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On Functional Determinants of Laplacians in Polygons and Simplicial Complexes

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Abstract: The functional determinant of an elliptic operator with positive, discrete spectrum may be defined as $e^{-Z'(0)}$, where $Z(s)$, the zeta function, is the sum $\sum_n \lambda_n^{-s}$ analytically continued in s. In this paper $Z'(0)$ is calculated for the Laplace operator with Dirichlet boundary conditions inside polygons with the topology of a disc in the Euclidean plane. Our results are complementary to earlier investigations of the determinants on smooth surfaces with smooth boundaries. Our expression can be viewed as the energy for a system of static point particles, corresponding to the corners of the polygon, with self-energy and pair interaction energy. We have completely explicit closed expressions for triangles and regular polygons with an arbitrary number of sides. Among these, there are five special cases (three triangles, the square and the circled), where the $Z'(0)$ are known by other means. One special case fixes an integration constant, and the other provide four independent analytical checks on our calculation.

1. Introduction

One of the basic integrals that arises in many parts in physics is

$$\int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^{1/2}} e^{-\frac{1}{2}xAx} = (\mathrm{Det}\, A)^{-1/2}, \tag{1}$$

where A is a real, symmetric matrix with positive eigenvalues. For instance, let (1) describe the integration of fluctuations around a classical solution in imaginary time quantum mechanics, where the Lagrangian has been expanded up to second order. The determinant of A then diverges, and both sides of (1) vanish. The equation is therefore undefined as it stands. As a basic example take a one-dimensional harmonic potential, and the fluctuations in the imaginary time interval $0 < \tau < L$. Then

$$xAx = \int_0^L d\tau((\partial_\tau x)^2 + \omega^2 x^2), \tag{2}$$

and we have Dirichlet boundary conditions for x at $\tau = 0$ and $\tau = L$. The most straightforward way to proceed is then to go back to the Gaussian integral (1), discretize the action by dividing the imaginary time interval into steps of length ε, and modify the integration measure depending on the cut-off ε so that the limit when ε goes to zero is finite. In quantum mechanics this is feasible: changing $(2\pi)^{-1/2}$ to $(2\pi\varepsilon)^{-1/2}$, and including one more factor $(2\pi\varepsilon)^{-1/2}$, turns (1) to a discrete approximation to Feynman's sum over paths, which in the limit gives

$$(\text{Renormalized}\,[\text{Det}\,A])^{-1/2} = \left(\frac{2\pi\sinh(\omega L)}{\pi}\right)^{-1/2}, \tag{3}$$

and this is the correct expression in the Greens function.

We may also observe that the eigenvalues of A are $\frac{\pi^2 n^2}{L^2} + \omega^2$, and the determinant is then formally

$$\det A \sim \prod_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{L^2} + \omega^2\right). \tag{4}$$

One way to regularize the determinant is to introduce a cut-off Λ in the product (4), check that in the limit of large Λ the result separates into one finite factor and one factor divergent with Λ, and keep the finite factor as the renormalized result. For (4), this gives the same result as (3) [26].

A regularization can also be found from the zeta function of the operator;

$$Z_A(s) = \sum \lambda_k^{-s}, \tag{5}$$

which converges when the real part of s is large enough. When this function can be analytically continued to be regular in a neighbourhood of the origin, then

$$\text{Renormalized}\,[\text{Det}\,A] = e^{-Z'_A(0)}. \tag{6}$$

For (4), this again gives the same renormalized result as (3), but the renormalization has been hidden in the analytic continuation. The zeta function method was first introduced in the context of regularizing expressions like (1) by Hawking [12], to study fluctuating fields in a background of curved space.

In general it is not evident that different regularizations give equivalent results. The proof of such equivalence is an important problem, usually involving a study of the symmetry properties of the quantity and its divergences and renormalization. One reason for the success of the zeta function regularization is its ability to leave symmetries intact. In this paper we will mostly leave such considerations aside and simply compute $Z'_A(0)$, with A the Laplace operator with Dirichlet's boundary conditions in a two-dimensional domain. The domains we consider are piecewise flat with corners at the boundary and in the interior, i.e. simplicial complexes.

Concerning alternative more direct reguarlizations of (1) in two dimensions, the most relevant result we are aware of works only for lattice laplacians, discretized on rectangular ($M \times N$) domains [8]:

$$\det A \sim 2^{5/4} e^{\frac{GMN}{\pi}} (1+\sqrt{2})^{-\frac{(M+N)}{2}} (MN)^{-1/4} \eta(q) \left(\frac{M}{N}\right)^{1/4} . \tag{7}$$

Here, in lattice units, MN is the area, $2(M+N)$ is the length of the boundary, $q = e^{-2\pi \frac{N}{M}}$ is the modular parameter, G is Catalan's constant and $\eta(q)$ is the modular form of Dedekind. There are now no less than three terms separately diverging with the size of the lattice. If, with hindsight, we use that for rectangular domains $Z_A(0) = 1/4$, we can rewrite (7) in terms of an explicit lattice spacing a:

$$\det A \sim \mu_A^{\frac{\text{Area}}{a^2}} \mu^{\frac{\text{Length}}{a}} (a^2)^{Z_A(0)} e^{-Z_A(0) \log \text{Area} - B} , \tag{8}$$

where e^{-B} are the various remaining terms in (7) which agree [1] with $e^{-Z'_A(0)}$ [16] (see Appendix C). If nothing else, it seems likely that a discretization on a rectangular grid, of a domain which is not itself of rectangular shape, will give rise to oscillating terms in the cut-off. If (8) is to be generally valid in two dimensions, it can probably only be of a smoothened discretized determinant, where the smoothening goes over cut-off scales. Assuming that this can be done, and considering that the area, the length of the boundary, and $Z_A(0)$ are all integrals of local distributions, it is possible to introduce local cut-off dependent counter-terms, such that the finite remaining piece is $e^{-Z'_A(0)}$.

It therefore at least makes sense to define the renormalized determinant to be $e^{-Z'_A(0)}$, and this is the view we take in the rest of this paper. We will use the notation $Z'_D(0)$ for our generic case: the zeta function of the laplacian with Dirichlet's boundary conditions in a simplicial domain D, with the topology of a disc. We will freely change the index of the zeta function to denote various special cases, and even contributions to the regularized determinant from some parts of the domain.

It is quite an old idea that hadrons are string-like objects [21, 23, 31] and that such excitations appear in field theories [24] such as non-abelian gauge theories [13, 34]. In lattice gauge theories, the statistical weight of a Wilson loop, when a quark and an anti-quark are taken apart for some time, is the area of the smallest area delimited by the loop. A model for strings was proposed, where the action of a surface is its area [11, 22]. It was modified [3, 6], to a form suitable for path integration [27], involving the embedding in d-dimensional external space, x^μ, and the internal two-dimensional metric, g^{ab},

$$Z \sim \int D[g^{ab}] D[x^\mu] e^{-\frac{1}{2}[\int \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\mu]} . \tag{9}$$

In addition other terms invariant under reparametrization may be included in the action (9), in particular an internal area term. Our computations are relevant to a part of the investigations of (9). with g^{ab} fixed, the integration over x^μ is just a quadratic integral with Dirichlet boundary conditions like (1). Our calculations hence give the finite piece of this determinant. As is well known, reparametrization invariance of the action in (9), gives rise to Faddeev-Popov determinants, which for smooth surfaces turn out to be determinants of laplacians acting on vector fields, with

[1] Up to a constant factor $2^{7/4}$

modified Dirichlet boundary conditions. We have not investigated these determinants. At least in a class of simplicial domains with a fixed number of corners, such a surface is its own model, and we would not have any more reparametrization invariance. In this respect, an approach closer to the simplicial discretization of (9) would seem to be more appropriate [2].

Let us now see why it may be interesting to investigate the determinants on simplicial disc-like domains. *Smooth* disc-like manifolds with *smooth boundary* can always be mapped conformally onto one another. If we denote the conformal factor $\sigma(x)$, the base metric and curvature by $\hat{g}$ and $\hat{R}$, and the base geodetic curvature of the boundary of $\hat{k}$, a celebrated result [1, 9, 10, 27] says that

$$Z'_{D'}(0) = Z'_D(0) - \frac{1}{4\pi} \int\limits_{\partial D} d\hat{s}\hat{n} \cdot \partial\sigma + \frac{1}{6\pi} \int\limits_{\partial D} d\hat{s}\hat{k}\sigma + \frac{1}{12\pi} \int\limits_{D} d^2 z \sqrt{\hat{g}} [\hat{g}^{ab} \partial_a \sigma \partial_b \sigma + \hat{R}\sigma] . \tag{10}$$

The integration constant can be computed from the upper half sphere [32]. In string theory σ is known as the Liouville field, and we will refer to (10) as the Liouville action, although we have not included the Liouville interaction.

On two-dimensional smooth surfaces, one can open up a corner with angle $2\pi\alpha$ (or $\pi\alpha$ at the boundary) with a coordinate transformation which is conformal and regular everywhere but at the corner, where it instead has a logarithmic singularity. The kinetic energy term in (10) will then be logarithmically divergent at the corner. In other words, the Liouville action (10) is divergent for domains with corners in the interior or on the boundary. On the other hand, $Z'_D(0)$ is a well defined mathematical object, and it is well known from special cases, such as rectangles and special triangles, to have a finite value, the surface being smooth or not.

It is instructive to consider the simpler quantity $Z_D(0)$. For smooth domains with smooth boundary it is a topological invariant, given by the Euler number as $\chi(D)/6$, (see for example [1, 19]). For a circle this is 1/6. For a polygon $Z_D(0)$ is a rational function of the corner angles, (see Eq. (37)), larger than 1/6. Thus if we try to regulate divergences associated with corners by rounding them off and taking the zero curvature radius limit in the end, we get the wrong result for $Z_D(0)$. We consider the unboundness of (10) for domains with corners a mirror of $Z_D(0)$'s dependence on the corner angles.

Our computation gives some more explicit results on determinants, to which one does not have access from smooth models. This can have some mathematical interest by itself. More speculatively, it is possible that a definition of determinants by a precise calculation of $Z'_D(0)$, may yield a better regularization of (9) than does (10) and its Faddeev-Popov ghosts. Certainly, such a result would go far beyond what is actually done here: we have not begun to address a computation of a sum over surfaces as in (9).

The rest of the paper is organized as follows. Section 2 contains standard results relating the zeta function to the heat kernel, in particular the variational formula crucial for our work. Section 3 contains explicit expressions for three different heat kernels we need. In Sect. 4 standard results on the short time behaviour of the heat kernel are discussed, and in Sect. 5 we compute $Z(0)$. Section 6, about the corner contribution to $Z'(0)$, contains the first hard new result in the paper. In Sect. 7 we put all pieces together and calculate $Z'(0)$ for a polygon; Eq. (62) is the general result.

In Sect. 8 our formula is applied to triangles, and checked with three earlier known cases. Section 9 gives a short outlook on further results that can be derived using the methods in this paper.

Appendix A contains asymptotic formulae for the corner contribution. In Appendix B we evaluate an integral which appears in the corner contribution to $Z'(0)$, in the special case when the opening angle is a rational fraction of π. In Appendix C we review special domains for which the spectrum is known explicitly, and $Z'_D(0)$ can be dedujced therefrom. In addition to fixing an integration constant, this provides us with four important checks of our formula.

2. Heat Kernel and Variational Formula

In this section we review for convenience some general results involving the heat kernel relevant to our investigations [1, 28, 29, 32].

The heat kernel for the Laplace operator on a domain D, and its trace, can be expressed, in terms of its eigenvalues and normalised eigenfunctions, as

$$K_D(x,y,t) = \theta(t)\,\langle x|\, e^{-\Delta t}\,|y\rangle = \theta(t)\sum_\nu \Psi_\nu(x)\Psi_\nu^*(y)e^{-\lambda_\nu t}\,, \tag{11}$$

$$\mathrm{Tr}(K_D(t)) = \theta(t)\,\mathrm{Tr}(e^{-\Delta t}) = \theta(t)\sum_\nu e^{-\lambda_\nu t}\,. \tag{12}$$

The Mellin transform of the trace is the zeta function of the operator

$$Z_D(s) = \sum_\nu \lambda_\nu^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}(e^{-\Delta t})\,. \tag{13}$$

This equation is especially convenient for computing the zeta function at $s = 0$. Since $1/\Gamma(0) = 0$, only parts of the integral singular at $s = 0$ contribute. In our case (Dirichlet boundary condition and compact domain) $\mathrm{Tr}(K(t))$ decreases monotonically and exponentially to zero as t tends to infinity, so such a singular contribution to the integral can arise only from the vicinity of $t = 0$.

This limiting behaviour of the heat kernel is easy to find. It depends on the local properties of the domain only. For an internal point on a smooth domain only the local curvature enters, and the behaviour can be found by perturbation theory around a flat domain. Likewise, for a point on a smooth boundary the curvature of the boundary enters, and the behaviour can be found by perturbation theory around a straight boundary of a flat domain. In this paper we are interested in corner contributions, so we need also the heat kernel in a sector. Fortunately, explicit expressions for the heat kernel exist in all three unperturbed cases, the plane, the half plane, and the sector.

Our main interest is in the derivative of the zeta function at $s = 0$. The sketched procedure does not work directly for this quantity since the derivative of $1/\Gamma(s)$ is nonzero at $s = 0$. Fortunately another trick is available. It is due to the simplicity of the expression for the laplacian in conformal coordinates ($g_{ab} = e^{2\sigma(x)}\delta_{ab}$),

$$\Delta = -e^{-2\sigma(x)}(\partial_1^2 + \partial_2^2)\,. \tag{14}$$

The variation of the laplacian under conformal variations of the domain is, then,

$$\delta\Delta = -2\delta\sigma\Delta\,. \tag{15}$$

Hence

$$\delta\,\mathrm{Tr}(K_D(t)) = -t\,\frac{d}{dt}\,\mathrm{Tr}(2\delta\sigma K_D(t))\,. \tag{16}$$

And, consequently, the variation of the zeta function can be written as

$$\begin{aligned}\delta Z_D(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^s\,\frac{d}{dt}\,\mathrm{Tr}(2\delta\sigma\Delta e^{-\Delta t}) \\ &= \frac{s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}(2\delta\sigma\, e^{-\Delta t})\,. \end{aligned}\tag{17}$$

Note that the variation of the eigenfunctions does not contribute since they are orthonormal, and that there are no convergence problems in connection with the partial integration since, by analyticity, one may always choose s big enough. The extra prefactor s makes it possible to apply the previous procedure also to the derivative at $s = 0$ of this expression. This is a powerful technique since it is possible to go between any two domains with this topology by conformal deformation. It is the key to obtaining analytic expressions for $Z'_D(0)$ for general domains.

3. The Sommerfeldt Kernel

Here we give explicit expressions for the three special heat kernels mentioned in Sect. 2.

The heat kernel (11) satisfies the differential equation

$$(\partial_t + \Delta_x)K_D(x,y,t) = 0 \qquad (t > 0)\,, \tag{18}$$

$$\mathrm{Limit}_{t\to 0}\, K_D(x,y,t) = \delta(x-y) \tag{19}$$

for $x \in D$, and Dirichlet's boundary conditions on ∂D. On an infinite flat plane the solution is the free heat kernel

$$K_P(x,y,t) = \theta(t)\,\frac{1}{4\pi t}\,e^{-(x-y)^2/4t}\,. \tag{20}$$

The heat kernel on the half plane $x_2 > 0$, with Dirichlet's boundary condition on $x_2 = 0$, can be obtained from the free case by the method of images,

$$K_{HP}(x,y,t) = \theta(t)\,\frac{1}{4\pi t}\,(e^{-(x-y)^2/4t} - e^{-((x_1-y_1)^2+(x_2+y_2)^2)/4t})\,. \tag{21}$$

The expression for the heat kernel on an infinite sector is due to Sommerfeld, who in 1896 solved the problem of diffraction of light by a perfectly conducting half-plane [30]. The solution takes the form of a kernel periodic in the angle variable with periodicity 4π; the difference of one "direct" and one "reflected" wave vanishes at 0 and 2π.

We need the solution to the diffusion problem in a sector with opening angle $\pi\alpha$, which is quite analogous. The solution of the diffusion problem at an interior corner with total angle $2\pi\alpha$ can be obtained in the same way, by keeping only the "direct" term. If the opening angle is of the form $\frac{\pi}{n}$ the sector can be reflected in its side $2n$ times to precisely over 2π, and the solutions to both the diffusion problem and the

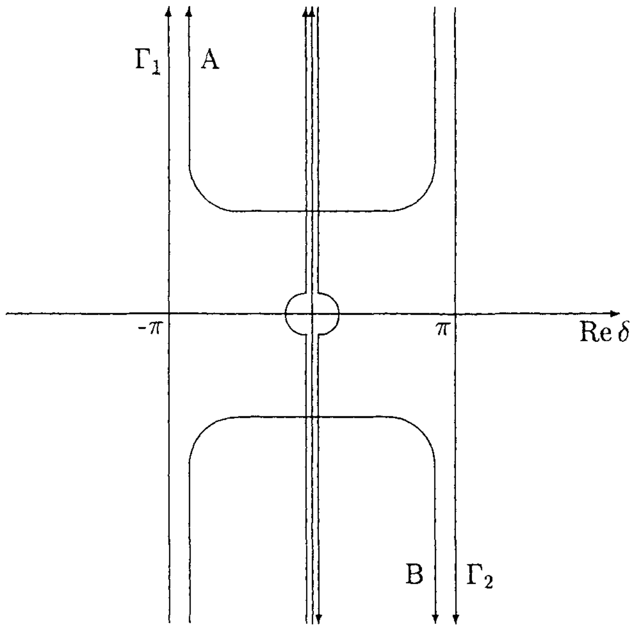

Figure. The complex δ plane with the integration controus A, B, Γ_1, Γ_2, and the shrunken integration contour used in Sect. 6. The shrunken contour is, for purpose of visualisation, slightly separated from the imaginary axis

diffraction problem are obtained by the method of images. Sommerfeldt's solution is a substitute when the reflections do not make up a full turn, and can be given (see (24)) by a certain finite number of image charges, and a correction term. It has the following integral representation:

$$K_S(r,\phi;r',\phi';t) = \frac{1}{4\pi t}\exp\left(-\frac{r^2+r'^2}{4t}\right) \times\left[\nu_\alpha\left(\frac{rr'}{2t},\phi-\phi'\right)-\nu_\alpha\left(\frac{rr'}{2t},\phi+\phi'\right)\right], \tag{22}$$

$$\nu_\alpha(a,\phi) = \frac{1}{2\pi\alpha}\int_{A+B}\exp(a\cos\delta)\frac{d\delta}{1-e^{-\frac{i(\delta+\phi)}{\alpha}}}. \tag{23}$$

A and B are paths in the plane of complex δ that go asymptotically to $\pm\pi\pm\infty$ (see the figure). Essentially this is a superposition of free heat kernels between x and y', where $|y'| = r'$ and $|x-y'|^2 = r^2+r'^2-2rr'\cos(\delta)$. In the bands $\frac{(4n+1)\pi}{2} < |\mathrm{Re}(\delta)| < \frac{(4n+3)\pi}{2}$ the contour integral can be taken to infinity since $\mathrm{Re}(x-x')^2 \to \infty$, but not in between.

K_S satisfies the heat equation because it is superposition of free heat kernels, it is symmetric in (x,y) and periodic in ϕ and ϕ' with period $2\pi\alpha$ by construction of ν_α, and it vanishes at the boundaries of the sector because it is the difference between a direct and a reflected term. Furthermore, away from the imaginary axis it is analytic in α.

By deforming A and B into the straight lines $\pi + iy$, $-\pi + iy$ and $[-\pi, \pi]$, ν_α can be written as

$$\nu_\alpha(a,\phi) = \sum_{k:-1<2\alpha k-\phi/\pi<1} \exp(a\cos(2\pi\alpha k - \phi)) - \frac{\sin(\pi/\alpha)}{2\pi\alpha} \int_{-\infty}^{\infty} \frac{\exp(-a\cosh y)}{\cosh(y/\alpha - i\phi/\alpha) - \cos\pi/\alpha}\, dy\,. \tag{24}$$

From the image charges of the "direct" term, it follows that the normalization of the kernel is correct. If for some k an image charge wanders through the line $\pm\pi$, we ought to take half the residue and the principal value of the integral in the usual fashion, but then α equals $\frac{\pi}{n}$, the prefactor of the integral is zero, and we have just a solution by images.

4. Short Times Behaviour of the Heat Kernel

The short time behaviour of the quantities appearing in Sect. 2 in the expressions for the zeta function and its variation are on the form [4, 5, 17, 19, 20]:

$$\mathrm{Tr}(K_D(t)) = \frac{A}{4\pi t} - \frac{C}{8\sqrt{\pi t}} + Z_D(0) + o(1)\,, \tag{25}$$

$$\mathrm{Tr}(2\delta\sigma K_D(t)) = \frac{\delta A}{4\pi t} - \frac{\delta C}{4\sqrt{\pi t}} - \delta Z_D(0)\log(t) + \delta Z_D'(0) - \gamma\delta Z_D(0) + o(1)\,, \tag{26}$$

These expressions are easily understood using the results of the last two sections.

For example the diagonal elements of the free heat kernel equal $1/(4\pi t)$. If the free heat kernel is used instead of the true one the trace operation produces a factor of area, so we get Eq. (25) with only the first term, and A equal to the area of D. Close to a smooth boundary it is better to compare with the heat kernel in the half plane (21). Compared to the free kernel it contains an extra term, the reflected term. When this term is integrated over D it produces, in the short time expansion, a contribution

$$-\frac{1}{4\pi t}\int_0^\infty dx_2\, e^{-x_2^2/t} = -\frac{1}{8\sqrt{\pi t}} \tag{27}$$

per unit length of smooth boundary. This explains the second term in (25); C is the length of the boundary of D. The next term in the asymptotic expansion, the t^0-term, receives contributions from several sources. For smooth domains, with smooth boundaries, surface and boundary curvatures contribute. It is explained in reference [1] how these are determined by perturbation theory. From that explanation it is also obvious that there are no other more singular terms for a smooth domain. When the domain is not smooth but has corners, either in the interior (conical singularities), or on the boundary (corners), there are additional contributions, which we determine in this paper. Finally, by inserting the asymptotic expansion in Eq. (13), and observing that only t close to zero contributes at $s = 0$, it is easy to verify that the constant term in the asymptotic expansion indeed equals $Z_D(0)$.

As for Eq. (26), applying $-t\frac{d}{dt}$ on both sides, and using Eq. (16), one verifies the singular t-dependent terms. The constant term can be verified as follows. Consider

Eq. (17). Split the integral into two by introducing a small cut-off, ε, and dividing the integration interval into two, $(0, \varepsilon)$ and (ε, ∞). Drop the second integral as it can contribute neither to $\delta Z(0)$ nor to $\delta Z'(0)$, since the prefactor has a double zero at $s = 0$. Insert the short time expansion (26) into the remaining term. Choose $s > 1$ so that the integral converges, and perform the integration, term by term. Then, by analytic continuation, let s tend to zero.

As a side remark we compare the zeta function definition of the determinant of the laplacian with its definition, in for example reference [1] as the integral

$$\log(\det \Delta)_\varepsilon = -\int_\varepsilon^\infty \frac{dt}{t} \operatorname{Tr}(K_D(t)) . \tag{28}$$

Using Eqs. (16) and (26), we see that the variation of this integral under conformal deformations is precisely the negative of (26), with t replaced by ε. In other words, this determinant equal our determinant, $\exp(-Z'(0))$, up to a simpler factor $\exp(\gamma Z(0))$, and cut-off, ε, dependent factors involving area, circumference, and $Z(0)$.

5. The Corner Contribution to $Z(0)$

According to Sect. 2, $Z_D(0)$, and the variation of $Z'_D(0)$ under an infinitesimal conformal variation of the domain, can be obtained as traces, i.e. as integrals over the domain, of quantities which only depend on the local properties of the domain. And according to Sect. 4 it is sufficient, for getting them, to determine the constant terms in the short time expansions of $\operatorname{Tr}(K_D(t))$ and $\operatorname{Tr}(2\delta\sigma K_D(t))$.

In this and the next section we consider a part of this problem. We assume that the domain has a corner with opening angle $\pi\alpha$, and that the infinitesimal conformal deformation changes this opening angle. We restrict our attention to what happens very close to this corner, and assume that we may approximate it with an infinite sector (with a large distance cut-off, when needed).

In this section we consider the heat kernel, i.e. we determine $Z_S(0)$, which was first given as an integral by Kac [17], and in explicit form by Ray (cited in [19]). We will here follow the procedure used for internal corners by Dowker [7].

Sommerfeldt's heat kernel (23) is the difference of an "image term" and a "direct term," and so is its trace. As in Sect. 3 we deform the integration contour, $A + B$, into two straight lines, $\Gamma_1 + \Gamma_2$, encircling the strip $|\operatorname{Im}\delta| < \pi$ in the extended complex δ-plane. Then we have to subtract the residues of the poles passed. In the direct term there is one such pole, at $\delta = 0$. In the reflected term, which depends on the angular coordinate φ, there is a pole at $\delta = -2\varphi$ if $-2\varphi > -\pi$, and a pole at $\delta = 2\pi\alpha - 2\varphi$ if $2\pi\alpha - 2\varphi < \pi$. If $\alpha < 1/2$ there are even more poles in the reflected term, but since the final result is analytic in α it is sufficient, to consider the case $\alpha > 1/2$, and we do so here. The density of the trace of Sommerfeldt's heat kernel is thus divided into four pieces (r and φ are polar coordinates in the sector)

$$K_S(r, \varphi, r, \varphi, t) = -I_\alpha(r, \varphi, t) + I_\alpha(r, 0, t) + P_{1\alpha}(r, \varphi, t) + P_{2\alpha}(r, 0, t) , \tag{29}$$

which we now consider one by one.

From the pole in the direct term comes the piece $P_{2\alpha}(r, 0, t) = 1/(4\pi t)$. This is precisely the first term in the asymptotic expansion (25). It makes no contribution to $Z_S(0)$.

$P_{1\alpha}(r,\varphi,t)$ comes from the two poles in the reflected term. Together the two residues manifest the $\varphi \to \pi/\alpha - \varphi$ symmetry of the system. They contribute the same to the trace, so for our purpose we may replace them by twice the first one, i.e.

$$P_{1\alpha}(r,\varphi,t) \to -\frac{2}{4\pi t}\, e^{-\frac{r^2}{2t}(1-\cos 2\varphi)} . \tag{30}$$

The pole is passed only when $\varphi < \pi/2$, so in taking the trace this term should be integrated over the first quadrant (although remember, the opening angle of the sector lies in the interval $\pi/2 < \pi\alpha < \pi$). For performing this integration it is convenient to go to cartesian Coordinates, $x = r\cos(\varphi)$, $y = r\sin(\varphi)$. The contribution from P_1 to the trace is then

$$\int_0^\infty dx \int_0^\infty dy\, \frac{-2}{4\pi t}\, e^{-\frac{y^2}{t}} = \int_0^\infty dx\, \frac{-2}{8\sqrt{\pi t}} . \tag{31}$$

This is recognised as the second term in the asymptotic expansion of the heat kernel; the divergent integral is the length of the boundary of the sector. It makes no contribution to $Z_S(0)$.

The reflected integral term is

$$I_\alpha(r,\varphi,t) = \frac{1}{4\pi t}\,\frac{1}{2\pi\alpha} \int_{\Gamma_1+\Gamma_2} \frac{d\delta}{1-e^{-i(\delta+2\varphi)/\alpha}}\, e^{-(1-\cos(\delta))\frac{r^2}{2t}} . \tag{32}$$

In forming the trace, φ is integrated from zero to $\alpha\pi$. The change of variable $z = \exp(2i\varphi/\alpha)$ transforms this φ integral into a contour integral around the unit circle in the complex z plane

$$\int_0^{\alpha\pi} \frac{d\varphi}{1-e^{-i(\delta+2\varphi)/\alpha}} = \frac{\alpha}{2i} \oint_{|z|=1} \frac{dz}{z-e^{-i\delta/\alpha}} . \tag{33}$$

This contour integral is zero or $\alpha\pi$ depending on if δ lies in the upper or lower half plane. But the δ integration, around the strip $|\mathrm{Re}\,\delta| < \pi$, should be performed first. Then the contributions from the two sides of the strip will cancel in the z integral. So $I_\alpha(r,\varphi,t)$ makes no contribution to $Z_S(0)$ either.

Finally, the direct integral term is

$$-I_\alpha(r,0,t) = -\frac{1}{4\pi t}\,\frac{1}{2\pi\alpha} \int_{\Gamma_1+\Gamma_2} \frac{d\delta}{1-e^{-i\delta/\alpha}}\, e^{-(1-\cos(\delta))\frac{r^2}{2t}} . \tag{34}$$

The trace operation means integrating over the sector. For convenience we integrate r up to infinity, but note that all relevant short time contributions come from r close to zero. We get

$$\mathrm{Tr}(K_S(t))_{I-\text{part}} = -\frac{1}{8\pi} \int_{\Gamma_1+\Gamma_2} \frac{d\delta}{1-e^{-i\delta/\alpha}}\,\frac{1}{1-\cos(\delta)} . \tag{35}$$

This integral equals the residue at zero, and it gives $Z(0)$ for the sector, which we denote $Z_\alpha(0)$ since it depends on the opening angle

$$Z_\alpha(0) = \mathrm{Tr}(K_S(t))_{t^0-\text{part}} = \frac{1}{24}\left(\frac{1}{\alpha} - \alpha\right). \tag{36}$$

Since the whole contribution comes from the corner, we can immediately write down $Z_P(0)$ for an arbitrary polygon with corner angles $\pi\alpha_i$, $i = 1, \ldots, n$,

$$Z_P(0) = \frac{1}{24}\sum_{i=1}^{n}\left(\frac{1}{\alpha_i} - \alpha_i\right). \tag{37}$$

6. The Corner Contribution to $Z'(0)$

In this section we calculate $Z'_D(0)$ for the infinite sector by repeating the calculation in the last section for the variation of $\mathrm{Tr}(K_D(t))$ under a conformal deformation.

We assume that the corner with opening angle $\pi\alpha$ is described by coordinates (ϱ, θ), $\varrho > 0$, $\theta \varepsilon [0, \pi]$, such that the ordinary polar coordinates r, φ are

$$r e^{i\varphi} = z = \frac{1}{\alpha} w^\alpha e^\lambda = \frac{1}{\alpha} e^{(\varrho + i\theta)\alpha} e^\lambda \tag{38}$$

and vary the opening angle α and the scale parameter λ. The scale factor e^σ is given by

$$\sigma = \alpha\varrho + \lambda. \tag{39}$$

And its variation is

$$\delta\sigma = \delta\alpha\varrho + \delta\lambda = \frac{\delta\alpha}{\alpha}(\log(\alpha r) - \lambda) + \delta\lambda. \tag{40}$$

According to Sect. 4, we have to determine the constant term in the short time expansion of the trace of the variation of σ times the heat kernel. We do this by repeating the steps in the last section. The trace density is divided into four pieces in the same manner as there, and we take the liberty to denote them the same way, although they refer to a different density

$$2\delta\sigma(r) K_S(r, \varphi, r, \varphi, t) = -I_\alpha(r, \varphi, t) + I_\alpha(r, 0, t) + P_{1\alpha}(r, \varphi, t) + P_{2\alpha}(r, 0, t). \tag{41}$$

We consider the four pieces one by one. As in last section, P_2 depends on t as $1/t$, so it does not contribute.

From last section we know that a constant term in $\delta\sigma$ makes no P_1 contribution. But $\delta\sigma$ also has a $\log r$ term. By the same procedure as in last section, and in addition a cut off, x_0, in x, we get a contribution to the trace,

$$\int_0^{x_0} dx \int_0^\infty dy \, \frac{\delta\alpha}{\alpha} \log(x^2 + y^2) \frac{-2}{4\pi t} e^{-\frac{y^2}{t}}. \tag{42}$$

Since only small y can give relevant contributions, we perform the x integral first, then expand in a power series in y and get

$$\int_0^\infty dy \, \frac{\delta\alpha}{\alpha} ((2x_0 \log(x_0) - 2x_0) + \pi y + O(y^2)) \frac{-2}{4\pi t} e^{-\frac{y^2}{t}}. \tag{43}$$

The second term in the expansion gives the relevant t independent contribution to the trace

$$\mathrm{Tr}(2\delta\sigma K_S(t))_{P_1\, t^0-\text{part}} = -\frac{\delta\alpha}{4\alpha}\,. \tag{44}$$

As in the last section, and by the same argument as there, the reflected integral term gives no contribution. This is so because the variation of σ does not depend on φ.

There remains the contribution

$$\begin{aligned}\mathrm{Tr}(2\delta\sigma K_S(t))_{I\text{direct}} &= \int_0^\infty dr\, r \int_0^{\pi\alpha} d\varphi \Big(\frac{\delta\alpha}{\alpha} (\log(\alpha r) - \lambda) + \delta\lambda \Big) \\ &\quad \times \frac{-1}{4\pi t} \frac{1}{2\pi\alpha} \int_{\Gamma_1+\Gamma_2} \frac{d\delta}{1 - e^{-i\delta/\alpha}} e^{-(1-\cos(\delta)) \frac{t^2}{2t}}\,. \end{aligned} \tag{45}$$

The x integration can be performed using the formula

$$\int_0^\infty dx \log(x) e^{-x} = \Gamma'(1) = -\gamma\,. \tag{46}$$

After r and φ integration we have

$$\begin{aligned}\mathrm{Tr}(2\delta\sigma K_S(t))_{I\,\text{direct}} &= \frac{-1}{8\pi} \int_{\Gamma_1+\Gamma_2} \frac{d\delta}{1 - e^{-i\delta/\alpha}} \frac{1}{1 - \cos(\delta)} \\ &\quad \times \left(\frac{\delta\alpha}{\alpha} \left(\log\left(\frac{2\alpha^2 t}{1 - \cos(\delta)} \right) - \gamma - 2\lambda \right) + 2\delta\lambda \right). \end{aligned} \tag{47}$$

Apart from the logarithm term, it is the same integral as in the last section. The logarithm is real for $\delta = \pm\pi$. Therefore its branches are different on different sides of the imaginary axis. We shrink the integration contour, $\Gamma_1 + \Gamma_2$, to a small circle of radius ε around the origin plus straight lines along the rest of the imaginary axis (see the figure). The straight lines make a contribution

$$\begin{aligned} &\frac{-2\pi i\delta\alpha}{-8\pi\alpha} \int_{i\varepsilon}^{i\infty} \frac{d\delta}{1 - \cos(\delta)} \left(\frac{1}{1 - e^{-i\delta/\alpha}} - \frac{1}{1 - e^{i\delta/\alpha}} \right) \\ &= \frac{-\delta\alpha}{8\alpha} \int_\varepsilon^\infty \frac{dy}{\sinh^2(y/2)} \coth\left(\frac{y}{2\alpha} \right). \end{aligned} \tag{48}$$

A partial integration, which makes integration with respect to α easy, transforms this into (terms tending to zero when ε tends to zero are dropped)

$$\delta\alpha \left(\frac{1}{4\alpha} - \frac{1}{12} \left(1 + \frac{1}{\alpha^2} \right) - \frac{1}{\varepsilon^2} + \frac{1}{8\alpha^2} \int_\varepsilon^\infty \frac{dy}{\sinh^2(y/(2\alpha))} \coth\left(\frac{y}{2} \right) \right). \tag{49}$$

Adding the contributions from the small circle we have

$$\begin{aligned}\mathrm{Tr}(2\delta\sigma K_S(t))_{I\,\mathrm{direct}} = \frac{1}{24}\left(\frac{1}{\alpha}-\alpha\right)\left(\frac{\delta\alpha}{\alpha}(\log(4\alpha^2 t)-\gamma-2\lambda)+2\delta\lambda\right. \\ +\delta\alpha\left(\frac{1}{4\alpha}-\frac{1}{12}\left(1+\frac{1}{\alpha^2}\right)-\frac{1}{24}+\frac{1}{12}\left(1-\frac{1}{\alpha^2}\right)\right. \\ \times\log(\varepsilon)-\frac{1}{2\varepsilon^2}+\frac{1}{8\alpha^2} \\ \left.\left.\times\int_{\varepsilon}^{\infty}\frac{dy}{\sinh^2(y/(2\alpha))}\coth\left(\frac{y}{2}\right)\right)\right). \qquad (50)\end{aligned}$$

Acording to the asymptotic expansion formulas (25, 26) the coefficient of the $\log t$ term in $\mathrm{Tr}(2\delta\sigma K(t))$ should equal $-\delta Z(0)$. The present results for the sector, $-\delta Z_S(0)) = \delta\alpha(1/\alpha^2+1)/24$, and $\log t$ coefficient in (50) equal to $\delta\alpha(1/\alpha^2-1)/24$, seem to disagree with this general result. However, as is clear from the discussion in Sect. 4, the domain must be bounded. Although we have computed the localized contributions from a corner, approximating the corner with an infinite sector, in reality we must think of these contributions from a compact domain, and of the sector as e.g. a corner in a polygon. The opening angles of the corners are then not independent, because the exterior angles sum to 2π. Our variation is hence only determined up to terms leaving this sum invariant, that is, up to terms linear in $\sum_{\nu}\delta\alpha_{\nu}$.

To get the variation of the total corner contribution to $Z'(0)$, we add the P_1 contribution (44) and subtract the $-\delta Z(0)(\log t+\gamma)$ piece. We drop the scale factor λ for the rest of this section. It is straightforward to write the result as a total variation. We choose a form which exhibits its behaviour under $\alpha \to 1/\alpha$. Finally removing the variation sign we get a quantity which we call $Z'_\alpha(0)$, since it depends on the opening angle $\alpha\pi$,

$$\begin{aligned}Z'_\alpha(0) = \left[J_S(\alpha)+\frac{\gamma}{12}\left(\frac{1}{\alpha}+3+\alpha\right)-\frac{1}{4}\log 2\pi\right] \\ -\left[\frac{1}{12}\left(\frac{1}{\alpha}-\alpha\right)\log 2+\frac{1}{24}\left(\frac{1}{\alpha}+3+\alpha\right)\log\alpha\right] \\ -\alpha\left[\frac{1}{4}\log 2\pi+\zeta'(-1)\right], \qquad (51)\end{aligned}$$

$$\begin{aligned}J_s(\alpha) = \lim_{\varepsilon\to 0}\left(\int_{\varepsilon}^{\infty}\frac{dy}{y}\frac{1}{e^{y/\sqrt{\alpha}}-1}\frac{1}{e^{y\sqrt{\alpha}}-1}-\frac{1}{2\varepsilon^2}\right. \\ \left.+\left(\frac{1}{\sqrt{\alpha}}+\sqrt{\alpha}\right)\frac{1}{2\varepsilon}+\frac{1}{12}\left(\frac{1}{\alpha}+\alpha\right)\log\varepsilon\right). \qquad (52)\end{aligned}$$

We stress again that terms constant and linear in α in (51) are not determined by our derivation, we have simply chosen them at our convenience. An alternative expression for the same quantity is

$$Z'_\alpha(0) = \frac{1}{12}\left(\frac{1}{\alpha} - \alpha\right)(\gamma - \log 2) - \frac{1}{12}\left(\frac{1}{\alpha} + 3 + \alpha\right)\log\alpha + \tilde{J}(\alpha), \qquad (53)$$

$$\tilde{J}(\alpha) = \int_0^\infty dy\, \frac{1}{e^y - 1} \times \left[\left(\frac{1}{2y}\right)\left(\coth\left(\frac{y}{2\alpha}\right) - \alpha\coth\left(\frac{y}{2}\right)\right) - \frac{1}{12}\left(\frac{1}{\alpha} - \alpha\right)\right]. \qquad (54)$$

This expression shows that our $Z'_\alpha(0)$ vanishes for opening angle π. In Appendix A we discuss the derivation of the alternative expression as well as the asymptotic behaviour of $Z'_\alpha(0)$ for α large small or close to one.

7. $Z'(0)$ for a Polygon

In this section we derive an expression for $Z'(0)$ for the Laplace operator in an arbitrary polygon. The Schwartz-Christoffel formula

$$z(u) = e^{\lambda_0} \int_0^u du' \prod_{\nu=1}^{n} (u' - e^{i\phi_\nu})^{-\beta_\nu} \qquad (55)$$

maps the unit circle in the complex u-plane on an n-sided polygon with exterior angles $\pi\beta_\nu$, provided only

$$\sum_\nu \beta_\nu = 2\,. \qquad (56)$$

Conversely, an arbitrary polygon can be described this way. We can interpolate smoothly between one polygon P_1 (at $a = 0$), and another polygon P_2 (at $a = 1$) by taking

$$z_a(u) = e^{\lambda_0} \int_0^u du' \left[\prod_{\nu \in P_1} (u' - e^{i\phi_\nu})^{-\beta_\nu}\right]^a \left[\prod_{\nu \in P_2} (u' - e^{i\phi_\nu})^{-\beta_\nu}\right]^{(1-a)}. \qquad (57)$$

Every intermediate figure is then again a polygon, which means that we may without restriction consider variations of λ_0 and the β_ν, but leaving the branch points $e^{i\phi_\nu}$ fixed. This simplifies somewhat the following argument.

Since the surface is flat, and the sides are straight, all contributions to $Z'(0)$ come from the corners, and are as given in last section. The opening angle and scale factor at corner μ are described by

$$\alpha_\mu = 1 - \beta_\mu\,, \qquad (58)$$

$$\lambda_\mu = \lambda_0 - \sum_{\nu \neq \mu} \beta_\nu \log|e^{i\phi_\mu} - e^{i\phi_\nu}|\,. \qquad (59)$$

According to last section, the variation of $Z'(0)$ for the polygon can be written

$$\delta Z'_P(0) = \sum_\mu \delta\left(Z'_{1-\beta_\mu}(0) + \frac{1}{12}\left(\frac{1}{1-\beta_\mu} - 1 + \beta_\mu\right)\lambda_\mu\right) - \frac{1}{6}\sum_\mu \delta\beta_\mu\lambda_\mu\,. \qquad (60)$$

Consistency requires that also the last term is a total variation. And, indeed, using (56) and (59) one finds

$$-\frac{1}{6}\sum_\mu \delta\beta\lambda_\mu = -\frac{1}{12}\,\delta\left(\sum_\mu \beta_\mu(\lambda_\mu - \lambda_0)\right). \tag{61}$$

So (60) integrates to

$$\begin{aligned} Z'_P(0) &= \sum_\mu \left(Z'_{1-\beta_\mu}(0) + \frac{1}{12}\frac{\beta_\mu\lambda_\mu}{1-\beta_\mu} + \frac{1}{12}\beta_\mu\lambda_0\right) + C \\ &= \sum_\mu Z'_{1-\beta_\mu}(0) + \frac{\lambda_0}{12}\sum_\mu\left(\frac{\beta_\mu}{1-\beta_\mu} + \beta_\mu\right) \\ &\quad - \frac{1}{12}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}} \frac{\beta_\mu\beta_\nu}{1-\beta_\mu}\log|e^{i\varphi_\mu} - e^{i\varphi_\nu}| + C \\ &= \sum_\mu Z'_{1-\beta_\mu}(0) - \frac{1}{12}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}} \frac{\beta_\mu\beta_\nu}{1-\beta_\mu}\log|e^{(i\varphi_\mu-\lambda_0)} - e^{(i\varphi_\nu-\lambda_0)}| + C\,. \end{aligned} \tag{62}$$

Note that the coefficient of λ_0 equals $2Z_P(0)$ (37). We will determine $C = 0$ below.

This is our final formula for $Z'(0)$ for an arbitrary polygon. We find it interesting to interpret $-\frac{1}{2}Z'(0)$, the result of a functional integration as in Eqs. (1) and (6), as an action for a static configuration of point masses, with self-energy and pairwise interaction energy. The interaction energy is proportional to the logarithm of the distance *in parameter space* between the point masses. We do not know if there is a similar expression involving instead the physical distance. We note that the self-energy is not quite physical: $-\frac{1}{2}Z'_\alpha(0)$ is positive for $\alpha > 1$, but negative for $0 < \alpha < 1$ (see Eq. (83)).

It may also be instructive to consider the behaviour of $Z'(0)$ as two corners approach, as $\varphi_1 \to \varphi_2$, let's say. $Z'(0)$ the increases indefinitely. This signals a new divergence appearing when the corners meet. But the zeta function regularisation is subtle, it automatically regularises the new divergence, replacing the would-be divergent expression with a finite one of different analytic form, containing $Z'_{1-\beta_1-\beta_2}(0)$.

To find more explicit expressions we consider more special domains. In the rest of this section we specialize to a regular polygon. Then the interaction term is

$$\begin{aligned} &-\frac{1}{12}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}} \frac{\beta_\mu\beta_\nu}{1-\beta_\mu}\log|e^{i\varphi_\mu} - e^{i\varphi_\nu}| \\ &= -\frac{1}{3}\frac{1}{(n-2)}\sum_{\nu=1}^{n-1}\log(1 - e^{2\pi i\nu/n}) = -\frac{1}{3}\frac{\log n}{(n-2)}, \end{aligned} \tag{63}$$

and the radius of the circumscribed circle is, from (62)

$$R = e^{\lambda_0}\int_0^1 du'(1-u'^n)^{-2/n} = e^{\lambda_0}\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Big/\Gamma\left(1-\frac{1}{n}\right). \tag{64}$$

$Z'(0)$ for a regular polygon is, then

$$Z'(0)_{\text{regular } n\text{-gon}} = nZ'_{1-2/n}(0) - \frac{\log n}{3(n-2)} + \frac{1}{3}\left(1 + \frac{1}{n-2}\right)\lambda_0 + C\,, \tag{65}$$

where λ_0 should be replaced by the radius by (64). In the limit $n \to \infty$ we get a circle

$$Z'(0)_{\text{circle}} = -2\frac{d}{d\alpha} Z'_\alpha(0)|_{\alpha=1} + \frac{1}{3}\log R + C\,. \tag{66}$$

Equation (89) contains an explicit expression for the first term. Comparing our expression with an existing result in the literature [32] we find that the integration constant C is zero.

Using the formulae in Appendix B, $Z'(0)$ for regular polygons can be expressed in terms of the gamma function. In particular we have used (103) to check that our formula for regular polygons agrees, for the square and the equilateral triangle, with calculations based on explicit expressions for the eigenvalues, see Appendix C.

8. $Z'(0)$ for a Triangle

For triangles the origins of the corners in parameter space in the Schwarz-Christoffel transformation (55) can be moved around the unit circle independently by Moebius transformations, so we may choose, without loss of generality, $\varphi_\nu = 2\pi i\nu/3$, $\nu = 1, 2, 3$. Then the interaction term simplifies

$$\begin{aligned} &-\frac{1}{12}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}} \frac{\beta_\mu\beta_\nu}{1-\beta_\mu}\log|e^{\imath\varphi_\mu} - e^{\imath\varphi_\nu}| \\ &= -\frac{\log 3}{24}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}} \frac{\beta_\mu\beta_\nu}{1-\beta_\mu} \\ &= -\frac{\log 3}{24}\sum_{\nu=1,2,3}\left(\frac{1}{\alpha_\nu} - \alpha_\nu\right) \equiv -\log 3 Z_T(0)\,, \end{aligned} \tag{67}$$

and $Z'(0)$ for the triangle becomes

$$Z'_T(0) = \sum_{\nu=1,2,3} Z'_{\alpha_\nu}(0) + (2\lambda_0 - \log 3) Z_T(0)\,. \tag{68}$$

The scale factor λ_0 may be expressed in terms of the area, A, of the triangle

$$A = e^{2\lambda_0 - \log 3}\frac{\pi\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{2\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)} \equiv e^{2\lambda_0 - \log 3} A(\alpha_1, \alpha_2, \alpha_3)\,. \tag{69}$$

For the three special triangles discussed in Appendix C, for which $Z'(0)$ has been determined by other means, we can take out expression for $Z'(0)$, (68), and, since the α's are rational, express the Z'_α's in terms of gamma functions using Eq. (103). It may be checked that indeed the expressions in Appendix C are reproduced.

$A(\alpha_1, \alpha_2, \alpha_3)$ is a normal area for which the second term in (68) vanishes. By numerically solving for about the first one thousand eigenvalues of the laplacian in isosceles triangles, and then estimating the analytic continuation of the zeta functions, Luck found that the quotient

$$\zeta_T = \frac{Z_T'(0)}{Z_T(0)} \tag{70}$$

varies surprisingly little over the triangles (taking the normal area) [18].

We can express this ζ_T as

$$\zeta_T = \frac{\sum\limits_{p=1,2,3} Z_{\alpha_p}'(0)}{\sum\limits_{p=1,2,3} Z_{\alpha_p}(0)} . \tag{71}$$

We find that it has a maximum for the equilateral triangle, where it equals 4.591151.... The minimum of ζ_T is obtained as an angle tends to zero, and the value follows from the asymptotic expansion (83):

$$\lim_{\alpha\to 0} \frac{Z_\alpha'(0)}{Z_\alpha(0)} = 2(1 - \log 2) - 24\zeta'(-1) = 4.583813\ldots . \tag{72}$$

It is interesting to note that extremal properties of $Z'(0)$ in classes of smooth surfaces have previously been used by Osgood et al. [25].

9. Outlook

In an earlier version of this work [2] we included an alternative derivation of the results of Sects. 5 and 6, based on the concept of a zeta function density, i.e. the Mellin transform of the diagonal elements of the heat kernel (the integral of the zeta function density over the domain then gives the usual zeta function). The separation of finite and diverging pieces is then performed once and for all, and the actual computations are somewhat easier. In the interest of not introducing non-standard quantities, and keeping simple things simple, we have chosen here to just present the calculations using the short-time properties of the heat kernel, along the lines of Dowker's computation of $Z(0)$.

The derivations in Sects. 5 and 6 can be repeated almost without change for an interior corner (a conical singularity). Apart from the simple contribution to $Z_\alpha'(0)$ from the reflected term (see (44)), $Z_\alpha(0)$ and $Z_\alpha'(0)$ from an interior corner of opening angle $2\pi\alpha$ will just be twice $Z_\alpha(0)$ and $Z_\alpha'(0)$ from a corner at the boundary with opening angle $\pi\alpha$. We expect that the separation of $Z_D'(0)$ into self-energies and pair interactions is also basically correct if there are corners in the interior.

It is finally interesting to compare, in the smooth boundary limit, our action, (62) with the Liouville action, (10). In this limit all corner angles are close to π, so (62) reduces to

$$Z_P'(0) \approx Z_{\text{circle}}'(0) + \frac{1}{3}\lambda_0 - \frac{1}{12}\sum_{\substack{\mu,\nu \\ \mu\neq\nu}} \beta_\mu\beta_\nu \log\left|e^{i\varphi_\mu} - e^{i\varphi_\nu}\right| , \tag{73}$$

where the double sum can be interpreted as the ordinary electrostatic interaction energy of a collection of charges on the unit circle. The base space is now the unit

[2] Göteborg ITP 93-6, available as hep-th/9304031 at hep-th@xxx.lanl.gov

disc, with $\hat{R}$ equal to zero and $\hat{k}$ equal to one. We use coordinates (ϱ, φ) such that $u = \exp(\varrho + i\varphi)$. The conformal factor is, according to (55),

$$\sigma(u) = \log \left| \frac{dz}{du} \right| = \lambda_0 - \sum_\nu \beta_\nu \log |u - e^{i\varphi_\nu}| . \tag{74}$$

Since the unit disc is mapped to a polygon, just inside the boundary of the unit disc the argument of $\frac{dz}{d\varphi}$ is a saw-tooth function, with steps of height $\pi\beta_\nu$ at $\varphi = \varphi_\nu$. Hence, at $\varrho = 0^-$,

$$\frac{d}{d\varphi} \operatorname{Im} \log \left(\frac{dz}{du} \right) \bigg|_{\varrho=0^-} = -1 + \pi \sum_\nu \beta_\nu \delta(\varphi - \varphi_\nu) . \tag{75}$$

By the Cauchy-Riemann equations, the left-hand side of (75) equals $-\hat{n} \cdot \partial\sigma$, with $\hat{n}$ the *inwardly* directed normal. The double sum in (73) can then be rewritten as

$$-\frac{1}{12\pi^2} \int d\varphi [1 - \hat{n} \cdot \partial\sigma(0^-, \varphi)] P \int d\varphi' [1 - \hat{n} \cdot \partial\sigma(0^-, \varphi')] \log |e^{i\varphi} - e^{i\varphi'}| , \tag{76}$$

where the principal part is taken on the inner integral, and the normal derivatives are evaluated just inside the unit circles.

In the limit when the charges tend to a smooth distribution, the inner integral in (76) may be closed, and we have

$$-\frac{1}{12\pi} \int d\varphi [1 - \hat{n} \cdot \partial\sigma(0^-, \varphi)] [\lambda_0 - \sigma(0, \varphi)] . \tag{77}$$

The line integral over σ and its normal derivative is by Gauss' law seen to be identical to the kinetic energy term in (10), while the other integrals in (77) vanish. Finally, the line integral in (10) over the normal derivative vanishes, and the line integral in (10) over σ is just equal to $\frac{1}{3}\lambda_0$. Hence we recover the Liouville action result for disc-like flat domains with a smooth boundary.

A. Asymptotics of $Z'_\alpha(0)$

In this apendix we collect some formulae and investigate $Z'_\alpha(0)$ for α large or small or close to one.

Large and small α. To get an asymptotic series for small α for the integral (52) in the expression for $Z'_\alpha(0)$ we first expand the factor involving $y\sqrt{\alpha}$ in a power series using the formula

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = 1 - \frac{x}{2} + \frac{x^2}{6 \cdot 2!} - \frac{x^4}{30 \cdot 4!} + \ldots . \tag{78}$$

Then we integrate term by using the formula

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{dx}{e^x - 1} x^{s-1} . \tag{79}$$

The first terms have $s \leq 0$. Then we use the same procedure as described in the penultimate paragraph of Sect. 4 and used on Eq. (26) to get an expression for the integral from ε to infinity. We get the following expressions:

$$\begin{aligned}
\int_{\varepsilon}^{\infty} \frac{dx}{e^x - 1} &= -\log\varepsilon + \mathcal{O}(\varepsilon)\,,\\
\int_{\varepsilon}^{\infty} \frac{dx}{e^x - 1}\,\frac{1}{x} &= \frac{1}{\varepsilon} + \frac{1}{2}\log\varepsilon + \frac{\gamma}{2} - \frac{1}{2}\log 2\pi + \mathcal{O}(\varepsilon)\,,\\
\int_{\varepsilon}^{\infty} \frac{dx}{e^x - 1}\,\frac{1}{x^2} &= \frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} - \frac{1}{12}\log\varepsilon + \frac{1}{12} - \frac{\gamma}{12} - \zeta'(-1) + \mathcal{O}(\varepsilon)\,,\\
\int_{\varepsilon}^{\infty} \frac{dx}{x}\,\frac{e^x}{(e^x - 1)^2} &= \frac{1}{2\varepsilon^2} + \frac{1}{12}\log\varepsilon + \frac{\gamma}{12} + \zeta'(-1) + \mathcal{O}(\varepsilon)\,.
\end{aligned} \tag{80}$$

These formulae may be used to compute the difference between the integrals in the two expressions for $Z'_\alpha(0)$ in Sect. 6,

$$\begin{aligned}
\tilde{J}(\alpha) = J_S(\alpha) &+ \frac{1}{24}\left(\alpha + 3 + \frac{1}{\alpha}\right)\log\alpha\\
&+ \frac{1+\alpha}{4}(\gamma - \log 2\pi) - \alpha\left(\frac{\gamma}{12} + \zeta'(-1)\right),
\end{aligned} \tag{81}$$

which proves the equivalence of the two alternative expressions, (51) and (53), for $Z'_\alpha(0)$ at the end of that section.

By using the formulae we also get the asymptotic expansion for the small opening angle for $J_S(\alpha)$,

$$\begin{aligned}
J_S(\alpha) \underset{\alpha\to 0}{\sim} &\frac{1}{24}\left(\frac{1}{\alpha} + 3 + \alpha\right)\log\alpha + \left(\frac{1-\gamma}{12} - \zeta'(-1)\right)\frac{1}{\alpha}\\
&+ \frac{1}{4}(\log 2\pi - \gamma) + \sum_{n=3}^{\infty} \frac{\zeta(n) B_{n+1}}{n(n+1)}\,\alpha^n\,,
\end{aligned} \tag{82}$$

and for the corner piece of $Z'(0)$,

$$\begin{aligned}
Z'_\alpha(0) \underset{\alpha\to 0}{\sim} &\frac{1}{\alpha}\left(\frac{1}{12}(1 - \log 2) - \zeta'(-1)\right)\\
&+ \alpha\left(\frac{\gamma + \log 2}{12} - \frac{1}{4}\log 2\pi - \zeta'(-1)\right)\\
&+ \sum_{n=3}^{\infty} \frac{\zeta(n) B_{n+1}}{n(n+1)}\,\alpha^n\,.
\end{aligned} \tag{83}$$

For small α the leading behaviour is $(0.190992\ldots)/\alpha$.

Using the symmetry under $\alpha \to \frac{1}{\alpha}$ we have the asymptotic expansion for large α:

$$\begin{aligned} Z'_\alpha(0) \underset{\alpha\to\infty}{\sim} & -\frac{1}{12}\left(\frac{1}{\alpha}+3+\alpha\right)\log\alpha \\ & +\alpha\left(\frac{1}{12}(1-\log 2)-\frac{1}{4}\log 2\pi - 2\zeta'(-1)\right) \\ & +\frac{1}{\alpha}\left(\frac{\gamma-\log 2}{12}\right)+\sum_{n=3}^{\infty}\frac{\zeta(n)B_{n+1}}{n(n+1)}\alpha^{-n}, \end{aligned} \tag{84}$$

for which the leading behaviour is $-\frac{1}{12}\alpha\log\alpha$.

α close to one. We find it convenient to express $J(\alpha)$ using yet another integral and write

$$\tilde{J}(\alpha) = J(\alpha) - \alpha\Delta J, \tag{85}$$

where J has the integral representation

$$J(\alpha) = \int_0^\infty \frac{1}{e^\mu - 1}\left[\frac{1}{2\mu}\coth\left(\frac{\mu}{2\alpha}\right) - \frac{\alpha}{4\sinh^2\left(\frac{\mu}{2}\right)} - \frac{1}{12}\left(\frac{1}{\alpha}+\alpha\right)\right] d\mu, \tag{86}$$

and the difference has the integral representation

$$\begin{aligned} \Delta J &= \int_0^\infty dx\left[\frac{1}{x}\left(\frac{1}{(e^x-1)^2}+\frac{1}{2(e^x-1)}\right) - \frac{e^x}{(e^x-1)^3} - \frac{1}{6(e^x-1)}\right] \\ &= -\frac{1}{6}\gamma - \frac{5}{24} + \frac{1}{4}\log(2\pi) + \zeta'(-1). \end{aligned} \tag{87}$$

The derivative of $J(\alpha)$ can be expanded around $\alpha = 1$, and the successive terms evaluated in Mathematica, which gives

$$J'(1+\varepsilon) = -\frac{1}{36}\varepsilon + \frac{1}{16}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{88}$$

and putting the various terms together we have for a corner on the boundary:

$$\begin{aligned} Z'_{1+\varepsilon}(0) = & \left(\frac{1}{6}\log 2 - \frac{5}{24} - \frac{1}{4}\log(2\pi) - \zeta'(-1)\right)\varepsilon \\ & + \left(\frac{14}{72} + \frac{\gamma - \log 2}{12}\right)\varepsilon^2 + \left(-\frac{29}{144} - \frac{\gamma-\log 2}{12}\right)\varepsilon^3 + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{89}$$

B. Corners with Opening Angle $\pi p/q$

The integral $\tilde{J}(\alpha)$ in the corner contribution to $Z'_\alpha(0)$ can be reduced to a finite sum when α is rational, $\alpha = p/q$. The derivation of this formula is the content of this appendix. We assume that p and q are relatively prime positive integers, and start from the expression (51), (52) for $Z_\alpha(0)$. The substitution $y = \sqrt{pq}x$ brings the integral $J_S(\alpha)$ into (as before, the limit $\varepsilon \to 0$ is tacitly understood)

$$J_S(\alpha) = -\frac{1}{2\varepsilon^2} + \left(\frac{1}{\sqrt{\alpha}} + \sqrt{\alpha}\right)\frac{1}{2\varepsilon} + \frac{1}{12}\left(\frac{1}{\alpha} + 3 + \alpha\right)\log(\varepsilon)$$
$$+ \int_{\varepsilon/\sqrt{pq}}^{\infty} \frac{dx}{x}\,\frac{1}{e^{px}-1}\,\frac{1}{e^{qx}-1}\,. \tag{90}$$

The integrand now contains a rational function of e^x. This rational function is first decomposed into partial fractions

$$\frac{1}{(e^{px}-1)}\,\frac{1}{(e^{qx}-1)} = \frac{1}{pq}\,\frac{e^x}{(e^x-1)^2} - \frac{p+q}{2pq}\,\frac{1}{e^x-1}$$
$$+\frac{1}{p}\sum_{\mu=1}^{p-1}\frac{1}{(e^{2\pi i\mu q/p}-1)}\,\frac{e^{2\pi i\mu/p}}{(e^x-e^{2\pi i\mu/p})}$$
$$+\frac{1}{q}\sum_{\nu=1}^{q-1}\frac{1}{(e^{2\pi i\nu p/q}-1)}\,\frac{e^{2\pi i\nu/q}}{(e^x-e^{2\pi i\nu/q})}\,, \tag{91}$$

and then the phase in the denominator is eliminated using the identity

$$\frac{e^{2\pi i\mu/p}}{e^x - e^{2\pi i\mu/p}} = \sum_{m=1}^{p} e^{2\pi i m\mu/p}\,\frac{e^{x(p-m)}}{e^{px}-1}\,. \tag{92}$$

We now have an expression containing double sums of integrals over x. The integrals can be performed using the formula

$$\int_{\varepsilon}^{\infty}\frac{dx}{x}\,\frac{e^{(1-a)x}}{e^x-1} = \frac{1}{\varepsilon} - \left(\frac{1}{2}-a\right)(\gamma+\log\varepsilon) + \log\Gamma(a) - \frac{1}{2}\log(2\pi)\,. \tag{93}$$

The next step is to simplify the double sums. Some pieces are easy. For example, the formula for the sum of a geometric series is sufficient for checking cancellation of $1/\varepsilon^2$ and $1/\varepsilon$ terms. Other double sums are on the form

$$\sum_{m=1}^{p} f(m)\sum_{\mu=1}^{p-1}\frac{e^{2\pi i\mu m/p}}{e^{2\pi i\mu q/p}-1}\,, \tag{94}$$

with f some function of m. In order to reduce them, we introduce a parameter a less than, but close to, $a = 1-\delta$, so that we can expand the denominator. In the end the limit $a \to 1$ is taken,

$$\sum_{\mu=1}^{p-1}\frac{e^{2\pi i\mu m/p}}{e^{2\pi i\mu q/p}-1} \approx \frac{1}{1-a} - \sum_{\mu=1}^{p}\frac{e^{2\pi i\mu m/p}}{e^{2\pi i\mu q/p}-a}$$
$$= \frac{1}{1-a} - \sum_{\mu=1}^{p}\sum_{r=0}^{\infty} a^r e^{2\pi i(m+qr)\mu/p}\,.$$

We now perform the μ summation first. It gives zero, except when $m + qr$ is a multiple of p; then it gives p. For each r there is precisely one m in the interval $[1, p]$ which fulfills this condition, namely the one for which $p - m$ is the least nonnegative remainder of the division of rq by p. Let us name this remainder $R(rq, p)$, that is

$$R(q,p) \equiv q - p\left[\frac{q}{p}\right]. \tag{95}$$

The m which makes the μ summation nonzero is, then, $m(r) = p - R(rq, p)$, and

$$\begin{aligned}\sum_{m=1}^{p} f(m) \sum_{\mu=1}^{p} \sum_{r=0}^{\infty} a^r e^{2\pi i(m+qr)\mu/p} &= \sum_{r=0}^{\infty} pa^r f(p - R(rq,p)) \\ &= \frac{p}{1-a^p} \sum_{r=0}^{p-1} a^r f(p - R(rq,p)) \\ &\approx \sum_{r=0}^{p-1} \left(\frac{1}{\delta} - r + \frac{(p-1)}{2}\right) f(p - R(rq,p)).\end{aligned}$$

In the limit $\delta \to 0$ we obtain the following formula for the double sum (94):

$$\sum_{m=1}^{p} f(m) \sum_{\mu=1}^{p-1} \frac{e^{2\pi i\mu/p}}{e^{2\pi i\mu q/p} - 1} = \sum_{r=0}^{p-1} \left(r - \frac{p-1}{2}\right) f(p - R(rq,p)). \tag{96}$$

Use of this formula brings $J_S(p/q)$ to the form

$$\begin{aligned} J_S(p/q) = {} & \frac{1}{12}\left(\frac{q}{p} + 3 + \frac{p}{q}\right)\log(\varepsilon) + \frac{1}{12pq}\left(\log\left(\frac{\varepsilon}{\sqrt{pq}}\right) + \gamma + 12\zeta'(-1)\right) \\ & + \frac{1}{4}\left(\frac{1}{p} + \frac{1}{q}\right)\left(\log\left(\frac{\varepsilon}{\sqrt{pq}}\right) + \gamma - \log(2\pi)\right) \\ & + \frac{1}{p}\sum_{r=0}^{p-1}\left(r - \frac{p-1}{2}\right)\left\{\left(1 - \frac{R(rq,p)}{p}\right)\right. \\ & \times \left.\left(\log\left(\varepsilon\sqrt{\frac{p}{q}}\right) + \gamma\right) + \log\left(\Gamma\left(1 - \frac{R(rq,p)}{p}\right)\right)\right\} \\ & + \frac{1}{q}\sum_{s=0}^{q-1}\left(s - \frac{q-1}{2}\right)\left\{\left(1 - \frac{R(sp,q)}{q}\right)\right. \\ & \times \left.\left(\log\left(\varepsilon\sqrt{\frac{q}{p}}\right) + \gamma\right) + \log\left(\Gamma\left(1 - \frac{R(sp,q)}{p}\right)\right)\right\}. \end{aligned} \tag{97}$$

The sums here, except those containing $\log(\Gamma)$, can be expressed in terms of Dedekind sums. The Dedekind sum $S(p, q)$ can be defined, when p and q are nonnegative relatively prime integers, as

$$S(p,q) \equiv \frac{1}{p}\sum_{r=0}^{p-1} r\left(\frac{R(rq,p)}{p} - \frac{1}{2}\right). \tag{98}$$

There is no more explicit expression for the Dedekind sums, but they satisfy an identity called Dedekind's reciprocity relation

$$S(p,q)+S(q,p)=\frac{1}{12}\left(\frac{p}{q}+\frac{1}{pq}+\frac{q}{p}-3\right). \tag{99}$$

(By the way, this relation, together with the identity $S(q+np,p)=S(q,p)$, provides an efficient method for calculating $S(q,p)$.) By Dedekind's reciprocity relation the $\log(\varepsilon)$ terms do indeed cancel. Using Dedekind's summation symbol, and Gauss' gamma function multiplication formula

$$\prod_{m=0}^{p-1}\Gamma\left(\frac{z+m}{p}\right)=(2\pi)^{\frac{p-1}{2}}p^{\frac{1}{2}-z}\Gamma(z), \tag{100}$$

the expression for $J_S(p/q)$ can be brought to our final form

$$\begin{aligned}J_S(p/q)=&\frac{1}{2}\log(2\pi)-\frac{1}{12}\left(\frac{p}{q}+3+\frac{q}{p}\right)\gamma+\frac{1}{pq}\zeta'(-1)\\&-\frac{1}{24pq}\log(pq)+\frac{1}{2}(S(p,q)-S(q,p))\log\left(\frac{p}{q}\right)\\&+\sum_{r=1}^{p-1}\left(\frac{1}{2}-\frac{r}{p}\right)\log\Gamma\left(\frac{R(rq,p)}{p}\right)\\&+\sum_{s=1}^{q-1}\left(\frac{1}{2}-\frac{s}{q}\right)\log\Gamma\left(\frac{R(sp,q)}{q}\right).\end{aligned} \tag{101}$$

This expression is inserted into (51) to get the desired expression for $Z'_D(0)$. Since $Z'_D(0)$ lacks $p\leftrightarrow q$ symmetry, we have chosen to eliminate one of the Dedekind sums using the reciprocity relation. The final result of this appendix is then

$$\begin{aligned}Z'_{p/q}(0)=&\frac{q-p}{4q}\log(2\pi)+\frac{p^2-q^2}{12pq}\log(2)-\frac{1}{q}\left(p-\frac{1}{p}\right)\zeta'(-1)\\&-\frac{1}{12pq}\log(q)+\left(\frac{1}{4}+S(q,p)\right)\log\left(\frac{q}{p}\right)\\&+\sum_{r=1}^{p-1}\left(\frac{1}{2}-\frac{r}{p}\right)\log\Gamma\left(\frac{R(rq,p)}{p}\right)\\&+\sum_{s=1}^{q-1}\left(\frac{1}{2}-\frac{s}{q}\right)\log\Gamma\left(\frac{R(sp,q)}{q}\right),\end{aligned} \tag{102}$$

where $R(p,q)$ and $S(p,q)$ are defined by (95) and (98) respectively. For the special case $p=1$ this reduces to

$$\begin{aligned}Z'_{1/n}(0)=&\frac{1}{4}\left(1-\frac{1}{n}\right)\log(\pi)-\left(\frac{n}{12}-\frac{1}{4}+\frac{1}{6n}\right)\log(2)\\&+\left(\frac{1}{4}-\frac{1}{12n}\right)\log(n)+\sum_{\nu=1}^{n-1}\left(\frac{1}{2}-\frac{\nu}{n}\right)\log\left(\Gamma\left(\frac{\nu}{n}\right)\right).\end{aligned} \tag{103}$$

C. Integrable Domains

In this appendix we collect the cases known to us, where one has deduced the derivative of the zeta function at zero directly from explicit knowledge of the spectrum.

In a rectangle with side lengths a and b, the eigenvalues of the laplacian with Dirichlet's boundary conditions are

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right). \tag{104}$$

The zeta function around the origin is

$$\begin{aligned} Z_{\text{rectangle}}(0) &= \frac{1}{4}, \\ Z'_{\text{rectangle}}(0) &= \frac{1}{4} \log(ab) - \log \left[2^{-\frac{1}{2}} \left(\frac{b}{a} \right)^{\frac{1}{4}} \eta(q) \right], \end{aligned} \tag{105}$$

where η is the modular form of Dedekind,

$$\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \qquad q = e^{-2\pi \frac{b}{a}}. \tag{106}$$

Three triangles tile the plane by reflections in the sides: the equilateral $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; the bisected equilateral $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$, and the right angle isosceles $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. Let us also include the square, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, in our list. In these domains (and in rectangles) one can express the eigenmodes of the laplacian as superpositions of plane waves [14, 15]. We normalize their areas such that a is the length of the sides of the square and the equilateral, the lengths of the legs in the right angle isosceles, and the length of the longest side in the bisected equilateral. The eigenvalues are, then,

$$\begin{aligned} E_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),n,m} &= \left(\frac{\pi}{a}\right)^2 (n^2 + m^2), & n > 0, m > 0, \\ E_{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right),n,m} &= \left(\frac{\pi}{a}\right)^2 (n^2 + m^2), & n > m > 0, \\ E_{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),n,m} &= \left(\frac{4\pi}{3a}\right)^2 (n^2 + m^2 + nm), & n > 0, m > 0, \\ E_{\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right),n,m} &= \left(\frac{4\pi}{3a}\right)^2 (n^2 + m^2 + nm), & n > m > 0. \end{aligned} \tag{107}$$

The corresponding zeta functions can be written in terms of Riemann's zeta-function and Dirichlet's L-series [15]

$$\begin{aligned} L_3(s) &= 1 - 2^{-s} + 4^{-s} - 5^{-s} + \dots, \\ L_4(s) &= 1 - 3^{-s} + 5^{-s} - 7^{-s} + \dots, \end{aligned}$$

as,

$$\begin{aligned} Z_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} &= \left(\frac{\pi}{a}\right)^{-2s} \left(L_4(s)\zeta(s) - \zeta(2s)\right), \\ Z_{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)} &= \frac{1}{2}\left(\frac{\pi}{a}\right)^{-2s} \left(L_4(s)\zeta(s) - (1+2^{-s})\zeta(2s)\right), \\ Z_{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)} &= \left(\frac{4\pi}{3a}\right)^{-2s} \left(L_3(s)\zeta(s) - \zeta(2s)\right), \\ Z_{\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)} &= \frac{1}{2}\left(\frac{4\pi}{3a}\right)^{-2s} \left(L_3(s)\zeta(s) - (1+3^{-s})\zeta(2s)\right). \end{aligned} \tag{108}$$

And these may in turn be resolved into sums of Hurwitz' zeta functions

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(a+k)^s},$$

with different arguments a. Using [33],

$$\zeta(0,a) = \frac{1}{2} - a, \qquad \frac{d}{ds}\zeta(s,a)|_{s=0} = \log\frac{\Gamma(a)}{\sqrt{2\pi}},$$

and the normal areas (69) determined by the Schwarz-Christoffel transformations,

$$A(\alpha_0,\alpha_1,\alpha_\infty) = \frac{\pi}{2}\,\frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_\infty)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1-\alpha_\infty)}, \tag{109}$$

one finds, after some algebra, the following expressions for the determinants for the triangles:

$$Z'_{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)}(0) = \frac{1}{3}\log\frac{\text{Area}}{\mathrm{A}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)} + \log\frac{\Gamma^{\frac{1}{2}}\left(\frac{1}{3}\right)\pi^{\frac{1}{2}}3^{\frac{2}{3}}2^{-\frac{1}{6}}}{\Gamma^{\frac{1}{2}}\left(\frac{2}{3}\right)}, \tag{110}$$

$$Z'_{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)}(0) = \frac{3}{8}\log\frac{\text{Area}}{\mathrm{A}\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)} + \log\frac{\Gamma^{\frac{1}{2}}\left(\frac{1}{4}\right)\pi^{\frac{1}{2}}2^{\frac{7}{8}}}{\Gamma^{\frac{1}{2}}\left(\frac{3}{4}\right)}, \tag{111}$$

$$Z'_{\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)}(0) = \frac{5}{12}\log\frac{\text{Area}}{\mathrm{A}\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)} + \log\frac{\Gamma\left(\frac{1}{3}\right)3^{\frac{11}{24}}2^{\frac{2}{9}}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{2}{3}\right)}, \tag{112}$$

and for the square

$$Z'_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}(0) = \frac{1}{4}\log a^2 + \frac{1}{4}\log\left[\pi 2^5\,\frac{\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)}\right]. \tag{113}$$

The eigenvalues of the laplacian in the upper half sphere of radius one are $l(l+1)$. If one imposes Dirichlet's boundary conditions at the equator, one selects out the eigenmodes odd under reflection in a plane through the equator. Each eigenvalue will then be l times degenerate. The zeta function of this spectrum is

$$Z_{\text{hemisphere}}(s) = \sum_{l=1}^{\infty} \frac{l}{(l(+1))^s}, \tag{114}$$

from which one can derive [32]

$$Z'_{\text{hemisphere}}(0) = 2\zeta'(-1) + \frac{1}{2}\log 2\pi - \frac{1}{4}. \tag{115}$$

The hemisphere can be mapped conformally to a disc with radius R. The difference of the regularized determinants on the hemisphere and on the disc can then be evaluated by computing the Liouville action (10) of the conformal factor [32]:

$$Z'_{\text{disc}}(0) = \frac{1}{6}\log 2 + \frac{1}{2}\log\pi + \frac{1}{3}\log R + 2\zeta'(-1) + \frac{5}{12}. \tag{116}$$

Equation (116) has also been checked numerically to great accuracy by computing the eigenvalues from the zeros of Bessel functions, and directly investigating the analytical continuation of the zeta function [18].

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References

1. Alvarez, O.: Nucl. Phys. B **216**, 125 (1983)
2. Boulatov, D.V., Kazakov, V.A., Kostov, I.K., Migdal, A.A.: Nucl. Phys. B **275** [FS17] 641–686 (1986)
3. Brink, L., Divecchia, P., Howe, P.: Phys. Lett. B **65**, 471 (1976)
4. Brüning, J., Seeley, R.: J. Funct. Anal. **73**, 369–429 (1987)
5. Cheeger, J.: Proc. Natl. Acad. Sci. **76**, 2103–2106 (1979)
6. Deser, S., Zumino, B.: Phys. Lett. B **65**, 369 (1976)
7. Dowker, J.S.: Phys. Rev. D **36**, 620 (1987)
8. Duplantier, B., David, F.: J. Stat. Phys. **51**, 327–434 (1988)
9. Durhuus, B., Nielsen, H.B., Olesen, P., Petersen, J.L.: Nucl. Phys. B **196**, 498 (1982)
10. Durhuus, B., Olesen, P., Petersen, J.L.: Nucl. Phys. B **201**, 176 (1982)
11. Goto, T.: Prog. Theor. Phys. **46**, 1560 (1971)
12. Hawkings, S.W.: Commun. Math. Phys. **55**, 133 (1977)
13. 't Hooft, G.: Nucl. Phys. B **72**, 461 (1973)
14. Itzykson, C.: Int. J. Mod. Phys. A **1**, 65–115 (1986)
15. Itzykson, C., Luck, J.-M.: J. Phys. A **19**, 211 (1986)
16. Itzykson, C., Zuber, J.-B.: Nucl. Phys. B **275**, 580 (1986)
17. Kac, M.: Am. Math. Monthly **753**, 1 (1966)
18. Luck, J.-M.: Private communication
19. McKean, H.P., Singer, I.M.: J. Diff. Geometry **1**, 43 (1967)
20. Nagasi, M.: Kodai Math. J. **7**, 382–455 (1984)
21. Nambu, Y.: In: Symmetries and the Quark Model. Chandler, R. (ed.) New York: Gordon and Breach 1970
22. Nambu, Y.: Lectures at Copenhagen Summer Symposium (1970)

23. Nielsen, H.B.: 15th Int. Conf. on High Energy Physics, Kiev (1970)
24. Nielsen, H.B., Olesen, P.: Nucl. Phys. B **61**, 45 (1973)
25. Osgood, B., Philips, R., Sarnak, P.: J. Funct. Anal. **80**, 148–211 (1988)
26. Parisi, G.: Statistical Field Theory, Reading, MA: Addison-Wesley 1988
27. Polyakov, A.: Phys. Lett. B **103**, 207 (1981)
28. Ray, D., Singer, I.: Adv. Math. **7**, 145 (1971)
29. Schwarz, A.S.: Commun. Math. Phys. **64**, 233 (1979)
30. Sommerfeldt, A.: Optics, Chapter 5:38, New York: Academic Press 1954
31. Susskind, L.: Nuovo Cimento A **69**, 457 (1970)
32. Weisberger, W.I.: Commun. Math. Phys. **112**, 633 (1987)
33. Whittaker, E.T., Watson, G.A.: A Course of Modern Analysis. Cambridge: Cambridge University Press
34. Wilson, K.G.: Phys. Rev. **10**, 2445 (1974)

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