

ON FUNCTIONS OF BOUNDED VARIATION RELATIVE TO A SET

P. C. BHAKTA

(Received 24 December 1968; revised 10 March 1970)

Communicated by E. Strzelecki

1. Introduction, Definitions and Notations

The present paper on functions of bounded variation relative to a set has its point of departure in the work of R. L. Jeffery [10]. Below we recapitulate Jeffery's class U of functions of bounded variation relative to a set, we state and prove a number of preliminary lemmas and theorems, we introduce a suitable pseudo-metric space (X, d) of such functions, and the analogous space (\bar{X}, ρ) , and prove that (X, d) is separable, that every closed sphere in (X, d) is compact and that (\bar{X}, ρ) is complete. These results extend known results of C. R. Adams, and C. R. Adams and A. P. Morse for the space of usual BV functions.

Let S be a subset of the closed interval $[a, b]$ such that S is dense in $[a, b]$. We define the class U of functions $F(x)$ in the following way: $F(x)$ is defined on $[a, b]$ such that for every point x_0 in $[a, b]$, $F(x)$ tends to finite limits as x tends to $x_0 +$ and to $x_0 -$ over the points of S ; these limits will be denoted by $F(x_0 +)$ and $F(x_0 -)$ respectively.

We now introduce the following definition:

DEFINITION 1.1. Let $F(x)$ belong to the class U and E be a subset of $[a, b]$ with α and β as its *g.l.b.* and *l.u.b.* Let

$$D : (\alpha < x_1 < x_2 < \cdots < x_p < \beta)$$

be any subdivision of $[\alpha, \beta]$ with $x_i \in E$. The *l.u.b.* of the sums V_D defined by

$$V_D = |F(\alpha) - F(x_1 -)| + \sum_{i=1}^{r-1} |F(x_i +) - F(x_{i+1} -)| + |F(x_r +) - F(\beta)|$$

for all possible subdivisions D is called the *total variation*, $V_S(F; E)$, of $F(x)$ on E relative to the set S . If $V_S(F; E) < +\infty$, then $F(x)$ is said to be $BV-S$ on E .

From theorem 3.1 and lemma 2.2 it follows that a function which is of bounded variation in the ordinary sense on $[a, b]$ is a $BV-S$ function and the $BV-S$ variation of a function $F(x)$ on a dense subset of $[a, b]$ is the same as that of the ordinary variation of the function $F(x-)$ [or $F(x+)$] on $[a, b]$. However,

there are examples of functions that are $BV-S$ on an infinite subset E of $[a, b]$ but not BV on E . Therefore the set of functions which are $BV-S$ on $[a, b]$ {that is, the set of functions $F(x)$ for which the total variation of $F(x-)$ [or $F(x+)$] on $[a, b]$ is finite} includes as a proper subset the functions which are BV on $[a, b]$. It may be noted in this connection that various authors have studied the properties of BV functions on a set. These studies can be found in most of the references appended in the list of the bibliography.

Throughout our discussion we suppose that S is a fixed set which is dense in $[a, b]$, and consequently U becomes a fixed class of functions as defined above. We denote the set of points x of S for which $F(x-) = F(x) = F(x+)$ by S_F , where S is as above and $F(x)$ is any function belonging to the class U . From theorem 3.4 onwards we suppose that S is Lebesgue measurable and $mS = b-a$.

2. Preliminary lemmas

LEMMA 2.1. *Let $F(x)$ belong to the class U . Then the set of points for which $F(x-) \neq F(x+)$ is countable. Also the subset of S for which we do not have $F(x-) = F(x) = F(x+)$ is countable.*

PROOF. For each positive integer n , let E_n denote the set of numbers x such that

$$|F(x-) - F(x+)| > \frac{1}{n}, \quad a + \frac{1}{n} < x < b - \frac{1}{n}.$$

The set E_n cannot have a cluster point, and hence it is finite. The set $\bigcup_{n=1}^{\infty} E_n$ is therefore countable. Similarly for the second part of the lemma and this completes the proof.

We now define the function $G(x)$ on $[a, b]$ as follows:

$$\begin{aligned} G(a) &= F(a), & G(b) &= F(b), \\ G(x) &= F(x-) \text{ for } a < x < b. \end{aligned}$$

It is easy to verify that

$$F(x+) = \lim G(\eta) \text{ as } (\eta > x, \eta \rightarrow x),$$

and

$$G(x) = \lim F(\xi+) \text{ as } (\xi < x, \xi \rightarrow x).$$

Clearly $G(x) = F(x)$ at each point of the set S_F .

LEMMA 2.2. *If E is dense in $[a, b]$ and if $F(x)$ belongs to U , then*

$$V_S(F; E) = V_a^b(G).$$

PROOF. The symbol \approx shall mean: can be made to differ by ε (> 0) by going far enough in the limiting process indicated by \rightarrow . The symbols x_0+ and $x_{r+1}-$ shall mean a and b respectively.

To prove $V_S(F; E) \leq V_a^b(G)$ we consider any subdivision

$$D : (a = x_0 < x_1 < x_2 < \dots < x_{r+1} = b)$$

of $[a, b]$ with $x_i \in E$ ($i = 1, 2, \dots, r$) and then take points η_i satisfying $x_i < \eta_i < x_{i+1}$ and $\eta_i \rightarrow x_i$ (except for $\eta_0 = a, \eta_{r+1} = b$). Denoting by V_D the sum

$$\sum_{i=0}^r |F(x_{i+}) - F(x_{i+1}-)|,$$

we have

$$V_D \approx \sum_{i=0}^r |F(\eta_i) - F(x_{i+1}-)| = \sum_{i=0}^r |G(\eta_i) - G(x_{i+1})| \leq V_a^b(G).$$

Hence

$$V_S(F; E) \leq V_a^b(G).$$

To prove the reverse inequality we consider any subdivision D of $[a, b]$, then take points ξ_i satisfying $\xi_i \in E, x_{i-1} < \xi_i < x_i$ (except for $\xi_0 = a, \xi_{r+1} = b$). Then

$$\sum_{i=0}^r |G(x_i) - G(x_{i+1})| \approx \sum_{i=0}^r |F(\xi_i+) - F(\xi_{i+1}-)| \leq V_S(F; E);$$

and hence

$$V_S(F; E) = V_a^b(G).$$

COROLLARY 2.2.1. *If E is dense in $[a, b]$, then $V_S(F; E) = V_S(F; [a, b])$.*

LEMMA 2.3. *Let $a < c < b$. If $F(x)$ is $BV-S$ on $[a, c]$ and on $[c, b]$, then it is so on $[a, b]$; further if $c \in S_F$ then*

$$V_S(F; [a, b]) = V_S(F; [a, c]) + V_S(F; [c, b]).$$

LEMMA 2.4. *If $F(x)$ is $BV-S$ on $[a, b]$, then $F(x+)$ is bounded on $[a, b]$. The proofs of these results are straightforward.*

Let $F(x)$ be $BV-S$ on $[a, b]$. We define the function $\pi(x)$ on $[a, b]$ as follows:

$$\pi(a) = 0 \text{ and } \pi(x) = V_a^x(G) \text{ for } a < x \leq b.$$

Clearly the function $\pi(x)$ is non-decreasing on $[a, b]$.

LEMMA 2.5. *If $F(x)$ is $BV-S$ on $[a, b]$, then $F(x)$ can be expressed as $F(x) = \pi(x) - v(x)$, where $v(x)$ is non-decreasing on S_F .*

PROOF. We define $v(x)$ by $v(x) = \pi(x) - F(x)$. Let x_1 and $x_2 (> x_1)$ be any two points of S_F . Then

$$\begin{aligned} v(x_2) - v(x_1) &= \{\pi(x_2) - F(x_2)\} - \{\pi(x_1) - F(x_1)\} \\ &= \pi(x_2) - \pi(x_1) - \{G(x_2) - G(x_1)\} \\ &\geq 0 \end{aligned}$$

and the lemma is proved.

3. Some results on BV-S functions

THEOREM 3.1. *If $F(x)$ is of bounded variation on $[a, b]$, then it is BV-S on $[a, b]$ and in any case*

$$V_S(F; [a, b]) \leq V_a^b(F), \quad F(x) \in U.$$

PROOF. We first suppose that $V_a^b(F)$ is finite. Let

$$D : (a = x_0 < x_1 < x_2 < \dots < x_{r+1} = b)$$

be any subdivision of $[a, b]$. Take points ξ_i, η_i of S with $x_i < \xi_i < \eta_i < x_{i+1}$ (except for $\xi_0 = a, \eta_{r+1} = b$). Then

$$\sum_{i=0}^r |F(\xi_i) - F(\eta_i)| \leq V_a^b(F).$$

Now letting $\xi_i \rightarrow x_i, \eta_i \rightarrow x_{i+1}$ over the points of S we obtain

$$|F(a) - F(x_1 -)| + \sum_{i=1}^{r-1} |F(x_i +) - F(x_{i+1} -)| + |F(x_r +) - F(b)| \leq V_a^b(F).$$

Since D is arbitrary, we have

$$(1) \quad V_S(F; [a, b]) \leq V_a^b(F).$$

So $F(x)$ is BV-S on $[a, b]$. If $V_a^b(F)$ is infinite, then clearly (1) holds.

NOTE. It is clear that if $F(x)$ is monotone or continuous on $[a, b]$ then

$$V_S(F; [a, b]) = V_a^b(F).$$

If (BV) denotes the set of all functions which are of bounded variation on $[a, b]$ and $(BV-S)$ the set of all functions which are BV-S, then by Theorem 3.1, $(BV) \subset (BV-S)$. The following example shows that (BV) is a proper subset of $(BV-S)$.

EXAMPLE. Let S be a dense subset of $[a, b]$ and let E be an infinite subset of $[a, b]$ with a and b as lower and upper bounds. Let $\phi(x)$ be any non-decreasing function on $[a, b]$ and $\sum \beta_n$ be a divergent series of positive terms with $\lim \beta_n = 0$. Choose a strictly monotone sequence $\{\alpha_n\}$ from E . Suppose that $\{\alpha_n\}$ is increasing. We define the function $F(x)$ on $[a, b]$ as follows:

$$F(x) = \phi(x) + \beta_n \text{ for } x = \alpha_{2n} (n = 1, 2, 3, \dots), \\ = \phi(x) \text{ elsewhere.}$$

It is easy to see that $F(x \pm) = \phi(x \pm)$ for all $x \in [a, b]$. For any two points $c, d (> c)$ of E

$$|F(c+) - F(d-)| = |\phi(c+) - \phi(d-)| \leq \phi(d) - \phi(c)$$

which shows that $F(x)$ is $BV-S$ on E . Now consider the subdivision $a \leq \alpha_1 < \alpha_2 < \dots < \alpha_{2m} < b$ and denote by V the sum

$$|F(a) - F(\alpha_1)| + \sum_{i=1}^{2m-1} |F(\alpha_i) - F(\alpha_{i+1})| + |F(\alpha_{2m}) - F(b)|.$$

Then

$$V \geq \sum_{i=1}^m |F(\alpha_{2i-1}) - F(\alpha_{2i})| \\ = \sum_{i=1}^m \{\phi(\alpha_{2i}) + \beta_i - \phi(\alpha_{2i-1})\} \\ \geq \beta_1 + \beta_2 + \dots + \beta_m.$$

Since $\sum \beta_n$ is divergent, it follows that $F(x)$ is not of bounded variation on E .

THEOREM 3.2. (cf. [2], Th. 2; [4], Lemma 1; [9], § 7).

Let $\{F_n(x)\}$ be a sequence of functions in the class U and $S_0 = \cap \{S_{F_n}; n = 1, 2, \dots\}$. If $F_n(x) \rightarrow F(x) \in U$ at each point of the set $E \cup \{a, b\}$ such that $E \subset S_0 \cap S_F$ and E is dense in $[a, b]$, then

$$\liminf_{n \rightarrow \infty} V_S(F_n; [a, b]) \geq V_S(F; [a, b]).$$

PROOF. We suppose that $V_S(F; E)$ is finite. If $V_S(F; E)$ is infinite the proof is analogous. Let $\epsilon > 0$ be arbitrary. There exists a subdivision

$$D : (a = x_0 < x_1 < x_2 < \dots < x_{r+1} = b)$$

with $x_i \in E$ ($i = 1, 2, \dots, r$) such that

$$|F(a) - F(x_1-)| + \sum_{i=1}^{r-1} |F(x_i+) - F(x_{i+1}-)| + |F(x_r+) - F(b)| > V_S(F; E) - \epsilon$$

or

$$V_D(F) = \sum_{i=0}^r |F(x_i) - F(x_{i+1})| > V_S(F; E) - \epsilon.$$

Since $V_D(F_n) \rightarrow V_D(F)$ as $n \rightarrow \infty$; a positive integer n_0 exists such that for $n \geq n_0$

$$V_S(F_n; [a, b]) \geq V_D(F_n) > V_S(F; E) - \epsilon.$$

So,

$$\liminf_{n \rightarrow \infty} V_S(F_n; [a, b]) \geq V_S(F; E) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain by using corollary 2.2.1,

$$\liminf_{n \rightarrow \infty} V_S(F_n; [a, b]) \geq V_S(F; [a, b]).$$

DEFINITION 3.1. Let $F(x) \in U$ and $D : (a = x_0 < x_1 < x_2 < \dots < x_{r+1} = b)$ be any subdivision of $[a, b]$ with $x_i \in S_F (i = 1, 2, \dots, r)$. We denote by $B(x) = B(x; F, D)$ the function whose graph is the polygonal line joining the points $(x_i, F(x_i)) (i = 0, 1, 2, \dots, r)$. $B(x)$ is said to be a *Polygonal function* associated with $F(x)$.

It is clear that

$$V_D(F) = \sum_{i=0}^r |F(x_i) - F(x_{i+1})| = V_D(B) = V_S(B; [a, b]).$$

So,

$$V_S(F; [a, b]) \geq V_S(B; [a, b]).$$

THEOREM 3.3. (cf. [3], § 2). *If $F(x)$ is BV-S on $[a, b]$ and S_F is dense in $[a, b]$, then it is possible to choose a sequence $\{B_n(x)\}$ of polygonal functions such that $B_n(x) \rightarrow F(x)$ at each point of S_F and*

$$\lim_{n \rightarrow \infty} V_S(B_n; [a, b]) = V_S(F; [a, b]).$$

PROOF. Let $\{D_n\}$ be a sequence of subdivisions

$$D_n : (a = x_0^{(n)} < x_1^{(n)} < x_2^{(n)} < \dots < x_{r_n+1}^{(n)} = b)$$

of $[a, b]$ with $x_i^{(n)} \in S_F (i = 1, 2, \dots, r_n)$ such that $D_n \subset D_{n+1}$ for each n and the set $E = \bigcup \{D_n; n = 1, 2, \dots\}$ is dense in $[a, b]$. Writing $B_n(x) = B(x; F, D_n)$ we have

$$(2) \quad V_S(B_n; [a, b]) \leq V_S(F; [a, b]).$$

Let $\epsilon > 0$ be arbitrary. Then a subdivision $D : (a = x_0 < x_1 < \dots < x_{r+1} = b)$ with $x_i \in E (i = 1, 2, \dots, r)$ exists such that

$$\sum_{i=0}^r |F(x_i) - F(x_{i+1})| > V_S(F; E) - \epsilon.$$

Since x_i 's are points of E , we can choose a positive integer n_0 such that $D \subset D_n$ for all $n \geq n_0$.

Then for $n \geq n_0$

$$(3) \quad V_S(B_n; [a, b]) \geq \sum_{i=0}^r |F(x_i) - F(x_{i+1})| > V_S(F; E) - \epsilon.$$

Combining (2) and (3) and noting corollary 2.2.1, we obtain

$$\lim_{n \rightarrow \infty} V_S(B_n; [a, b]) = V_S(F; [a, b]).$$

It is clear that $B_n(x) \rightarrow F(x)$ at each point of the set E . Let ξ be any point of $S_F - E$. Choose points ξ', ξ'' of E with $\xi' < \xi < \xi''$ such that $\pi(\xi'') - \pi(\xi') < \frac{1}{2}\epsilon$.

Let m be a positive integer such that $\xi', \xi'' \in D_n$ for all $n \geq m$. Then for all $n \geq m$

$$\begin{aligned} |F(\xi) - B_n(\xi)| &\leq |F(\xi) - F(\xi')| + |F(\xi') - B_n(\xi')| + |B_n(\xi') - B_n(\xi)| \\ &\leq |G(\xi) - G(\xi')| + V_S(F; [\xi', \xi'']) \\ &\leq 2[\pi(\xi'') - \pi(\xi')] < \varepsilon. \end{aligned}$$

This proves the theorem.

THEOREM 3.4. (cf. [13], p. 222). *Let $\mathcal{F} = \{F(x)\}$ be a sequence of functions in the class U . If there is a positive K such that $|F(x \pm)| \leq K$, $a < x < b$; $|F(a)|$, $|F(b)| \leq K$ and $V_S(F; [a, b]) \leq K$ for every $F(x) \in \mathcal{F}$, then there exist a subsequence in \mathcal{F} which converges to a function $\phi(x)$ almost everywhere in $[a, b]$, where $\phi(x)$ is of bounded variation in $[a, b]$.*

To prove the theorem we require the following lemma:

LEMMA 3.1. (cf. [13], p. 221). *Let $\mathcal{F} = \{F(x)\}$ be a sequence of functions in the class U and $S_0 = \cap \{S_F; F \in \mathcal{F}\}$. If each $F(x)$ is non-decreasing on S_0 and if there is a positive K such that $|F(x \pm)| \leq K$, $a < x < b$; $|F(a)|$, $|F(b)| \leq K$ for each $F \in \mathcal{F}$, then there is a subsequence $\{F_n(x)\}$ of functions in \mathcal{F} which converges to a function $\phi(x)$ almost everywhere in $[a, b]$, where $\phi(x)$ is non-decreasing on $[a, b]$.*

The lemma can be proved in the usual way.

PROOF OF THEOREM 3.4. By lemma 2.5, each $F(x)$ in \mathcal{F} can be expressed as $F(x) = \pi(x) - v(x)$, where $\pi(x)$ is non-decreasing on $[a, b]$ and $v(x)$ is non-decreasing on $S_0 = \cap \{S_F; F \in \mathcal{F}\}$. Clearly $\pi(x)$ belongs to the class U . Also $V_S(\pi; [a, b]) = \pi(b)$. So $\pi(x) \leq k$ for all $x \in [a, b]$. Since $v(x) = \pi(x) - F(x)$, $v(x)$ belongs to the class U and $|v(x \pm)| \leq 2k$, $a < x < b$; $|v(a)|$, $|v(b)| \leq 2k$. By lemma 2 ([13], p. 221) there is a subsequence $\{\pi_n(x)\}$ of $\{\pi(x)\}$ which converges to a non-decreasing function $\alpha(x)$ everywhere in $[a, b]$.

Let E_n denote the set of points in $[a, b]$, where

$$v_n(x-) = v_n(x) = v_n(x+) \text{ and } E_0 = \cap \{E_n; n = 1, 2, \dots\}.$$

Then Lebesgue measure of E_0 is $b - a$. Applying lemma 3.1 to the sequence $\{v_n(x)\}$ (where S_0 is to be replaced by $S_0 \cap E_0$) we obtain a subsequence $\{v_{n_i}(x)\}$ which converges to a non-decreasing function $\beta(x)$ almost everywhere in $[a, b]$. Let $\phi(x) = \alpha(x) - \beta(x)$. Then $\phi(x)$ is of bounded variation on $[a, b]$ and the sequence $\{F_{n_i}(x)\}$ converges to $\phi(x)$ almost everywhere in $[a, b]$. This proves the theorem.

4. The space (X, d)

Let X denote the set of all functions $x(t)$ in the class U which are $BV - S$ on $[0, 1]$. To each pair x, y of functions in X we associate the real number $d(x, y)$

defined by

$$(4) \quad d(x, y) = \int_0^1 |x(t) - y(t)| dt + |T(x) - T(y)|,$$

where the integral is taken in Lebesgue sense and $T(x)$ stands for $V_S(x; [0, 1])$. Since $d(x, y) = 0$ implies $T(x) = T(y)$ and $x(t) = y(t)$ almost everywhere in $[0, 1]$, it follows that d is a pseudo-metric for X and therefore (X, d) is a pseudo-metric space.

The pseudo-metric (4) is analogous to that introduced by C. R. Adams [1] and C. R. Adams & A. P. Morse [4] to study the properties of the space (BV) of functions of bounded variation. The above two papers contain interesting and elaborate discussions of the space (BV) . Here we wish to mention only two properties of the space (X, d) leaving out, of course, possible scope of further study.

THEOREM 4.1. *The space (X, d) is separable.*

PROOF. Let E denote the set of all polygonal functions in X with rational corners. Then clearly E is countable. Let $x(t)$ be any function in X . By theorem 3.3, it is possible to choose a sequence of polygonal functions $\{B_n(t)\}$ in X such that $B_n(t) \rightarrow x(t)$ almost everywhere in $[0, 1]$ and $T(B_n) \rightarrow T(x)$. For each $B_n(t)$ we can choose a polygonal function $P_n(t)$ in X with rational corners such that $|B_n(t) - P_n(t)| < 1/n$ for all $t \in [0, 1]$ and $|T(B_n) - T(P_n)| < 1/n$. So the sequence $\{P_n(t)\}$ converges to $x(t)$ almost everywhere in $[0, 1]$ and $T(P_n) \rightarrow T(x)$. Therefore $d(P_n, x) \rightarrow 0$ as $n \rightarrow \infty$ and hence x is an accumulation point of E . Thus the set E is dense in X . This completes the proof.

THEOREM 4.2. *Every closed sphere in (X, d) is compact.*

PROOF. Let x_0 be an element of X and

$$Y = \{x; x \in X \text{ and } d(x, x_0) \leq r\},$$

where r is a positive number. Since X is separable, Y considered as a subspace of X is a Lindelöf space. Let $\theta(t) \equiv 0$ in $[0, 1]$. Then θ is an element of X . For any x in Y ,

$$d(x, \theta) \leq d(x, x_0) + d(x_0, \theta) \leq r + d(x_0, \theta).$$

So

$$(5) \quad d(x, \theta) = \int_0^1 |x(t)| dt + T(x) \leq M,$$

where M denotes the constant $r + d(x_0, \theta)$. If t is a point in $(0, 1)$, then

$$(6) \quad |x(t \pm)| \leq \max \{|x(0)|, |x(1)|\} + T(x).$$

We show that $\max \{|x(0)|, |x(1)|\} \leq 2M$. If possible, assume that $\max \{|x(0)|, |x(1)|\} > 2M$. If $|x(t)| \geq M$ almost everywhere in $[0, 1]$, then $\int_0^1 |x(t)| dt \geq M$

which with (5) gives $T(x) = 0$. So $|x(t \pm)| = |x(0)| = |x(1)| > 2M$. This contradicts (5). Hence there is a subset E of $[0, 1]$ of positive measure such that $|x(t \pm)| < M$ for all $t \in E$. Let t be any point of $E \cap S_x$. Then

$$|x(0) - x(t)| + |x(t) - x(1)| > M.$$

So, $T(x) > M$ which contradicts (5). Therefore $\max\{|x(0)|, |x(1)|\} \leq 2M$. Combining this with (6) we get $|x(t \pm)| \leq 3M$ for all $t \in (0, 1)$.

Let $\{x_n(t)\}$ be any sequence of points in Y . Then

$$|x_n(t \pm)| \leq 3M, 0 < t < 1; |x_n(0)|, |x_n(1)| < 3M$$

and $V_S(x_n; [0, 1]) < 3M$. By theorem 3.4, there is a subsequence $\{x_{n_i}(t)\}$ which converges to a function $x(t)$ in X almost everywhere in $[0, 1]$. We may choose $\{x_{n_i}(t)\}$ and take the function $x(t)$ such that $\{x_{n_i}(t)\}$ converges to $x(t)$ also at $t = 0, 1$. Let $\tau = \lim_{i \rightarrow \infty} \inf T(x_{n_i})$. We choose a subsequence $\{T(x_{m_i})\}$ of $\{T(x_{n_i})\}$ which converges to τ . By theorem 3.2, $\tau \geq T(x)$. Let $K = \tau - T(x)$. We define the function $y(t)$ on $[0, 1]$ as follows:

$$\begin{aligned} y(t) &= x(t), 0 < t \leq 1, \\ &= x(0) + K \text{ for } t = 0 \text{ if } x(0) > x(0+), \\ &= x(0) - K \text{ for if } x(0) \leq x(0+). \end{aligned}$$

It is clear that $y \in X$ and $T(y) = T(x) + K$. Further

$$d(x_{m_i}, y) = \int_0^1 |x_{m_i}(t) - x(t)| dt + |T(x_{m_i}) - \tau|.$$

Then $d(x_{m_i}, y) \rightarrow 0$ as $i \rightarrow \infty$. We have

$$d(y, x_0) \leq d(x_{m_i}, y) + d(x_{m_i}, x_0) \leq r + d(x_{m_i}, y).$$

Letting $i \rightarrow \infty$ we get $d(y, x_0) \leq r$. Thus every sequence in Y has a cluster point in Y . So by lemma ([12], Ch. 5, § 4) Y is compact.

5. The space (\bar{X}, ρ)

Let \bar{X} denote the family of all sets

$$\{x\}^- = \{y; y \in X \text{ and } d(x, y) = 0\}$$

for $x \in X$. For convenience, we write \bar{x} for $\{x\}^-$. For any two members \bar{x}, \bar{y} of \bar{X} , let

$$\rho(\bar{x}, \bar{y}) = \inf \{d(\alpha, \beta); \alpha \in \bar{x} \text{ and } \beta \in \bar{y}\}.$$

Then (\bar{X}, ρ) is a metric space ([12], Ch. 4, § 15).

THEOREM 5.1. *The space (\bar{X}, ρ) is complete.*

PROOF. Let $\{\bar{x}_n\}$ be any Cauchy sequence in \bar{X} . Then there is a positive number M such that $\rho(\bar{x}_n, \theta) \leq M$ for all n , where $\theta(t) = 0$ in $[0, 1]$. Let α_n be any member of \bar{x}_n . Then $d(\alpha_n, \theta) = \rho(\bar{x}_n, \theta) \leq M$ for all n . Following the method of theorem 4.2, we obtain a subsequence $\{\alpha_{n_i}(t)\}$ which converges to a function $x(t)$ in X almost everywhere in $[0, 1]$ such that $d(\alpha_{n_i}, x) \rightarrow 0$ as $i \rightarrow \infty$. Since $d(\alpha_{n_i}, x) = \rho(\bar{x}_{n_i}, \bar{x})$ it follows that the sequence $\{\bar{x}_{n_i}\}$ converges to \bar{x} which implies that the sequence $\{\bar{x}_n\}$ converges to \bar{x} . This completes the proof.

Finally, the author is thankful to Dr. B. K. Lahiri for his kind help and suggestions in the preparation of the paper. Also, the author is thankful to the referee for his helpful suggestions.

References

- [1] C. R. Adams, 'The space of functions of bounded variation and certain general spaces', *Trans. Amer. Math. Soc.*, 40 (1936), 421—438.
- [2] C. R. Adams and J. A. Clarkson, 'On convergence in Variation', *Bull. Amer. Math. Soc.*, 40 (1934), 413—417.
- [3] C. R. Adams and H. Lewy, 'On convergence in length', *Duke Math. Journal* 1 (1935), 19—26.
- [4] C. R. Adams and A. P. Morse, 'On the space (BV)', *Trans. Amer. Math. Soc.* 42 (1937), 194—205.
- [5] L. Cesari, 'Sulle funzioni a Variazione limitata', *Annali R. Scuola Normale Sup. Pisa, Series II*, 5 (1936).
- [6] L. Cesari, 'Funzioni continue a variazine limitata in Un insieme', *Mem. Accad. Sci. Ist. Bologna, cl. Sci. Fis* (10) 2 (1946), 127—145.
- [7] L. Cesari, 'Invertibilita in piccolo delle funzioni continue e teorema di derivazione', *Mem. Accad. Sci. Ist. Bologna cl. Sci. Fis* (10) 3 (1947), 99—112.
- [8] L. Cesari, 'Sul teorema di derivazione dulle funzioni variazione limitata in Un insieme', *Mem. Accad. Sci. Ist. Bologna cl. Sci. Fis* (10) 8 (1950—51), 223—229.
- [9] L. Cesari, 'Variation, multiplicity and semi-continuity', *Amer. Math. Monthly* 65 (1958), 317—332.
- [10] R. L. Jeffery, 'Generalised integrals with respect to functions of bounded variation', *Can. J. Math.* 10 (1958), 617—628.
- [11] K. Krickeberg, 'Distributionen functionen beschränkter variation und Lebesguescher inhalt nichtparamätischer flachen', *Annali Matem. Pura appl.* (4) 44 (1957), 105—133.
- [12] J. H. Kelley, *General Topology* (Van Nostrand, New York, 1955).
- [13] I. P. Natanson, *Theory of functions of a real variable. Vol. I*, (Unger, New York, 1960).
- [14] S. Saks, *Theory of the integral* (Warsaw, 1937; Dover, New York, 1964).

Department of Mathematics
Jadavpur University
Calcutta-32, India