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On function spaces with fractional Fourier transform in weighted Lebesgue spaces

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Abstract

Let w and ω be weight functions on $\mathbb{R}^d$. In this work, we define $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ to be the vector space of $f \in L^1_w(\mathbb{R}^d)$ such that the fractional Fourier transform $F_\alpha f$ belongs to $L^p_\omega(\mathbb{R}^d)$ for $1 \le p < \infty$. We endow this space with the sum norm $\|f\|_{A^{w,\omega}_{\alpha,p}} = \|f\|_{1,w} + \|F_\alpha f\|_{p,\omega}$ and show that $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ becomes a Banach space and invariant under time-frequency shifts. Further we show that the mapping $y \to T_y f$ is continuous from $\mathbb{R}^d$ into $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$, the mapping $z \to M_z f$ is continuous from $\mathbb{R}^d$ into $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ and $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is a Banach module over $L^1_w(\mathbb{R}^d)$ with Θ convolution operation. At the end of this work, we discuss inclusion properties of these spaces.

Keywords: fractional Fourier transform; convolution; Banach module

1 Introduction

In this work, for any function $f : \mathbb{R}^d \to \mathbb{C}$, the translation and modulation operator are defined as $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{iwt} f(t)$ for all $y, w \in \mathbb{R}^d$, respectively. Also we write the Lebesgue space $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, for $1 \le p < \infty$. Let w be a weight function on $\mathbb{R}^d$, that is, a measurable and locally bounded function w satisfying $w(x) \ge 1$ and $w(x+y) \le w(x)w(y)$ for all $x, y \in \mathbb{R}^d$. We define, for $1 \le p < \infty$,

$$L^p_w(\mathbb{R}^d) = \{ f | fw \in L^p(\mathbb{R}^d) \}.$$

It is well known that $L^p_w(\mathbb{R}^d)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$.

Let w_1 and w_2 are two weight functions. We say that $w_1 \prec w_2$ if there exists $c > 0$, such that $w_1(x) \le c w_2(x)$ for all $x \in \mathbb{R}^d$ [1, 2].

The Fourier transform $\hat{f}$ (or $\mathcal{F}f$) of $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-iwt}\, dt.$$

The fractional Fourier transform is a generalization of the Fourier transform with a parameter α and can be interpreted as a rotation by an angle α in the time-frequency plane. The fractional Fourier transform with angle α of a function f is defined by

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} K_\alpha(u,t) f(t)\, dt,$$

where

$$K_\alpha(u,t) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{i(\frac{u^2+t^2}{2})\cot\alpha - iut\operatorname{cosec}\alpha}, & \text{if } \alpha \text{ is not multiple of } \pi, \\ \delta(t-u), & \text{if } \alpha = 2k\pi, k \in \mathbb{Z}, \\ \delta(t+u), & \text{if } \alpha = (2k+1)\pi, k \in \mathbb{Z}, \end{cases}$$

and δ is a Dirac delta function. The fractional Fourier transform with $\alpha = \frac{\pi}{2}$ corresponds to the Fourier transform [3–9].

The fractional Fourier transform can be extended to higher dimensions as [9]:

$$\begin{aligned} &(\mathcal{F}_{\alpha_1,\ldots,\alpha_n} f)(u_1,\ldots,u_n) \\ &\quad = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_{\alpha_1,\ldots,\alpha_n}(u_1,\ldots,u_n;t_1,\ldots,t_n) f(t_1,\ldots,t_n)\,dt_1\cdots dt_n, \end{aligned}$$

or shortly

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} K_\alpha(u,t) f(t)\,dt,$$

where

$$K_\alpha(u,t) = K_{\alpha_1,\ldots,\alpha_n}(u_1,\ldots,u_n;t_1,\ldots,t_n) = K_{\alpha_1}(u_1,t_1)K_{\alpha_2}(u_2,t_2)\cdots K_{\alpha_n}(u_n,t_n).$$

In this work we define the function spaces with fractional Fourier transform in weighted Lebesgue spaces and discuss some properties of these spaces.

2 On function spaces with fractional Fourier transform in weighted Lebesgue spaces

Definition 1 Let w and ω be weight functions on $\mathbb{R}^d$ and $1 \le p < \infty$. The space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ consist of all $f \in L_w^1(\mathbb{R}^d)$ such that $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$. The norm on the vector space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is

$$\|f\|_{A_{\alpha,p}^{w,\omega}} = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p,\omega}.$$

Theorem 2 *$(A_{\alpha,p}^{w,\omega}(\mathbb{R}^d), \|\cdot\|_{A_{\alpha,p}^{w,\omega}})$ is a Banach space for $1 \le p < \infty$.*

Proof Let $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Thus $(f_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_\alpha f_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L_w^1(\mathbb{R}^d)$ and $L_\omega^p(\mathbb{R}^d)$, respectively. Since $L_w^1(\mathbb{R}^d)$ and $L_\omega^p(\mathbb{R}^d)$ are Banach spaces, there exist $f \in L_w^1(\mathbb{R}^d)$ and $g \in L_\omega^p(\mathbb{R}^d)$ such that $\|f_n - f\|_{1,w} \to 0$, $\|\mathcal{F}_\alpha f_n - g\|_{p,\omega} \to 0$ and hence $\|f_n - f\|_1 \to 0$ and $\|\mathcal{F}_\alpha f_n - g\|_p \to 0$. Then $(\mathcal{F}_\alpha f_n)_{n\in\mathbb{N}}$ has a subsequence $(\mathcal{F}_\alpha f_{n_k})_{n_k\in\mathbb{N}}$ that converges pointwise to g almost everywhere. Also it is easy to see that $\|f_{n_k} - f\|_1 \to 0$. Then we have

$$\begin{aligned} \left|\mathcal{F}_\alpha f(u) - g(u)\right| &\le \left|\mathcal{F}_\alpha (f_{n_k} - f)(u)\right| + \left|\mathcal{F}_\alpha f_{n_k}(u) - g(u)\right| \\ &\le \prod_{j=1}^{d} \left|\sqrt{\frac{1 - i\cot\alpha_j}{2\pi}}\right| \end{aligned}$$

$$\times \int_{\mathbb{R}^d} \left|(f_{n_k} - f)(t)\right| \left|e^{\sum_{j=1}^d (\frac{i}{2}(u_j{}^2 + t_j{}^2)\cot\alpha_j - iu_jt_j \operatorname{cosec}\alpha_j)}\right| dt$$
$$+ \left|\mathcal{F}_\alpha f_{n_k}(u) - g(u)\right|$$
$$= \prod_{j=1}^d \left|\sqrt{\frac{1 - i\cot\alpha_j}{2\pi}}\right| \|f_{n_k} - f\|_1 + \left|\mathcal{F}_\alpha f_{n_k}(u) - g(u)\right|.$$

From this inequality, we obtain $\mathcal{F}_\alpha f = g$ almost everywhere. Thus $\|f_n - f\|_{A^{w,\omega}_{\alpha,p}} \to 0$ and $f \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. Hence $(A^{w,\omega}_{\alpha,p}(\mathbb{R}^d), \|\cdot\|_{A^{w,\omega}_{\alpha,p}})$ is a Banach space. □

The following proposition is generalization of the one-dimensional and two-dimensional versions. The proof of this proposition is very similar to the proofs of one-dimensional and two-dimensional versions in [3, 5, 10, 11], and we omit the details.

Proposition 3 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \le i \le d$ and $k \in \mathbb{Z}$. Then*

$$(1) \quad \mathcal{F}_\alpha(T_y f)(u) = e^{\sum_{j=1}^d (\frac{i}{2}y_j^2 \sin\alpha_j \cos\alpha_j - iu_jy_j \sin\alpha_j)} \mathcal{F}_\alpha f(u_1 - y_1\cos\alpha_1, \ldots, u_d - y_d\cos\alpha_d) \quad (1)$$

for all $f \in L^1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$;

$$(2) \quad \mathcal{F}_\alpha(M_v f)(u) = e^{\sum_{j=1}^d (-\frac{i}{2}v_j^2 \sin\alpha_j \cos\alpha_j + iu_jv_j \cos\alpha_j)} \mathcal{F}_\alpha f(u_1 - v_1\sin\alpha_1, \ldots, u_d - v_d\sin\alpha_d)$$

for all $f \in L^1(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$.

Theorem 4 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \le i \le d$ and $k \in \mathbb{Z}$.*

(1) *Let $1 \le p < \infty$, w and ω be weight functions on $\mathbb{R}^d$. Then the space $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is translation invariant.*
(2) *Let ω be a bounded weight function on $\mathbb{R}^d$. Then the mapping $y \to T_y f$ of $\mathbb{R}^d$ into $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is continuous.*

Proof (1) Let $f \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. Then $f \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha f \in L^p_\omega(\mathbb{R}^d)$. It is well known that the space $L^1_w(\mathbb{R}^d)$ is translation invariant and holds $\|T_y f\|_{1,w} \le w(y)\|f\|_{1,w}$ for all $y \in \mathbb{R}^d$ [12]. Let $b = (y_1\cos\alpha_1, \ldots, y_d\cos\alpha_d)$. By using the equality (1), we get

$$\left\|\mathcal{F}_\alpha(T_y f)\right\|_{p,\omega} = \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha(T_y f)(u)\right|^p \omega^p(u)\,du\right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha f(u_1 - y_1\cos\alpha_1, \ldots, u_d - y_d\cos\alpha_d)\right|^p \right.$$
$$\left. \times \left|e^{\sum_{j=1}^d (\frac{i}{2}y_j{}^2 \sin\alpha_j\cos\alpha_j - iu_jy_j\sin\alpha_j)}\right|^p \omega^p(u)\,du\right)^{\frac{1}{p}}$$
$$\le \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha f(u - b)\right|^p \omega^p(u - b)\omega^p(b)\,du\right)^{\frac{1}{p}}$$
$$= \omega(b)\|\mathcal{F}_\alpha f\|_{p,\omega}$$

for all $y \in \mathbb{R}^d$. Hence, we have

$$\|T_y f\|_{A^{w,\omega}_{\alpha,p}} \le w(y)\|f\|_{1,w} + \omega(b)\|\mathcal{F}_\alpha f\|_{p,\omega} < \infty.$$

This means that $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is translation invariant.

(2) Let $f \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. We will show that if $\lim_{n\to\infty} y_n = 0$ for any sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$, then $\lim_{n\to\infty} T_{y_n} f = f$, which will complete the proof. It is well known that the mapping $y \to T_y f$ is continuous from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ (see [12]). Thus, we have

$$\|T_{y_n} f - f\|_{1,w} \to 0 \tag{2}$$

as $n \to \infty$. Also,

$$\begin{aligned}
\|\mathcal{F}_\alpha(T_{y_n} f - f)\|_{p,\omega} &= \|\mathcal{F}_\alpha(T_{y_n} f) - \mathcal{F}_\alpha f\|_{p,\omega} \\
&= \left\| e^{\sum_{j=1}^d (\frac{i}{2}(y_n^j)^2 \sin\alpha_j\cos\alpha_j - iu_j y_n^j \sin\alpha_j)} T_{(y_n^1\cos\alpha_1,\dots,y_n^d\cos\alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f \right\|_{p,\omega} \\
&\le \left\| \left(T_{(y_n^1\cos\alpha_1,\dots,y_n^d\cos\alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f\right)\right\|_{p,\omega} \\
&\quad + \left\| \left(e^{\sum_{j=1}^d (\frac{i}{2}(y_n^j)^2 \sin\alpha_j\cos\alpha_j - iu_j y_n^j \sin\alpha_j)} - 1\right)\mathcal{F}_\alpha f \right\|_{p,\omega}.
\end{aligned}$$

Since $\mathcal{F}_\alpha f \in L^p_\omega(\mathbb{R}^d)$, the mapping $y \to T_y(\mathcal{F}_\alpha f)$ is continuous from $\mathbb{R}^d$ into $L^p_\omega(\mathbb{R}^d)$ for all $y \in \mathbb{R}^d$ [12]. Then we obtain $\|T_{(y_n^1\cos\alpha_1,\dots,y_n^d\cos\alpha_d)}(\mathcal{F}_\alpha f) - \mathcal{F}_\alpha f\|_{p,\omega} \to 0$ as $n \to \infty$. Now let $h_{y_n}(u) = |e^{\sum_{j=1}^d (\frac{i}{2}(y_n^j)^2 \sin\alpha_j\cos\alpha_j - iu_j y_n^j \sin\alpha_j)} - 1||\mathcal{F}_\alpha f(u)|$. Since $\lim_{n\to\infty} y_n = 0$ and ω is a bounded weight function on $\mathbb{R}^d$, we see that $\lim_{n\to\infty} h^p_{y_n}(u)\omega^p(u) = 0$ for all $u \in \mathbb{R}^d$. Also, since

$$h_{y_n}(u) = \left|e^{\sum_{j=1}^d (\frac{i}{2}(y_n^j)^2 \sin\alpha_j\cos\alpha_j - iu_j y_n^j \sin\alpha_j)} - 1\right|\left|\mathcal{F}_\alpha f(u)\right| \le 2\left|\mathcal{F}_\alpha f(u)\right|$$

and $\mathcal{F}_\alpha f \in L^p_\omega(\mathbb{R}^d)$, we can write $h^p_{y_n}(u)\omega^p(u) \le 2^p|\mathcal{F}_\alpha f(u)|^p\omega^p(u)$. Thus, by the Lebesgue dominated convergence theorem,

$$\left\| \left(e^{\sum_{j=1}^d (\frac{i}{2}(y_n^j)^2 \sin\alpha_j\cos\alpha_j - iu_j y_n^j \sin\alpha_j)} - 1\right)\mathcal{F}_\alpha f \right\|_{p,\omega} \to 0$$

as $\lim_{n\to\infty} y_n = 0$. Hence,

$$\|T_{y_n} f - f\|_{A^{w,\omega}_{\alpha,p}} \to 0 \tag{3}$$

as $n \to \infty$. Combining (2) and (3),

$$\|T_{y_n} f - f\|_{A^{w,\omega}_{\alpha,p}} = \|T_{y_n} f - f\|_{1,w} + \left\|\mathcal{F}_\alpha(T_{y_n} f - f)\right\|_{p,\omega} \to 0$$

as $n \to \infty$. This is the desired result. □

Theorem 5 *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \ne k\pi$ for each index i with $1 \le i \le d$ and $k \in \mathbb{Z}$.*

(1) *Let* $1 \le p < \infty$, *w and* ω *be weight functions on* $\mathbb{R}^d$. *Then* $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ *is invariant under modulations.*

(2) *Let* ω *be a bounded weight function on* $\mathbb{R}^d$. *Then the mapping* $z \to M_z f$ *is continuous from* $\mathbb{R}^d$ *into* $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$.

Proof (1) Let $f \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. Then $f \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha f \in L^p_\omega(\mathbb{R}^d)$. It is easy to see that $\|M_\eta f\|_{1,w} = \|f\|_{1,w}$ and $M_\eta f \in L^1_w(\mathbb{R}^d)$. Let $c = (\eta_1 \sin\alpha_1, \dots, \eta_d \sin\alpha_d) \in \mathbb{R}^d$. Thus by Proposition 3, we have

$$\begin{aligned}
\|\mathcal{F}_\alpha(M_\eta f)\|_{p,\omega} &= \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha(M_\eta f)(u)\right|^p \omega^p(u)\,du\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha f(u_1 - \eta_1 \sin\alpha_1, \dots, u_d - \eta_d \sin\alpha_d)\right|^p \right. \\
&\quad \left. \times \left|e^{\sum_{j=1}^d (-\frac{i}{2}\eta_j^2 \sin\alpha_j \cos\alpha_j + iu_j\eta_j \cos\alpha_j)}\right|^p \omega^p(u)\,du\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^d} \left|\mathcal{F}_\alpha f(u-c)\right|^p \omega^p(u-c)\omega^p(c)\,du\right)^{\frac{1}{p}} \\
&= \omega(c)\|\mathcal{F}_\alpha f\|_{p,\omega}
\end{aligned}$$

for all $\eta \in \mathbb{R}^d$. Hence, we get

$$\|M_\eta f\|_{A^{w,\omega}_{\alpha,p}} \le \|f\|_{1,w} + \omega(c)\|\mathcal{F}_\alpha f\|_{p,\omega} < \infty.$$

(2) The proof technique of this part is the same as that of Theorem 4(2). So, for the sake of brevity, we will not prove it. □

The following definition is an extension of the convolution in [13, 14] of two functions to n dimensions.

Definition 6 Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \le i \le d$ and $k \in \mathbb{Z}$. Then the convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ is the function $f\Theta g$ defined by

$$(f\Theta g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)e^{\sum_{j=1}^d iy_j(y_j - x_j)\cot\alpha_j}\,dy.$$

It is easy to see that $f\Theta g$ belongs to $L^1(\mathbb{R}^d)$ by Fubini's theorem.

Theorem 7 *Let* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, *where* $\alpha_i \neq k\pi$ *for each index i with* $1 \le i \le d$ *and* $k \in \mathbb{Z}$, *and* $f, g \in L^1(\mathbb{R}^d)$. *Then*

$$\mathcal{F}_\alpha(f\Theta g)(u) = \left[\prod_{j=1}^d \sqrt{\frac{2\pi}{1 - i\cot\alpha_j}}\right] e^{\sum_{j=1}^d -\frac{i}{2}u_j^2 \cot\alpha_j} \mathcal{F}_\alpha f(u)\mathcal{F}_\alpha g(u),$$

where $\mathcal{F}_\alpha f$ *and* $\mathcal{F}_\alpha g$ *are the fractional Fourier transforms of functions f and g, respectively.*

Proof Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$, and $f, g \in L^1(\mathbb{R}^d)$. We can write from the definition of the fractional Fourier transform

$$\begin{aligned}\mathcal{F}_\alpha(f\Theta g)(u) &= \left[\prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right] \int_{\mathbb{R}^d} (f\Theta g)(t) e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2+t_j^2)\cot\alpha_j - iu_jt_j \operatorname{cosec}\alpha_j)}\, dt \\ &= \left[\prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right] \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(t-y) e^{\sum_{j=1}^d iy_j(y_j-t_j)\cot\alpha_j} \\ &\quad \times e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2+t_j^2)\cot\alpha_j - iu_jt_j \operatorname{cosec}\alpha_j)}\, dt\, dy.\end{aligned}$$

We make the substitution $t - y = k$ and obtain

$$\begin{aligned}\mathcal{F}_\alpha(f\Theta g)(u) &= \left[\prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right] \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2+y_j^2)\cot\alpha_j - iu_jy_j \operatorname{cosec}\alpha_j)}\, dy\right) \\ &\quad \times g(k) e^{\sum_{j=1}^d (\frac{i}{2}k_j^2\cot\alpha_j - iu_jk_j \operatorname{cosec}\alpha_j)}\, dk \\ &= \left[\prod_{j=1}^d \sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right] e^{\sum_{j=1}^d -\frac{i}{2}u_j^2\cot\alpha_j} \left[\prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right]^2 \\ &\quad \times \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(y) e^{\sum_{j=1}^d (\frac{i}{2}(u_j^2+y_j^2)\cot\alpha_j - iu_jy_j \operatorname{cosec}\alpha_j)}\, dy\right) \\ &\quad \times g(k) e^{\sum_{j=1}^d (\frac{i}{2}(k_j^2+u_j^2)\cot\alpha_j - iu_jk_j \operatorname{cosec}\alpha_j)}\, dk \\ &= \left[\prod_{j=1}^d \sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right] e^{\sum_{j=1}^d -\frac{i}{2}u_j^2\cot\alpha_j} \left[\prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right] \\ &\quad \times \int_{\mathbb{R}^d} \mathcal{F}_\alpha f(u) g(k) e^{\sum_{j=1}^d (\frac{i}{2}(k_j^2+u_j^2)\cot\alpha_j - iu_jk_j \operatorname{cosec}\alpha_j)}\, dk \\ &= \left[\prod_{j=1}^d \sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right] e^{\sum_{j=1}^d -\frac{i}{2}u_j^2\cot\alpha_j} \mathcal{F}_\alpha f(u)\mathcal{F}_\alpha g(u).\end{aligned}$$

□

Theorem 8 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. $L^1_w(\mathbb{R}^d)$ is a Banach algebra under Θ convolution.*

Proof It is well known that $L^1_w(\mathbb{R}^d)$ is a Banach space [2]. Let $f, g \in L^1_w(\mathbb{R}^d)$, then we have

$$\begin{aligned}\|f\Theta g\|_{1,w} &= \int_{\mathbb{R}^d} |f\Theta g| w(x)\, dy \\ &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} f(y)g(x-y) e^{\sum_{j=1}^d iy_j(y_j-x_j)\cot\alpha_j}\, dy\right| w(x)\, dx \\ &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|g(x-y)\right| w(x-y)\, dx\right) \left|f(y)\right| w(y)\, dy \\ &= \|g\|_{1,w} \int_{\mathbb{R}^d} \left|f(y)\right| w(y)\, dy \\ &= \|g\|_{1,w}\|f\|_{1,w}. \end{aligned} \tag{4}$$

It is easy to show that the other conditions of the Banach algebra are satisfied. □

Theorem 9 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is a Banach Θ-convolution module over $L_w^1(\mathbb{R}^d)$.*

Proof It is well known that $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is a Banach space by Theorem 2. Let $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ and $g \in L_w^1(\mathbb{R}^d)$. By using the inequality (4), we get

$$\begin{aligned}
\left\|\mathcal{F}_\alpha(f\Theta g)\right\|_{p,\omega} &= \left\|\left[\prod_{j=1}^{d}\sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right]e^{\sum_{j=1}^{d}-\frac{i}{2}u_j^2\cot\alpha_j}\mathcal{F}_\alpha f(u)\mathcal{F}_\alpha g(u)\right\|_{p,\omega} \\
&= \left|\prod_{j=1}^{d}\sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right|\left(\int_{\mathbb{R}^d}\left|\mathcal{F}_\alpha f(u)\right|^p\left|\mathcal{F}_\alpha g(u)\right|^p\omega^p(u)\,du\right)^{\frac{1}{p}} \\
&= \left|\prod_{j=1}^{d}\sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right|\left(\int_{\mathbb{R}^d}\left|\mathcal{F}_\alpha f(u)\right|^p\left|\prod_{j=1}^{d}\sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right|^p\right. \\
&\quad \times\left.\left|\int_{\mathbb{R}^d}g(t)e^{\sum_{j=1}^{d}(\frac{i}{2}(u_j^2+t_j^2)\cot\alpha_j-iu_jt_j\operatorname{cosec}\alpha_j)}\,dt\right|^p\omega^p(u)\,du\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^d}\left|\mathcal{F}_\alpha f(u)\right|^p\left(\int_{\mathbb{R}^d}\left|g(t)\right|dt\right)^p\omega^p(u)\,du\right)^{\frac{1}{p}} \\
&= \|g\|_1\left(\int_{\mathbb{R}^d}\left|\mathcal{F}_\alpha f(u)\right|^p\omega^p(u)\,du\right)^{\frac{1}{p}} \\
&\leq \|g\|_{1,w}\|\mathcal{F}_\alpha f\|_{p,\omega}. \qquad (5)
\end{aligned}$$

Combining (4) and (5), we obtain

$$\begin{aligned}
\|f\Theta g\|_{A_{\alpha,p}^{w,\omega}} &= \|f\Theta g\|_{1,w} + \left\|\mathcal{F}_\alpha(f\Theta g)\right\|_{p,\omega} \\
&\leq \|g\|_{1,w}\|f\|_{1,w} + \|g\|_{1,w}\|\mathcal{F}_\alpha f\|_{p,\omega} \\
&= \|f\|_{A_{\alpha,p}^{w,\omega}}\|g\|_{1,w}.
\end{aligned}$$

This is the desired result. It is easy to see that the other conditions of the module are satisfied. □

3 Inclusion properties of the space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$

Proposition 10 *For every $0 \neq f \in A_{\alpha,p}^{w,1}(\mathbb{R}^d)$ there exists $c(f) > 0$ such that*

$$c(f)w(x) \leq \|T_x f\|_{A_{\alpha,p}^{w,1}} \leq w(x)\|f\|_{A_{\alpha,p}^{w,1}}.$$

Proof Let $0 \neq f \in A_{\alpha,p}^{w,1}(\mathbb{R}^d)$. By [12], there exists $c(f) > 0$ such that

$$c(f)w(x) \leq \|T_x f\|_{1,w} \leq w(x)\|f\|_{1,w}. \qquad (6)$$

By using (6) and the equality $\|\mathcal{F}_\alpha(T_x f)\|_p = \|\mathcal{F}_\alpha f\|_p$, we obtain

$$\begin{aligned}
c(f)w(x) &\leq \|T_x f\|_{1,w} \leq \|T_x f\|_{1,w} + \left\|\mathcal{F}_\alpha(T_x f)\right\|_p \\
&\leq w(x)\|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_p
\end{aligned}$$

$$\leq w(x)\|f\|_{1,w} + w(x)\|\mathcal{F}_\alpha f\|_p$$
$$= w(x)\|f\|_{A^{w,1}_{\alpha,p}}$$

for all $f \in A^{w,1}_{\alpha,p}(\mathbb{R}^d)$. □

Lemma 11 *Let w_1, w_2, ω_1 and ω_2 be weight functions on $\mathbb{R}^d$. If $A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d)$, then $A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d)$ is a Banach space under the norm $|||f||| = \|f\|_{A^{w_1,\omega_1}_{\alpha,p}} + \|f\|_{A^{w_2,\omega_2}_{\alpha,p}}$.*

Proof Let $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d), |||\cdot|||)$. Then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d), \|\cdot\|_{A^{w_1,\omega_1}_{\alpha,p}})$ and $(A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d), \|\cdot\|_{A^{w_2,\omega_2}_{\alpha,p}})$. As these spaces are Banach spaces, there exist $f \in A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d)$ and $g \in A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d)$ such that $\|f_n - f\|_{A^{w_1,\omega_1}_{\alpha,p}} \to 0$, $\|f_n - g\|_{A^{w_2,\omega_2}_{\alpha,p}} \to 0$. Using the inequalities $\|\cdot\|_1 \leq \|\cdot\|_{1,w_1} \leq \|\cdot\|_{A^{w_1,\omega_1}_{\alpha,p}}$ and $\|\cdot\|_1 \leq \|\cdot\|_{1,w_2} \leq \|\cdot\|_{A^{w_2,\omega_2}_{\alpha,p}}$, we obtain $\|f_n - f\|_1 \to 0$ and $\|f_n - g\|_1 \to 0$. Also $\|f - g\|_1 \leq \|f_n - f\|_1 + \|f_n - g\|_1$, we have $f = g$. Hence $|||f_n - f||| \to 0$ and $f \in A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d)$. That means $(A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d), |||\cdot|||)$ is a Banach space. □

Theorem 12 *Let w_1 and w_2 be weight functions on $\mathbb{R}^d$. Then $A^{w_1,1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,1}_{\alpha,p}(\mathbb{R}^d)$ if and only if $w_2 \prec w_1$.*

Proof Suppose that $w_2 \prec w_1$. Thus there exists $c_1 > 0$ such that $w_2(x) \leq c_1 w_1(x)$ for all $x \in \mathbb{R}^d$. Also let $f \in A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$. Then we write

$$\|f\|_{1,w_2} \leq c_1\|f\|_{1,w_1} < \infty.$$

Hence we have

$$\|f\|_{A^{w_2,1}_{\alpha,p}} = \|f\|_{1,w_2} + \|\mathcal{F}_\alpha f\|_p \leq c_1\|f\|_{1,w_1} + c_1\|\mathcal{F}_\alpha f\|_p = c_1\|f\|_{A^{w_1,1}_{\alpha,p}}.$$

Therefore, $A^{w_1,1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,1}_{\alpha,p}(\mathbb{R}^d)$.

Conversely, suppose that $A^{w_1,1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,1}_{\alpha,p}(\mathbb{R}^d)$. For every $f \in A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$, we have $f \in A^{w_2,1}_{\alpha,p}(\mathbb{R}^d)$. By Proposition 10, there are constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 w_1(x) \leq \|T_x f\|_{A^{w_1,1}_{\alpha,p}} \leq c_2 w_1(x) \tag{7}$$

and

$$c_3 w_2(x) \leq \|T_x f\|_{A^{w_2,1}_{\alpha,p}} \leq c_4 w_2(x) \tag{8}$$

for all $x \in \mathbb{R}^d$. It is well known from Lemma 11 that the space $A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$ is a Banach space under the norm $|||f|||$, $f \in A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$. Then by the closed graph theorem the norms $\|\cdot\|_{A^{w_1,1}_{\alpha,p}}$ and $\|\cdot\|_{A^{w_2,1}_{\alpha,p}}$ are equivalent on $A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$. So, there exists $c > 0$ such that $\|f\|_{A^{w_2,1}_{\alpha,p}} \leq \|f\|_{A^{w_1,1}_{\alpha,p}}$ for all $f \in A^{w_1,1}_{\alpha,p}(\mathbb{R}^d)$. Moreover, as $T_x f \in A^{w_2,1}_{\alpha,p}(\mathbb{R}^d)$, we have

$$\|T_x f\|_{A^{w_2,1}_{\alpha,p}} \leq c\|T_x f\|_{A^{w_1,1}_{\alpha,p}}. \tag{9}$$

Then, combining (7), (8), and (9), we obtain

$$c_3 w_2(x) \le \|T_x f\|_{A^{w_2,1}_{\alpha,p}} \le c\|T_x f\|_{A^{w_1,1}_{\alpha,p}} \le cc_2 w_1(x).$$

Thus, $w_2(x) \le \frac{cc_2}{c_3} w_1(x)$. Let $\frac{cc_2}{c_3} = k$. Then we find $w_2(x) \le k w_1(x)$ for all $x \in \mathbb{R}^d$. □

Proposition 13 *Let w_1, w_2, ω_1 and ω_2 be weight functions on $\mathbb{R}^d$. If $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$, then $A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d)$.*

Proof Assume that $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$. Then there exist $c_1, c_2 > 0$ such that $w_2(x) \le c_1 w_1(x)$ and $\omega_2(x) \le c_2 \omega_1(x)$ for all $x \in \mathbb{R}^d$. Let $f \in A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d)$. As $f \in L^1_{w_1}(\mathbb{R}^d)$ and $\mathcal{F}_\alpha f \in L^p_{\omega_1}(\mathbb{R}^d)$, we have $\|f\|_{1,w_2} \le c_1\|f\|_{1,w_1} < \infty$ and $\|\mathcal{F}_\alpha f\|_{p,\omega_2} \le c_2\|\mathcal{F}_\alpha f\|_{p,\omega_1} < \infty$. Hence, we obtain $f \in A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d)$, and then $A^{w_1,\omega_1}_{\alpha,p}(\mathbb{R}^d) \subset A^{w_2,\omega_2}_{\alpha,p}(\mathbb{R}^d)$. □

4 Duality

Let the mapping $\Phi : A^{w,\omega}_{\alpha,p}(\mathbb{R}^d) \to L^1_w(\mathbb{R}^d) \times L^p_\omega(\mathbb{R}^d)$ be defined by $\Phi(f) = (f, \mathcal{F}_\alpha f)$ for $1 \le p < \infty$ and let $H = \Phi(A^{w,\omega}_{\alpha,p}(\mathbb{R}^d))$. Then

$$\|\Phi(f)\| = \|(f, \mathcal{F}_\alpha f)\| = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p,\omega}$$

is a norm on H for all $f \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. Moreover, we define a set K as

$$K = \biggl\{(\varphi,\psi) : \bigl((\varphi,\psi) \in L^\infty_{w^{-1}}\bigl(\mathbb{R}^d\bigr) \times L^{p'}_{\omega^{-1}}\bigl(\mathbb{R}^d\bigr)\bigr), \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx + \int_{\mathbb{R}^d} \mathcal{F}_\alpha f(y)\psi(y)\,dy = 0 \text{ for all } (f, \mathcal{F}_\alpha f) \in H\biggr\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The following proposition is proved by the duality theorem, Theorem 1.7 in [15].

Proposition 14 *Let $1 \le p < \infty$, and w and ω be weight functions on $\mathbb{R}^d$. The dual space of $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is isomorphic to $L^\infty_{w^{-1}}(\mathbb{R}^d) \times L^{p'}_{\omega^{-1}}(\mathbb{R}^d)/K$ where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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