On function-theoretic conditions characterizing compact composition operators on H^2

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Abstract: For a holomorphic self-map φ of the unit disk of the complex plane, the compactness of the composition operator $C_\varphi(f) = f \circ \varphi$ on the Hardy spaces is known to be equivalent to the various function theoretic conditions on φ, such as Shapiro's Nevanlinna counting function condition, MacCluer's Carleson measure condition, Sarason condition and Yanagihara-Nakamura condition, etc. A direct function-theoretic proof of Shapiro's condition and Sarason's condition was recently given by Cima and Matheson. We give another direct function-theoretic proof of the equivalence of these conditions by use of Stanton's integral formula.

Key words: Composition operator; Nevanlinna counting function; Sarason condition; outer function.

1. Introduction. Let φ be a holomorphic self-map of the open unit disk D in the complex plane. We will assume $\varphi(0) = 0$, only for the simplicity, throughout the paper. There have been discovered several differently looking conditions on φ characterizing the compactness of the composition operator $C_\varphi(f) = f \circ \varphi$ on the spaces like the Hardy space H^p, $L^1(\partial D)$, the harmonic Hardy space h^1, the Smirnov class N^+, and the Nevanlinna class N, etc. See [10, 9, 13, 14, 1, 2, 3]. The compactness of C_φ on these spaces turns out to be equivalent to the compactness of C_φ on H^2 [10, 9, 13, 14, 3]. Consequently, all the function theoretic conditions, different each other in appearance, are equivalent. Cima and Matheson [4] has recently given a function-theoretic proof of the equivalence of Shapiro's Nevanlinna counting function condition and Sarason's condition, (a) and (d) below, by use of Aleksandrov measures.

The main purpose of the paper is to provide another direct function-theoretic proof of the equivalence of the conditions on φ without recourse to the compactness of the composition operator C_φ. The Stanton's integral formula plays an essential role in our proof. The precise equivalent conditions are stated in the following theorem. We added (b) to the list as a new condition to have a smooth stream of proofs.

Theorem. *For a holomorphic self-map φ of D with $\varphi(0) = 0$, the following conditions are equivalent*:

(a) $\displaystyle\lim_{|w|\nearrow 1} \frac{N_\varphi(w)}{\log(1/|w|)} = 0.$

(b) $\exp\left(\dfrac{\eta+\varphi}{\eta-\varphi}\right)$ *is an outer function for the Nevanlinna class N for every $\eta \in \partial D$.*

(c) *For every $\eta \in \partial D$ there exists $k_\eta(\zeta) \geq 0$ in $L^1(\partial D)$ such that*

$$\frac{\eta+\varphi(z)}{\eta-\varphi(z)} = \int_{\partial D} \frac{\zeta+z}{\zeta-z} k_\eta(\zeta)\, d\sigma(\zeta).$$

In this case, $k_\eta(\zeta) = (1-|\varphi^*(\zeta)|^2)/(|\eta-\varphi^*(\zeta)|^2)$ a.e. $\zeta \in \partial D$, where $\varphi^*(\zeta) = \lim_{r\nearrow 1} \varphi(r\zeta)$ is the radial limit which exists a.e. $\zeta \in \partial D$, and σ is the normalized Lebesgue measure on ∂D.

(d) $\displaystyle\int_{\partial D} \frac{1-|\varphi^*(\zeta)|^2}{|\eta-\varphi^*(\zeta)|^2} d\sigma(\zeta) = 1, \quad \forall \eta \in \partial D.$

(e) $\displaystyle\lim_{|a|\nearrow 1} \int_{\partial D} \frac{1-|a|^2}{|1-\bar{a}\varphi^*(\zeta)|^2} d\sigma(\zeta) = 0.$

(f) *The pull-back measure $\sigma \circ (\varphi^*)^{-1}$ is a vanishing Carleson measure on $\overline{D}$.*

The condition (a) was shown to be a necessary and sufficient condition for C_φ to be compact on H^2 by Shapiro [10], where N_φ is the Nevanlinna counting function for φ, given by $N_\varphi(w) = \sum_{\varphi(z)=w} \log(1/|z|)$, $w \in \varphi(D)$, 0 if $w \notin \varphi(D)$. The condition (d) was proved by Sarason [9] to be equivalent to the compactness of C_φ on $L^1(\partial D)$, acting

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on the Poisson integral of functions of $L^1(\partial D)$. The compactness of C_φ on $L^1(\partial D)$ and H^2 was shown to be equivalent by Shapiro and Sundberg [13]. Yanagihara and Nakamura [14] showed that the condition (c) is equivalent to the compactness of C_φ on the Smirnov class N^+, which was shown to be equivalent to the compactness of C_φ on H^2 by Choa, Kim and Shapiro [3]. The condition (e) is known to be equivalent to the vanishing Carleson measure condition (f)(See [6, 15]) which was shown to be equivalent to the compactness of C_φ on H^2 by MacCluer [7]. From this short history we know indriectly the equivalence of all the conditions above except for (b). Our direct function theoretic proofs go as follows: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (a). A direct function theoretic proof of (e) $\Longleftrightarrow$ (f) is well-known and is omitted here. See [6, 15].

2. Prerequisites. The materials in 2.1 and 2.2 are well known and summarized shortly.

2.1. H^p, N and N^+. The Nevanlinna class N is the algebra of holomorphic functions f on D for which

$$\sup_{0\le r<1}\int_{\partial D}\log^+|f(r\zeta)|d\sigma(\zeta)<\infty,$$

or equivalently,

$$\|f\|_N \overset{\text{def}}{=} \sup_{0\le r<1}\int_{\partial D}\log(1+|f(r\zeta)|)d\sigma(\zeta)<\infty. \tag{2.1}$$

For $f\in N$, the radial limit

$$f^*(\zeta)\overset{\text{def}}{=}\lim_{r\nearrow 1}f(r\zeta)$$

is known to exist a.e. $\zeta\in\partial D$. The Smirnov class N^+ is the subalgebra of N consisting of holomorphic functions f for which

$$\lim_{r\nearrow 1}\int_{\partial D}\log^+|f(r\zeta)|d\sigma(\zeta)=\int_{\partial D}\log^+|f^*(\zeta)|d\sigma(\zeta). \tag{2.2}$$

The Hardy space $H^p(0<p\le\infty)$ is defined as the collection of holomorphic functions f on D for which

$$\|f\|_{H^p}\overset{\text{def}}{=}\begin{cases}\sup\limits_{0\le r<1}M_p(r,f), & 0<p<\infty,\\ \sup\limits_{z\in D}|f(z)|, & p=\infty\end{cases} \tag{2.3}$$

is finite, where

$$M_p(r,f)=\left(\int_{\partial D}|f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}.$$

We have the following containment relations: $N\supsetneqq N^+\supsetneqq H^p\supsetneqq H^q(0<p<q\le\infty)$.

2.2. Canonical factorization theorem. For a function f in N, we have the unique canonical factorization:

$$f(z)=B(z)\cdot\frac{S_{\mu_1}(z)}{S_{\mu_2}(z)}\cdot F(z), \tag{2.4}$$

where B is the Blaschke product with zeros α_n of f in D:

$$B(z)=\prod_n\frac{|\alpha_n|}{\alpha_n}\frac{\alpha_n-z}{1-\overline{\alpha_n}z},$$

$S_{\mu_i}(i=1,2)$ is the singular inner function defined by

$$S_{\mu_i}(z)=\exp\left(-\int_{\partial D}\frac{\zeta+z}{\zeta-z}d\mu_i(\zeta)\right),$$

where $\mu_i\ge 0$, $\mu_i\perp d\sigma$, $d\mu_1\perp d\mu_2$ and F is the outer function

$$F(z)=\lambda\exp\left(\int_{\partial D}\frac{\zeta+z}{\zeta-z}\log h(\zeta)d\sigma(\zeta)\right),$$

where $|\lambda|=1$, $h\ge 0$, $\log h\in L^1(\partial D)$. In this case, $h(\zeta)=|f^*(\zeta)|$ a.e. $\zeta\in\partial D$. It is interesting and useful to know that for $f\in N$, $\mu_2(\partial D)=\lim_{a\to 0}\|af\|_N$ and

$$\|f\|_N=\int_{\partial D}\log(1+|f^*(\zeta)|)d\sigma(\zeta)+e\mu_2(\partial D). \tag{2.5}$$

We also note that $f\in N^+$ iff $S_{\mu_2}\equiv 1$, or $\mu_2(\partial D)=0$ and that the outer functions are the only invertible elements in the algebra N^+. See [12].

2.3. Stanton's formula [5]. *If g is a subharmonic function on D and φ is a holomorphic selfmap of D, then*

$$\lim_{r\nearrow 1}\int_{\partial D}g(\varphi(r\zeta))d\sigma(\zeta)=g(\varphi(0))+\frac{1}{2}\int_D\triangle g(w)N_\varphi(w)dA(w) \tag{2.6}$$

where $dA(w)=du\,dv/\pi$ with $w=u+iv$ and $\triangle$ is the Laplacian in distributional sense.

A couple of special cases of Stanton's formula are essential in our proof of the equivalences of the function-theoretic conditions on φ. By taking $g(w)=|w|^2$, we have

$$(2.7)\qquad \begin{aligned}||f\circ\varphi||_{H^2} &= |f(\varphi(0))|^2+\\ &2\int_{\partial D}|f'(w)|^2N_\varphi(w)dA(w).\end{aligned}$$

If we take $g(w)=\log(1+|w|)$, we get

$$(2.8)\qquad \begin{aligned}||f\circ\varphi||_N &= \log(1+|f(\varphi(0))|)+\\ &\frac{1}{2}\int_D\frac{|f'(w)|^2}{(1+|f(w)|)^2|f(w)|}N_\varphi(w)dA(w).\end{aligned}$$

3. Proofs of equivalences. We now proceed to the proofs of equivalences.

(a)$\Longrightarrow$(b): Let $f_\eta(z)=\exp((\eta+z)/(\eta-z))$ with $\eta\in\partial D$. Then

$$\frac{1}{f_\eta\circ\varphi(z)}=\exp\left(-\frac{\eta+\varphi(z)}{\eta-\varphi(z)}\right),$$

as a bounded holomorphic function, belongs to N^+ and has the following canonical representation $1/(f_\eta\circ\varphi(z))=S_\mu(z)F(z)$, or $f_\eta\circ\varphi(z)=\tilde{F}(z)/S_\mu(z)$, where $\tilde{F}(z)=1/F(z)$ is the outer function and S_μ is the singular inner function associated to $f\circ\varphi$. Since $f_\eta\circ\varphi\in N$, it suffices to show $S_\mu\equiv1$, or $\mu(\partial D)=0$. We note from (2.5) that $\mu(\partial D)=\lim_{a\to0}||a(f\circ\varphi)||_N$. By (2.8) we have the following representation:

$$(3.1)\qquad \begin{aligned}\|a(f_\eta\circ\varphi)\|_N &= \log(1+|a||f_\eta\circ\varphi(0)|)+\\ &\frac{1}{2}\int_D\frac{|a||f'_\eta(w)|^2}{(1+|a||f_\eta(w)|)^2|f_\eta(w)|}N_\varphi(w)dA(w).\end{aligned}$$

Now let $\varepsilon>0$ and choose $r\in(0,1)$ so that $N_\varphi(w)<\varepsilon\log(1/|w|)$ $(r\le|w|<1)$. We write $\int_D=\int_{rD}+\int_{D\setminus rD}=$ (I) + (II) for the integral in (3.1).

(3.2)

$$\begin{aligned}\text{(II)} &\le \varepsilon\int_D\frac{|a||f'_\eta(w)|^2}{(1+|a||f_\eta(w)|)^2|f_\eta(w)|}\log\frac{1}{|w|}dA(w)\\ &= \varepsilon\cdot2\pi(\|af_\eta\|_N-\log(1+|a||f_\eta(0)|))\\ &\le \varepsilon\cdot2\pi\|af_\eta\|_N\le\varepsilon\cdot2\pi(|a|+1)\|f_\eta\|_N\\ &\le \varepsilon\cdot2\pi(|a|+1)(2\log2+1).\end{aligned}$$

The equality on the second line is the representation (3.1) for $\|af_\eta\|_N$. For the estimate of (I), we note that $N_\varphi(w)\le\log(1/|w|)$ since $\varphi(0)=0$.

(3.3)

$$\begin{aligned}\text{(I)} &\le |a|\int_{rD}\frac{|f'_\eta(w)|^2}{|f_\eta(w)|}N_\varphi(w)dA(w)\\ &\le |a|\int_{rD}\frac{|f'_\eta(w)|^2}{|f_\eta(w)|}\log\frac{1}{|w|}dA(w)\\ &= 4|a|\int_{rD}\exp\left(\frac{1-|w|^2}{|\eta-w|^2}\right)\frac{\log(1/|w|)}{|\eta-w|^4}dA(w)\\ &\le 4|a|\exp\left(\frac{2}{1-r}\right)\frac{1}{(1-r)^4}\int_{rD}\log\frac{1}{|w|}dA(w)\\ &= 4|a|\exp\left(\frac{2}{1-r}\right)\frac{1}{(1-r)^4}\left(\frac{r^2}{4}+\frac{r^2}{2}\log\frac{1}{r}\right).\end{aligned}$$

From these estimates (3.2) and (3.3), we have $\lim_{a\to0}\|a(f_\eta\circ\varphi)\|_N=0$.

(b)$\Longrightarrow$(c): The condition (b) implies the following canonical representation:

$$\exp\left(\frac{\eta+\varphi(z)}{\eta-\varphi(z)}\right)=$$
$$\exp\left(\int_{\partial D}\frac{\zeta+z}{\zeta-z}\cdot\frac{1-|\varphi^*(\zeta)|^2}{|\eta-\varphi^*(\zeta)|^2}d\sigma(\zeta)\right).$$

Therefore, (c) obviously follows from this.

(c)$\Longrightarrow$(d): Take the real parts of both sides of (c) and evaluate at the origin.

(d)$\Longrightarrow$(e): If $|a|<1$, $f_a(z)=\exp((1+\overline{a}z)/(1-\overline{a}z))$ is a bounded holomorphic function; so $f_a\circ\varphi$ is bounded. $f_a\circ\varphi\in N^+$, in particular. Therefore, by (2.2),

$$\lim_{r\nearrow1}\int_{\partial D}\log^+|f_a\circ\varphi(r\zeta)|d\sigma(\zeta)=$$
$$\int_{\partial D}\log^+|f_a\circ\varphi^*(\zeta)|d\sigma(\zeta).$$

This implies

$$(3.4)\qquad \int_{\partial D}\frac{1-|a|^2|\varphi^*(\zeta)|^2}{|1-\overline{a}\varphi^*(\zeta)|^2}d\sigma(\zeta)=1$$

for all a with $|a|<1$. The formula (3.4) still holds for all $|a|\le1$ because the case $|a|=1$ in (3.4) is Sarason's condition (d). Since

$$\frac{1-|a|^2}{|1-\overline{a}\varphi^*(\zeta)|^2}\le\frac{1-|a|^2|\varphi^*(\zeta)|^2}{|1-\overline{a}\varphi^*(\zeta)|^2},$$

we have

$$\lim_{|a|\to1}\int_{\partial D}\frac{1-|a|^2}{|1-\overline{a}\varphi^*(\zeta)|^2}d\sigma(\zeta)=0$$

by the Generalized Dominated Convergence Theorem [8, p. 89].

(e) $\Longrightarrow$ (a): The proof is essentially the same as Proof of Necessity in [11, p. 192] but it is included here for the sake of completeness. For a function $f_a(z)=(1-|a|^2)^{1/2}/(1-\overline{a}z)$, its H^2-norm $||C_\varphi\circ f_a||_{H^2}$ has the following form as in (2.7)

$$(3.5)\qquad \int_{\partial D} \frac{1-|a|^2}{|1-\overline{a}\varphi^*(\zeta)|^2} d\sigma(\zeta) = ||C_\varphi \circ f_a||_{H^2}^2 = |f_a \circ \varphi(0)|^2 + 2\int_D |f_a'(w)|^2 N_\varphi(w) dA(w).$$

The last integral in (3.5) has the following equivalent formulations by use of the automorphism $\varphi_a(z) = (a-z)/(1-\overline{a}z)$;

$$(3.6)\qquad \int_D |f_a'(w)|^2 N_\varphi(w) dA(w) = \frac{|a|^2}{1-|a|^2}\int_D |\varphi_a'(w)|^2 N_\varphi(w) dA(w) = \frac{|a|^2}{1-|a|^2}\int_D N_{\varphi_a\circ\varphi}(w) dA(w).$$

For $a \in D$ with $|a|$ sufficiently close to 1, the sub-averaging property [11, p. 190] implies that

$$(3.7)\qquad \int_{(1/2)D} N_{\varphi_a\circ\varphi}(w) dA(w) \geq 4 \cdot N_{\varphi_a\circ\varphi}(0) = 4N_\varphi(a).$$

Combining all the estimates (3.5)-(3.7), we have

$$N_\varphi(a) \leq \frac{1}{4}\cdot\frac{1}{2|a|^2}(1-|a|^2)\int_{\partial D} \frac{1-|a|^2}{|1-\overline{a}\varphi^*(\zeta)|^2} d\sigma(\zeta) = o(1-|a|)$$

by the hypothesis of (e).

4. Remarks. 4.1. The original Yanagihara-Nakamura condition is the condition (c) with $|\varphi^*(\zeta)| < 1$ a.e. $\zeta \in \partial D$ and the equi-absolute continuity of $\{k_\eta | \eta \in \partial D\}$. It is not hard to see that both $|\varphi^*(\zeta)| < 1$ a.e. $\zeta \in \partial D$ and the equi-absolute continuity follow from the condition (c).

4.2. It is of interest to note that the condition (b) is easily seen to be equivalent to the condition $S \circ \varphi$ is an outer function for the Nevanlinna class N for every singular inner function S.

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