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ON FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. Let $U=\{z=re^{i\theta}|r<1\}$. For $k\geqq 2$ let V_k be the class of normalized analytic functions $f(z)$ such that the boundary rotation of $f(U)$ is at most $k\pi$. Let $A(r)$ be the integral

$$\int_0^{2\pi}\int_0^r |f'(\rho e^{i\theta})|^2\rho d\rho d\theta,$$

$L(r)$ the length of the image of the circle $|z|=r$ under the mapping $f(z)$. In this paper the author proves that for $z\in U$ if $f(z)\in V_k$ then

$$\limsup_{r\to 1}\left(\operatorname*{Sup}_{f\in V_k} L(r)\right)\left(\pi A(r)\log\left(\frac{1+r}{1-r}\right)\right)^{-1/2}\leqq k.$$

This generalizes to arbitrary $k\geqq 2$ the recent result of Nunokawa for the case $k=2$.

Let U be the unit disk, $|z|<1$. For fixed $k\geqq 2$, by V_k we denote the class of functions $f(z)$ satisfying the following conditions:

(i) $f(z)$ is analytic and $f'(z)\neq 0$ for $z\in U$.

(ii) $f(z)$ is normalized by the requirements $f(0)=0$ and $f'(0)=1$.

(iii) The boundary rotation of $f(U)$ is at most $k\pi$, that is

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|d\theta\leqq k\pi, \tag{1}$$

where $re^{i\theta}=z\in U$.

The concept of bounded boundary rotation originates from Löwner [1]. Extensive investigations are due to Paatero [3].

Let K denote the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, convex in U, i.e. $\operatorname{Re}\{1+zf''(z)/f'(z)\}>0$. It is clear that $V_2=K$.

Let $C(r)$ denote the image of the circle $|z|=r<1$ under some mapping $f(z)$, $L(r)$ the length of $C(r)$. By $A(r)$ we denote the integral

$$\int_0^{2\pi}\int_0^r |f'(t)|^2\rho d\rho d\theta \qquad (t=\rho e^{i\theta})$$

which is the generalized area of the image of the set $|z|\leqq r$ under the mapping $f(z)$.

Recently, Nunokawa [2, p. 332] obtained:

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THEOREM N. *If* $f(z)\in K\ (=V_2)$ *then*

$$L(r)=O\left(A(r)\log\frac{1}{1-r}\right)^{1/2}, \tag{2}$$

as $r\to 1$.

In this paper we shall extend Theorem N to all classes V_k by proving

THEOREM. *Let* $2\leqq k<\infty$. *If* $f(z)\in V_k$, *then*

$$\limsup_{r\to 1}\left(\sup_{f\in V_k}L(r)\right)\left(\pi A(r)\log\left(\frac{1+r}{1-r}\right)\right)^{-1/2}\leqq k. \tag{3}$$

PROOF. Let a be a fixed constant, $0<a<r=|z|<1$. Nunokawa [**2**, p. 333] has shown that if $f(z)$ has a nonvanishing derivative $f'(z)$ in U then

$$L(r)=\int_0^{2\pi}|zf'(z)|\,d\theta\leqq C+(A(r)J/a)^{1/2}\qquad(z=re^{i\theta}),$$

where

$$C=\int_0^a\int_0^{2\pi}|tf''(t)+f'(t)|\,d\theta d\rho\qquad(t=\rho e^{i\theta})$$

and

$$J=\int_0^r\int_0^{2\pi}\left|1+\frac{tf''(t)}{f'(t)}\right|^2d\theta d\rho.$$

We observe that since the functions $f(z)$ of the class V_k comprise a normal family there exists a constant $A=A(a,k)$ depending only upon a and k such that $C\leqq A$. An estimate for J for the class V_k follows from the following result due to Robertson [**4**, p. 1480]:

$$\int_0^{2\pi}\left|1+t\frac{f''(t)}{f'(t)}\right|^2d\theta=2\pi+\rho^2\int_0^{2\pi}\left|\frac{f''(t)}{f'(t)}\right|^2d\theta$$
$$\leqq 2\pi\left(1+\frac{k^2\rho^2}{(1-\rho^2)}\right)<\frac{2\pi k^2}{1-\rho^2}.$$

Then

$$J<2\pi k^2\int_0^r\frac{d\rho}{1-\rho^2}=\pi k^2\log\left(\frac{1+r}{1-r}\right).$$

Since $\int_0^{2\pi} |f'(z)|^2 d\theta \geqq 2\pi$ it follows that $A(r) \log ((1+r)/(1-r))$ is unbounded as $r \to 1$. Since a can be chosen arbitrarily close to one the inequality (3) follows at once.

REMARK. It is interesting to note that recently, M. S. Robertson [5, p. 97] has considered a family of functions that includes the class K as a proper subfamily. For $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, we say that $f(z) \in S_\alpha$ if

(i) $f(z)$ is analytic and $f'(z) \neq 0$ for $z \in U$.

(ii) $f(0)=0$ and $f'(0)=1$.

(iii) For $z \in U$,

$$\operatorname{Re}\left\{e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0.$$

Clearly, $S_0 = K$. We can extend inequality (2) from the class K to all classes S_α ($|\alpha| < \frac{1}{2}\pi$) as follows:

For $f(z) \in S_\alpha$ ($|\alpha| < \frac{1}{2}\pi$) we have

$$\limsup_{r\to 1}\left(\operatorname*{Sup}_{f\in S_\alpha} L(r)\right)\left(\pi A(r) \log\left(\frac{1+r}{1-r}\right)\right)^{-1/2} \leqq 2 \cos \alpha. \tag{4}$$

The proof of (4) is similar to that in our Theorem. Let $f(z) \in S_\alpha$. Then there is a function $P(z)$ with positive real part and regular in U, $P(0)=1$, $|P^{(n)}(0)| \leqq 2(n!)$, $n = 1, 2, 3, \cdots$, such that

$$e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right) = P(z) \cos \alpha + i \sin \alpha.$$

Then

$$\begin{aligned}\int_0^{2\pi}\left|1+\frac{zf''(z)}{f'(z)}\right|^2 d\theta &= 2\pi + \int_0^{2\pi}\left|\frac{zf''(z)}{f'(z)}\right|^2 d\theta \\ &= 2\pi + \int_0^{2\pi} |(P(z)-1)\cos\alpha|^2 d\theta \\ &\leqq 2\pi + \frac{8\pi r^2 \cos^2 \alpha}{1-r^2} = \frac{8\pi \cos^2\alpha}{1-r^2}(1+o(1))\end{aligned}$$

as $r \to 1$. Then (4) follows.

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REFERENCES

1. K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden*, Ber. Kön. Sächs. Ges. Wiss. Leipzig **69** (1917), 89–106.

2. M. Nunokawa, *A note on convex and Bazilevič functions*, Proc. Amer. Math. Soc. **24** (1970), 332–335. MR **40** #4437.

3. V. Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Ann. Acad. Sci. Fenn. Ser. A **33** (1931), 77 pp.

4. M. S. Robertson, *Coefficients of functions with bounded boundary rotation.* Canad. J. Math. **21** (1969), 1477–1482.

5. ———, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. **16** (1969), 97–101. MR **39** #5785.

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