nu+1}$ and $P_{j+\nu}(t, \varepsilon)^{-1} \sim P(t, \varepsilon)^{-1}$ in $S_{j+\nu}$ and $\alpha_j^\nu(\varepsilon) \sim 1$ as $\varepsilon \to 0$, we see that

$$\hat{x}_j^{\nu+1}(t, \varepsilon) - \hat{x}_j^\nu(t, \varepsilon) \sim 0 \quad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{in} \quad S_{j+\nu} \cap S_{j+\nu+1}.$$

Putting $\mathscr{V}_j^\nu(t, \varepsilon) = x_j^{\nu+1}(t, \varepsilon) - x_j^\nu(t, \varepsilon)$ and defining $\hat{\mathscr{V}}_j^\nu(t, \varepsilon)$ in a similar manner to (4.5), we investigate the behavior of $\hat{\mathscr{V}}_j^\nu(t, \varepsilon)$.

Let t move from the sector $S_{j+\nu} \cap S_{j+\nu+1}$ in the positive direction and lie in $S_{j+\nu+1}$. The solution $\mathscr{V}_j^\nu(t, \varepsilon)$ of (1.1) can be expressed in terms of a linear combination of n solutions of the $x_k^{j+\nu+1-k}(t, \varepsilon)$ $(k=0, 1,\dots, n)$, excluding one solution which has its Stokes line in $S_{j+\nu+1}$, as follows:

$$\mathscr{V}_j^\nu(t, \varepsilon) = \sum_{\substack{k=0 \\ k \neq j+\nu+1}}^{n} A_k(\varepsilon) x_k^{j+\nu+1-k}(t, \varepsilon)$$

with the coefficients $A_k(\varepsilon)$ depending on ε alone, whence it yields that

$$\hat{\mathscr{V}}_j^\nu(t, \varepsilon) = \sum_{\substack{k=1 \\ k \neq j+\nu+1}}^{n} A_k(\varepsilon) \hat{x}_k^{j+\nu+1-k}(t, \varepsilon) \exp(\tilde{q}_j(t, \varepsilon) - \tilde{q}_k(t, \varepsilon)). \tag{4.6}$$

Here we point out that all the functions $\hat{x}_k^{j+\nu+1-k}(t, \varepsilon)$ are asymptotically known and bounded in $S_{j+\nu+1}$. Let τ be an arbitrary point in $S_{j+\nu} \cap S_{j+\nu+1}$. Setting $t=\tau$ in (4.6), we seek the coefficients $A_k(\varepsilon)$ by the usual Cramer rule. For that purpose, we need to calculate the determinant

$$\begin{aligned} \Delta_j^\nu(\tau) &= \det[\hat{x}_0^{j+\nu+1}(\tau, \varepsilon), \hat{x}_1^{j+\nu}(\tau, \varepsilon), \dots, \overset{\vee}{\hat{x}}{}_{j+\nu+1}^0(\tau, \varepsilon), \dots, \hat{x}_n^{j+\nu+1-n}(\tau, \varepsilon)] \\ &= \det[x_0^{j+\nu+1}(\tau, \varepsilon), x_1^{j+\nu}(\tau, \varepsilon), \dots, \overset{\vee}{x}{}_{j+\nu+1}^0(\tau, \varepsilon), \dots, x_n^{j+\nu+1-n}(\tau, \varepsilon)] \\ &\quad \times \exp\Big(\sum_{\substack{k=0 \\ k \neq j+\nu+1}}^{n} \tilde{q}_k(\tau, \varepsilon)\Big), \end{aligned}$$

where the symbol $\vee$ on a vector indicates that the vector is to be omitted. Taking account of the definition (4.3–4), we have

$$\begin{aligned} \sum_{\substack{k=0 \\ k \neq j+\nu+1}}^{n} \tilde{q}_k(\tau, \varepsilon) &= \sum_{k=0}^{j+\nu} q_k(\tau, \varepsilon) + \sum_{k=j+\nu+2}^{n} q_{k-1}(\tau, \varepsilon) \\ &= \frac{n}{n+1} \frac{1}{\varepsilon} \tau^{\frac{n+1}{n}} \Big(\sum_{k=0}^{j+\nu} \omega_n^{-k} + \sum_{k=j+\nu+2}^{n} \omega_n^{-k+1}\Big) \\ &= 0. \end{aligned}$$

We therefore obtain

$$\begin{aligned}\Delta_j^\nu(\tau) &= \Big|\prod_{\substack{k=0\\k\neq j+\nu+1}}^{n} \alpha_k^{j+\nu+1-k}(\varepsilon)\Big| \det|P_{j+\nu+1}(\tau,\varepsilon)^{-1}| \\ &\quad \times \det[y_0(\tau,\varepsilon), y_1(\tau,\varepsilon),\dots, \check{y}_{j+\nu+1}(\tau,\varepsilon),\dots, y_n(\tau,\varepsilon)] \\ &= \Big|\prod_{\substack{k=0\\k\neq j+\nu+1}}^{n} \alpha_k^{j+\nu+1-k}(\varepsilon)\Big| \det|P_{j+\nu+1}(\tau,\varepsilon)^{-1}| \\ &\quad \times \det[y_0(0,\varepsilon), y_1(0,\varepsilon),\dots, \check{y}_{j+\nu+1}(0,\varepsilon),\dots, y_n(0,\varepsilon)] \\ &= \Big|\prod_{\substack{k=0\\k\neq j+\nu+1}}^{n} \alpha_k^{j+\nu+1-k}(\varepsilon)\beta_k\Big| \varepsilon^{\frac{n(n-1)}{2(n+1)}} \det|P_{j+\nu+1}(\tau,\varepsilon)^{-1}| \\ &\quad \times W[Ai(z), Ai(\omega^{-1}z),\dots, \check{Ai}(\omega^{-j-\nu-1}z),\dots, Ai(\omega^{-n}z): z] \\ &= \varepsilon^{\frac{n(n-1)}{2(n+1)}} K(\varepsilon),\end{aligned}$$

where $K(\varepsilon)$ is bounded away from zero for $0<\varepsilon\leqq\varepsilon_1$. Since $\hat{\mathscr{V}}_j^\nu(\tau,\varepsilon)\sim 0$ as $\varepsilon\to 0$, we consequently obtain

$$A_k(\varepsilon) = \exp(-\tilde{q}_j(\tau,\varepsilon) + \tilde{q}_k(\tau,\varepsilon))\hat{A}_k(\tau,\varepsilon), \tag{4.7}$$

$$\hat{A}_k(\tau,\varepsilon) \sim 0 \quad \text{as} \quad \varepsilon \longrightarrow 0 \tag{4.8}$$

$$(k = 0, 1,\dots, n;\ k \neq j+\nu+1).$$

We have to pay attention to the fact that the relations (4.8) hold for every τ in $S_{j+\nu}\cap S_{j+\nu+1}$. The substitution of (4.7) into (4.6) yields

$$\hat{\mathscr{V}}_j^\nu(t,\varepsilon) = \sum_{\substack{k=0\\ \nu\neq j+\nu+1}}^{n} \{\hat{A}_k(\tau,\varepsilon)\hat{x}_k^{j+\nu+1-k}(t,\varepsilon) \times \exp(\tilde{q}_j(t,\varepsilon) - \tilde{q}_k(t,\varepsilon) - \tilde{q}_j(\tau,\varepsilon) + \tilde{q}_k(\tau,\varepsilon))\} \tag{4.9}$$

in $S_{j+\nu+1}$.

Now, if necessary, we divide the sectorial domain $S_{j+\nu+1}-(S_{j+\nu}\cap S_{j+\nu+1})$ into a finite number of subsectorial domains, i.e., $T_1\cup T_2\cup\cdots\cup T_p=S_{j+\nu+1}-(S_{j+\nu}\cup S_{j+\nu+1})$, such that in each T_μ ($\mu=1, 2,\dots, p$) there always hold

$$\text{(i)}\ \operatorname{Re}(\tilde{q}_j(t,\varepsilon)-\tilde{q}_k(t,\varepsilon))\leqq 0 \quad \text{or} \quad \text{(ii)}\ \operatorname{Re}(\tilde{q}_j(t,\varepsilon)-\tilde{q}_k(t,\varepsilon))>0$$

for all k ($k=0, 1,\dots, n$; $k\neq j+\nu+1$). Let t move from $S_{j+\nu}\cap S_{j+\nu+1}$ to the neighboring sectorial domain T_1. For k for which (i) holds, we take $\tau=0$ in (4.9), and for k for which (ii) holds, we take as τ in (4.9) a point which is lying in $S_{j+\nu}\cap S_{j+\nu+1}$ and is close to the ray of boundary of T_1, i.e.,

$$\tau = t_1 \exp\left\{\left(-\pi + \frac{2\pi}{n+1}(j+\nu+1) - \delta - \delta'\right)i\right\},$$

δ' being an arbitrarily small positive number, to obtain $\mathrm{Re}(\tilde{q}_j(\tau,\varepsilon) - \tilde{q}_k(\tau,\varepsilon)) > 0$. Then it follows that the exponential factors in (4.9) remain bounded as $\varepsilon \to 0$ for $t \in \mathrm{T}_1$, $|t| \leqq t_1' < t_1$. Here, taking account of the boundedness of the $\hat{x}_k^{j+\nu+1-k}(t,\varepsilon)$ and (4.8), we can conclude that

$$\hat{\mathscr{V}}_j^\nu(t,\varepsilon) \sim 0 \quad \text{as} \quad \varepsilon \longrightarrow 0 \tag{4.10}$$

uniformly in $t \in \mathrm{T}_1$, $|t| \leqq t_1'$. Next, if t lies in T_2, then using the relation (4.10) just derived, we can prove by means of entirely the same argument stated above that $\hat{\mathscr{V}}_j^\nu(t,\varepsilon) \sim 0$ as $\varepsilon \to 0$ uniformly in $t \in \mathrm{T}_2$, $|t| \leqq t_1'' < t_1'$. We continue this process in $\mathrm{T}_3, \mathrm{T}_4, \ldots, \mathrm{T}_p$ to obtain the required result in $S_{j+\nu+1}$. Similarly in other sectorial domains we have only to follow the above argument. Thus we have completed the proof of Lemma 5.

From the above two lemmas we now derive the main theorem of this paper.

THEOREM 3. *There exists a matrix function $Q(t,\varepsilon)$, which admits an asymptotic expansion*

$$Q(t,\varepsilon) \sim P(t,\varepsilon) \quad \textit{as} \quad \varepsilon \longrightarrow 0$$

uniformly in $|t| \leqq t_2$, $-\pi + \delta \leqq \arg t \leqq \pi + \frac{2\pi}{n+1} - \delta$, δ being a sufficiently small positive number, such that the linear transformation $Y = Q(t,\varepsilon)X$ changes $\varepsilon\frac{dX}{dt} = A(t,\varepsilon)X$ into $\varepsilon\frac{dY}{dt} = A_0(t)Y$.

PROOF. Let an integer ν be taken from $1, 2, \ldots, n$ and be fixed. We put

$$\mathscr{X}_k(t,\varepsilon) = [x_1^\nu(t,\varepsilon), x_2^\nu(t,\varepsilon), \ldots, \check{x}_k^\nu(t,\varepsilon), \ldots, x_{n+1}^\nu(t,\varepsilon)]$$

$$\mathscr{Y}_k(t,\varepsilon) = [y_1(t,\varepsilon), y_2(t,\varepsilon), \ldots, \check{y}_k(t,\varepsilon), \ldots, y_{n+1}(t,\varepsilon)]$$

$$(k = 1, 2, \ldots, n+1),$$

where $x_{n+1}^\nu(t,\varepsilon) = x_0^\nu(t,\varepsilon)$ and $y_{n+1}(t,\varepsilon) = y_0(t,\varepsilon)$. It is easily seen that the matrices $\mathscr{X}_k(t,\varepsilon)$ and $\mathscr{Y}_k(t,\varepsilon)$ $(k = 1, 2, \ldots, n+1)$ are fundamental matrix solutions of (1.1) and (2.1), respectively. We now define the matrices $Q_k(t,\varepsilon)$ $(k = 1, 2, \ldots, n+1)$ by

$$Q_k(t,\varepsilon) = \mathscr{Y}_k(t,\varepsilon)\mathscr{X}_k(t,\varepsilon)^{-1}. \tag{4.11}$$

This definition implies that for each fixed k we have

$$Q_k(t,\varepsilon)x_j^\nu(t,\varepsilon) = y_j(t,\varepsilon) \qquad (j = 1, 2, \ldots, n+1;\ j \neq k).$$

But, considering the relation proved in Lemma 4

$$x_k^\nu(t, \varepsilon) = -\sum_{\substack{j=1\\ j\neq k}}^{n+1} x_j^\nu(t, \varepsilon) \qquad (x_{n+1}^\nu(t, \varepsilon) = x_0^\nu(t, \varepsilon))$$

together with

$$y_k(t, \varepsilon) = -\sum_{\substack{j=1\\ j\neq k}}^{n+1} y_j(t, \varepsilon) \qquad (y_{n+1}(t, \varepsilon) = y_0(t, \varepsilon)),$$

we also have

$$\begin{aligned} Q_k(t, \varepsilon)x_k^\nu(t, \varepsilon) &= -\sum_{\substack{j=1\\ j\neq k}}^{n+1} Q_k(t, \varepsilon)x_j^\nu(t, \varepsilon) \\ &= -\sum_{\substack{j=1\\ j\neq k}}^{n+1} y_j(t, \varepsilon) = y_k(t, \varepsilon). \end{aligned}$$

Hence we can conclude that all $n+1$ linear transformations $Q_k(t, \varepsilon)$ ($k=1, 2, \dots, n+1$) are identical from the fact that each $Q_k(t, \varepsilon)$ is a linear transformation from the n-dimensional vector space of solutions of (1.1) onto the n-dimensional vector space of solutions of (2.1) in the same coordinate system. We put

$$Q(t, \varepsilon) \equiv Q_k(t, \varepsilon) \qquad (k = 1, 2, \dots, n+1).$$

We now need only to prove that

$$Q(t, \varepsilon) \sim P_k(t, \varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{in} \quad S_k.$$

From (4.11) it immediately follows that

$$\begin{aligned} Q_k(t, \varepsilon) = [&\hat{y}_1(t, \varepsilon), \hat{y}_2(t, \varepsilon), \dots, \check{\hat{y}}_k(t, \varepsilon), \dots, \hat{y}_{n+1}(t, \varepsilon)] \\ &\times [\hat{x}_1^\nu(t, \varepsilon), \hat{x}_2^\nu(t, \varepsilon), \dots, \check{\hat{x}}_k^\nu(t, \varepsilon), \dots, \hat{x}_{n+1}^\nu(t, \varepsilon)]^{-1}. \end{aligned} \tag{4.12}$$

On the other hand, an application of Lemma 5 yields that

$$\begin{aligned} &[\hat{x}_1^\nu(t, \varepsilon), \hat{x}_2^\nu(t, \varepsilon), \dots, \check{\hat{x}}_\nu^k(t, \varepsilon), \dots, \hat{x}_{n+1}^\nu(t, \varepsilon)]^{-1} \\ &\sim [\hat{x}_1^{k-1}(t, \varepsilon), \hat{x}_2^{k-2}(t, \varepsilon), \dots, \check{\hat{x}}_k^0(t, \varepsilon), \dots, \hat{x}_{n+1}^{k-(n+1)}(t, \varepsilon)]^{-1} \\ &= [\prod_{\substack{j=1\\ j\neq k}}^{n+1} \alpha_i^{k-j}(\varepsilon)][\hat{y}_1(t, \varepsilon), \hat{y}_2(t, \varepsilon), \dots, \check{\hat{y}}_k(t, \varepsilon), \dots, \hat{y}_{n+1}(t, \varepsilon)]^{-1} P_k(t, \varepsilon) \\ &\sim [\hat{y}_1(t, \varepsilon), \hat{y}_2(t, \varepsilon), \dots, \check{\hat{y}}_k(t, \varepsilon), \dots, \hat{y}_{n+1}(t, \varepsilon)]^{-1} P_k(t, \varepsilon) \end{aligned}$$

as $\varepsilon \to 0$ in S_k. Combining this with (4.12), we consequently obtain

$$Q_k(t, \varepsilon) \sim P_k(t, \varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{in} \quad S_k \qquad (k = 1, 2, \ldots, n + 1).$$

Thus the proof of our main theorem has been completed.

References

[1] M. Kohno: An extended Airy function of the first kind, Hiroshima Math. J., **9** (1979), 473–489.

[2] Roy Y. Lee: On uniform simplification of linear differential equation in a full neighborhood of a turning point, J. Math. Anal. Appl., **27** (1969), 501–510.

[3] K. Okubo: On certain reduction theorems for systems of differential equations which contain a turning point, Proc. Japan Acad., **37** (1961), 544–549.

[4] Y. Sibuya: Uniform simplification in a full neighborhood of a transition point, Memoirs Amer. Math. Soc., No. 149, Providence, Rhode Island, 1974.

[5] W. Wasow: Simplification of turning point problems for system of linear differential equations, Trans. Amer. Math. Soc., **106** (1963), 100–114.

[6] W. Wasow: Turning point problems for systems of linear equations, I. The formal theory, II. The analytic theory, Comm. Pure Appl. Math., **14** (1961), 657–673, **15** (1962), 173–187.

[7] W. Wasow: Asymptotic expansions for ordinary differential equations, Robert E. Krieger Publishing Co., Inc., New York, 1976 (Original edition 1965).

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