On functions of finite variation, depending on a parameter

by

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1. Let X be a Banach space, $x(v)$ a function from an interval $a\leqslant v\leqslant b$ to X. If A is a subset of $[a,b]$, we shall denote by $[v_i, A]$ any system of real numbers

$$v_1 < v_2 < \ldots < v_n,$$

where $v_i \epsilon A$.

By $V_A x(v)$ we shall denote the supremum of the sums

$$\Big\| \sum_{i=1}^{n-1} \varepsilon_i \{x(v_{i+1}) - x(v_i)\} \Big\|$$

taken for all $[v_i, A]$ and all possible choices of $\varepsilon_i = \pm 1$. $V_A x(v)$ will be called the *variation* of $x(v)$ on A. We will denote by $W_A x(v)$ the supremum of the sums

$$\sum_{i=1}^{n-1} \|x(v_{i+1}) - x(v_i)\|$$

taken for all $[v_i, A]$; $W_A x(v)$ will be called the *strong variation* of $x(v)$ on A.

If $A=[a,b]$, we shall omit the subscript A in $W_A x(v)$ and $V_A x(v)$.

By *essential variation* of $x(v)$ we shall understand the infimum $V^* x(v)$ of the variations $V_A x(v)$ taken for all subsets A of $[a,b]$ of measure $b-a$. $W^* x(v)$ will be defined analogously.

It is obvious that there exists a set A such that $|A| = b-a$ and $V^* x(v) = V_A x(v)$.

Every function $x(v)$ for which either $V_A x(v)$, or $W_A x(v)$, or $V^* x(v)$, or $W^* x(v)$ is finite will be called of *finite variation over* A[1]), *finite strong variation over* A, *finite essential variation*, and *finite essential strong variation*, respectively.

[1]) I. Gelfand gives in the paper *Abstrakte Funktionen und lineare Operatoren*, Mat. Sbornik (1938), p. 235-285, also another, equivalent, definition of functions of finite variation.

If X is identical with the space of real numbers, we shall write $\mathrm{var}^* x(v)$ for $V^* x(v)$. In this case we shall also denote

$$\sup^* x(v) = \inf_{|A|=b-a} \sup_{A} x(v).$$

A set of linear functionals will be called *fundamental* if for every $x \epsilon X$

$$\sup_{\xi\epsilon\Gamma,\, \|\xi\|=1} \xi(x) = \|x\|.$$

In the sequel Γ will denote a fundamental set of linear functionals. A sequence $\{x_n\}$ of elements will be said to *Γ-converge to* x_0 if $\lim\limits_{n\to\infty} \xi(x_n) = \xi(x_0)$ for every $\xi \epsilon \Gamma$. If $\{x_n\}$ Γ-converges to x_0, then

$$\|x_0\| \leqslant \varliminf_{n\to\infty} \|x_n\|. \tag{$*$}$$

The function $x(v)$ will be said to be *Γ-continuous at* v_0 if $v_n \to v_0$ implies the Γ-convergence of $x(v_n)$ to $x(v_0)$; any function Γ-continuous at every v will be called *Γ-continuous*.

1.1. *If $x(v)$ is Γ-continuous, then*

$$Vx(v) = V^* x(v), \qquad Wx(v) = W^* x(v).$$

Proof. It suffices to prove for every set $A \subset [a,b]$ of measure $b-a$, the inequalities $V_A x(v) \geqslant Vx(v)$ and $W_A x(v) \geqslant Wx(v)$. Given any $\varepsilon > 0$, choose $[v_i, [a,b]]$ and $\varepsilon_i = \pm 1$ so that

$$Vx(v) - \varepsilon < \Big\| \sum_{i=1}^{n-1} \varepsilon_i \{x(v_{i+1}) - x(v_i)\} \Big\|;$$

now choose $v_i^{(p)} \epsilon A$ so that $\lim\limits_{n\to\infty} v_i^{(p)} = v_i$ and $v_1^{(p)} < v_2^{(p)} < \ldots < v_n^{(p)}$. Then

$$\Big\| \sum_{i=1}^{n-1} \varepsilon_i \{x(v_{i+1}^{(p)}) - x(v_i^{(p)})\} \Big\| \leqslant V_A x(v).$$

The elements

$$y_p = \sum_{i=1}^{n-1} \varepsilon_i \{x(v_{i+1}^{(p)}) - x(v_i^{(p)})\}$$

Γ-converge to

$$y = \sum_{i=1}^{n-1} \varepsilon_i \{x(v_{i+1}) - x(v_i)\}.$$

Hence by $(*)$

$$Vx(v)-\varepsilon<\|y\|\leqslant\varliminf_{p\to\infty}\|y_p\|\leqslant V_A x(v);$$

$\varepsilon>0$ being arbitrary, we get $Vx(v)\leqslant V_A x(v)$.

The proof that $W_A x(v)\geqslant Wx(v)$ is similar.

Given any set A denote by A^+ and A^- respectively the sets of accumulation points on the right and on the left of the set A.

1.2. *If $x(v)$ is of finite strong variation over A, then for every $v\in A^+$ resp. $v\in A^-$ there exists the limit*

$$\lim_{w\to v+} {}_A\, x(w)=x^*(v) \qquad \textit{resp.} \qquad \lim_{w\to v-} {}_A\, x(w)=x_*(v),$$

and the function $x(v)$ is continuous (relatively to A) in A, except a denumerable set.

Proof. Suppose, if possible, that there exists a v_n in A such that $v_n\to v$, $v_n>v$, and that $\lim_{n\to\infty} x(v_n)$ does not exist. We can assume freely that

$$\|x(v_n)-x(v_{n+1})\|\geqslant\varepsilon,$$

ε being a positive constant, and that $v_1>v_2>\ldots$ Then for every N

$$W_A x(v)\geqslant\sum_{k=1}^{N}\|x(v_{k+1})-x(v_k)\|\geqslant\varepsilon N,$$

hence $W_A x(v)=\infty$, contrarily to our hypothesis.

To prove the second part of 1.2, denote by $\omega(v)$ the oscillation relative to the set A of $x(w)$ at v; it suffices to prove that the set Q_k of the points $v\in A$ for which $\omega(v)>k$ is finite for every $k>0$. Suppose that $v_i\in Q_k$ $(i=1,2,\ldots,m)$, $v_1<v_2<\ldots<v_m$. Then there exist $v_i', v_i''\in A$ such that $v_i'<v_i''<v_{i+1}'$ and $\|x(v_i')-x(v_i'')\|>k$, and this implies in turn $km\leqslant W_A x(v)$; hence the set Q_k is finite.

If $W^*x(v)$ is finite, we choose a set $A\subset\langle a,b\rangle$ of measure $b-a$ such that $W_A x(v)=W^*x(v)$. The function $x^*(v)$ is then defined everywhere in $[a,b)$; we complete its definition by writing $x^*(b)=\lim_{v\to b-} x^*(v)$. Then $x^*(v)$ is continuous on the right, $x(v)=x^*(v)$, except in a set of measure 0 and $Wx^*(v)=W^*x(v)$, $Vx^*(v)=V^*x(v)$. There exists only one function with the above continuity properties, equivalent to $x(v)$. The function $x^*(v)$ will be called the *reduced function* of $x(v)$.

2. We shall be concerned now with functions $x(v)$ taking their values from the space M — this is the space of the functions $x=x(t)$ defined in $[a,b]$, measurable and essentially bounded, the norm being defined as $\|x\|=\sup^*|x(t)|$. By Γ we shall denote the set of linear functionals of the form

$$\xi(x)=\pm\frac{1}{q-p}\int_p^q x(t)\,dt.$$

Let $D(t,v)$ be a function defined in $[a,a;b,b]$, measurable and bounded for $v=\mathrm{const}$. Hence the formula $x(v)=D(\cdot,v)$ defines a vector valued function from the interval $[a,b]$ to M.

The sequence $\{x_n\}$ of elements of M is said to *l-converge* to x_0 if $\sup_n\|x_n\|<\infty$ and

$$\int_a^b|x_n(t)-x_0(t)|\,dt\to 0.$$

If the sequence $\{x_n\}$ l-converges to x_0, then it is Γ-convergent to x_0.

The l-limit of x_n will be written $l\text{-}\lim_{n\to\infty} x_n$. If for $w_n\in A$, $w_n\to v$, $v<w_n$ implies $l\text{-}\lim x(w_n)=x_0$, then we shall write $x_0=l\text{-}\lim_{w\to v+}{}_A\, x(v)$.

2.1. *Suppose $V_A x(v)<\infty$; then $V_A x(v)$ is equal to the supremum S of the expressions*

$$\sup_t{}^*\sum_{i=1}^{n}|D(t,v_i)-D(t,v_{i-1})|,$$

the supremum being taken for all systems $[v_i,A]$.

Proof. Since $V_A x(v)$ is the supremum of the expressions

$$\sup_t{}^*\Big|\sum_{i=1}^{n}\varepsilon_i[D(t,v_i)-D(t,v_{i-1})]\Big|$$

taken for all $[v_i,A]$ and $\varepsilon_i=\pm1$, we have $V_A x(v)\leqslant S$. On the other hand choose $[v_i,A]$ so that

$$S-\varepsilon\leqslant\sup_t{}^*\sum_{i=1}^{n}|D(t,v_i)-D(t,v_{i-1})|$$

$$=\sup_t{}^*\sum_{i=1}^{n}\varepsilon_i(t)[D(t,v_i)-D(t,v_{i-1})],$$

where

$$\varepsilon_i(t) = \text{sign}\,[D(t,v_i) - D(t,v_{i-1})].$$

The vector valued function $\varphi(t) = \{\varepsilon_1(t), \ldots, \varepsilon_n(t)\}$ can assume at most 2^n values, hence if B denotes a set of positive measure, then one value $\{\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n\}$ is assumed on a set of positive measure $B' \subset B$. Hence for $t \in B'$

$$\sum_{i=1}^{n} \bar{\varepsilon}_i [D(t,v_i) - D(t,v_{i-1})] = \sum_{i=1}^{n} |D(t,v_i) - D(t,v_{i-1})|,$$

which implies

$$S - 2\varepsilon \leqslant \sup_t{}^* \Big| \sum_{i=1}^{n} \bar{\varepsilon}_i [D(t,v_i) - D(t,v_{i-1})] \Big| \leqslant V_A x(v).$$

2.2. *If $x(v) = D(\cdot, v)$ and $V_A x(v) < \infty$, then for every $v \in A$, except a denumerable set, $l\text{-}\lim_{A,\, w\to v+} x(w) = x(v)$ exists.*

Proof. Consider $x(v)$ as a function with values in the space L. Then

$$\sum_{i=1}^{n} \|x(v_i) - x(v_{i-1})\|_L = \sum_{i=1}^{n} \int_a^b |D(t,v_i) - D(t,v_{i-1})|\, dt$$

$$= \int_a^b \sum_{i=1}^{n} |D(t,v_i) - D(t,v_{i-1})|\, dt \leqslant (b-a)\, V_A x(v).$$

By 1.2 for every $v \in A$, except a denumerable set, the limit $\lim_{A,\, w\to v+} x(w) = x(v)$ exists, and since the function $x(v)$ is bounded on A as a function to M, this implies in turn that $l\text{-}\lim_{A,\, w\to v+} x(w) = x(v)$ exists.

2.3. Theorem 1. *If $D(t,v)$ is measurable in $[a,a;b,b]$ and $x(v) = D(\cdot,v) \in M$ for every v, then*

$$V^* x(v) = \sup_t{}^* [\operatorname{var}_v{}^* D(t,v)]. \tag{1}$$

Proof. Suppose that the left hand side of the formula (1) is finite. Choose the set A so that $|A| = b-a$, and $V_A x(v) = V x^*(v)$, then let $R = \{v_i\}$ be a sequence of elements of A, dense in A. The sets

$$E_{nk} = \mathop{E}_{(t,v)} \left\{ |D(t,v) - D(t,v_n)| < \frac{1}{k} \right\}$$

are measurable. The set H of the points (t,v) for which there exists a sequence $\{v_{n_i}\}$ such that $v_{n_i} \to v$, $v_{n_i} > v$ and

$$\lim_{i\to\infty} D(t, v_{n_i}) = D(t,v) \tag{2}$$

is measurable, for we have

$$H = \prod_{k=1}^{\infty} \sum_{k=1}^{\infty} E_{nk} R_{nk},$$

where $R_{nk} = \mathop{E}_{(t,v)} \{v < v_n < v + 1/k\}$.

By 2.2 for every $v \in A$, except a denumerable set, there exists a sequence $\{v_{n_i}\}$ such that $v_{n_i} > v$, $v_{n_i} \to v$ and $l\text{-}\lim_{i\to\infty} x(v_{n_i}) = x(v)$, hence there exists a subsequence $\{v_{m_i}\}$ such that $\lim_{i\to\infty} D(t, v_{m_i}) = D(t,v)$ for almost every t. Hence, the set H being measurable, $|H| = (b-a)^2$. By the theorem of Fubini, there is a set $T_0 \subset [a,b]$ of measure $b-a$ and for every $t \in T_0$ there is a set $Q_t \subset [a,b]$ of measure $b-a$ such that (2) is fulfilled for every $v \in Q_t$.

Given a $[v_i^{(n)}, R]$ we have by 2.1

$$\sum_{i=1}^{m} |D(t, v_i^{(n)}) - D(t, v_{i-1}^{(n)})| \leqslant V_A x(v) = V^* x(v) \tag{3}$$

for almost any t. Hence there is a set N of measure 0 such that (3) holds for every $t \in (T_0 - N)$ and every $[v_i^{(n)}, R]$.

Now let $t_0 \in (T_0 - N)$; for $[v_i, AQ_{t_0}]$ we choose $[v_i^{(n)}, R]$ such that $D(t_0, v_i^{(n)}) \to D(t_0, v_i)$; hence, by (3),

$$\sum_{i=1}^{m} |D(t_0, v_i) - D(t_0, v_{i-1})| \leqslant V x^*(v),$$

$$\operatorname{var}^* D(t_0, v) \leqslant V^* x(v),$$

whence

$$\sup_t{}^* \operatorname{var}_v D(t,v) \leqslant V x^*(v).$$

Suppose now that the right hand side in formula (1) is finite. There is a set $T_0 \subset [a,b]$ of measure $b-a$ such that

$$\sup_t{}^* [\operatorname{var}_v{}^* D(t,v)] = \sup_{t\in T_0} \operatorname{var}_v{}^* D(t,v) < \infty;$$

then for every $t \in T_0$ we replace the function $D(t,v)$ by the reduced function $\bar{D}(t,v)$. This function is measurable in $T_0 \times [a,b]$, for

$$\bar{D}(t,v) = \lim_{h\to 0+} \frac{1}{h} \int_v^{v+h} D(t,w)\,dw.$$

If $t \in T_0$, $D(t,v) = \bar{D}(t,v)$ for almost any v, then since both functions are measurable, there exists a set $S \subset [a,b]$ of measure $b-a$ such that $v \in S$ implies $D(t,v) = \bar{D}(t,v)$ for almost any t.

Since $t \in T_0$ implies

$$\operatorname*{var}_v \bar{D}(t,v) = \operatorname*{var^*}_v D(t,v) \leqslant \sup_{t\in T_0} \operatorname*{var^*}_v D(t,v),$$

we have

$$\sum_{i=1}^{n} |\bar{D}(t,v_i) - \bar{D}(t,v_{i-1})| \leqslant \sup_{t\in T_0} \operatorname*{var^*}_v D(t,v);$$

hence for $[v_i, S]$ we have almost everywhere

$$\sum_{i=1}^{n} |D(t,v_i) - D(t,v_{i-1})| \leqslant \sup_{t\in T_0} \operatorname*{var^*}_v D(t,v);$$

in virtue of 2.1 this implies immediately

$$V^* x(v) \leqslant \sup_t{}^* \operatorname*{var^*}_v D(t,v).$$

2.4. In the next theorem the following condition will play an essential role:

$$\text{(J)} \qquad \int_a^u D(t,v)\,dt = \int_a^v D(t,u)\,dt \quad \text{for every } u,v \in [a,b].$$

2.41. This condition implies the Γ-continuity of the function $x(v) = D(:,v)$. In fact for

$$\xi(x) = \pm \frac{1}{q-p} \int_p^q x(t)\,dt,$$

$$\xi\big(x(v) - x(v_0)\big) = \pm \frac{1}{q-p} \int_p^q [D(t,v) - D(t,v_0)]\,dt$$

$$= \pm \frac{1}{q-p} \int_{v_0}^{v} [D(t,q) - D(t,p)]\,dt,$$

hence the right hand side tends to 0 if $v \to v_0$.

The last formula is valid also if $D(t,v) \in L$ for $v \in [a,b]$. This implies immediately:

If $x(v) = D(:,v) \in L$ for $v \in [a,b]$ and

$$\text{(o)} \qquad \sup_t{}^* |D(t,v)| \leqslant k \quad \text{for} \quad v \in S, \quad |S| = b-a,$$

then the inequality (o) is satisfied for every v.

If the condition (J) is not satisfied by the function $D(t,v)$ but by $\bar{D}(t,v) = D(t,v) + h(t)$, where $h(t)$ is a fixed integrable function, the above remarks remain valid for the function $D(t,v)$.

2.5. Theorem 2. *Under the hypothesis of Theorem 1 suppose, moreover, that the condition (J) is satisfied, then*

(a) $Vx(v) < \infty$ *implies the following generalization of the Lipschitz condition:*

$$(*) \qquad \int_a^b |D(t,v+h) - D(t,v)|\,dt \leqslant K|h| \quad \textit{for} \quad a-v \leqslant h \leqslant b-v,$$

and this holds certainly if $K = Vx(v)$;

(b) *If* $(*)$ *is satisfied, then* $Vx(v) < \infty$, *and* $Vx(v)$ *is the best constant* K *in formula* $(*)$;

(c) $Vx(v) = V^* x(v)$.

Proof. The statement (c) holds by the Γ-continuity of the function $x(v) = D(:,v)$ and by 1.1.

To prove (a) choose two partitions:

$$a = u_1 < u_2 < \ldots < u_n = b$$

and

$$a = w_1 < w_2 < \ldots < w_m = b,$$

and set $u(t) = a_i$ for $u_i \leqslant t < u_{i+1}$, $w(t) = b_i$ for $w_i \leqslant t < w_{i+1}$; then by the condition (J)

$$\int_a^b u(t)\,D(t,v)\,dt = \int_a^v \left\{ \sum_{i=1}^{n-1} a_i [D(t,u_{i+1}) - D(t,u_i)] \right\} dt,$$

whence

$$(4) \qquad \begin{aligned} &\int_a^b u(t) \left\{ \sum_{i=1}^{m-1} b_i [D(t,w_{i+1}) - D(t,w_i)] \right\} dt \\ &= \int_a^b w(t) \left\{ \sum_{j=1}^{n-1} a_j [D(t,u_{j+1}) - D(t,u_j)] \right\} dt; \end{aligned}$$

$$\int_a^b |w(t)|\,dt = \sum_{i=1}^{m-1} |b_i|(w_{i+1}-w_i);$$

$$\sup{}^* |u(t)| = \max |a_i|.$$

If $Vx(v)<\infty$, then

$$\int_a^b w(t)\left\{\sum_{j=1}^{n-1} a_j\,[D(t,u_{j+1})-D(t,u_j)]\right\}dt$$

$$\leqslant \sup{}^* |u(t)|\, Vx(v) \int_a^b |w(t)|\,dt,$$

and by (4)

$$\int_a^b u(t)\left\{\sum_{i=1}^{m-1} b_i\,[D(t,w_{i+1})-D(t,w_i)]\right\}dt$$

$$\leqslant \sup{}^* |u(t)|\, Vx(v) \int_a^b |w(t)|\,dt;$$

this being valid for every simple function, it results the inequality

$$\int_a^b \left|\sum_{i=1}^{m-1} b_i\,[D(t,w_{i+1})-D(t,w_i)]\right| dt \leqslant Vx(v)\int_a^b |w(t)|\,dt.$$

Choosing $w(t)$ in an appropriate manner we get

$$\int_a^b [D(t,v+h)-D(t,v)]\,dt \leqslant |h|\, Vx(v)$$

for $a-v\leqslant h\leqslant b-v$. Hence (a) is proved.

To prove (b) suppose that (*) is satisfied; then the expression on the left hand side of (4) is not greater than

$$K\int_a^b |w(t)|\,dt\,\sup{}^*|u(t)|\,,$$

whence

$$\int_a^b w(t)\left\{\sum_{j=1}^{n-1} a_j\,[D(t,u_{j+1})-D(t,u_j)]\right\}dt$$

$$\leqslant K\sup{}^*|u(t)|\int_a^b |w(t)|\,dt,$$

and this inequality is valid for every integrable function $w(t)$. Hence

$$\sup{}^*\left|\sum_{j=1}^{n-1} a_j[D(t,u_{j+1})-D(t,u_j)]\right| \leqslant K\sup{}^*|u(t)| = K\max|a_i|.$$

Choosing $a_i=\varepsilon_i=\pm 1$ we get (b) and $K\geqslant Vx(v)$.

Remark 1. In the statement (b) of Theorem 2 we can replace the hypothesis that $D(:,v)\in M$ by the hypothesis that $D(:,v)\in L$ and that $D(:,a)\in M$; then $x(v)\in M$ for every $v\in[a,b]$. This follows from the application of the inequality at the end of the proof above.

As corollary of Theorems 1 and 2 we get

Theorem 3. *Under the hypotheses of Theorem 2 the following conditions are equivalent:*

$$Vx(v)<\infty,$$

$$\int_a^b |D(t,v+h)-D(t,v)|\,dt \leqslant K|h| \quad \textit{for} \quad a-v\leqslant h\leqslant b-v,$$

$$\sup_t{}^*\operatorname{var}_v{}^* D(t,v)<\infty.$$

Remark 2. In order to deduce the first two of the inequalities in Theorem 3 from

$$\sup_t{}^*\operatorname{var}_v{}^* D(t,v)<\infty,$$

it suffices to suppose $D(t,v)\in L$ for $v\in(a,b]$, $D(t,a)\in M$ for these hypotheses enable us to repeat the arguments of 2.3. Hence it follows that in a set S of measure $b-a$ the inequality

$$\sum_{i=1}^{n} |D(t,v_i)-D(t,v_{i+1})| \leqslant C$$

is satisfied for arbitrary $[v_i,S]$.

Choosing $v_0\in S$ put $\bar D(t,v)=D(t,v)-D(t,v_0)$. If $v\in S$ then $\sup{}^*|\bar D(t,v)|\leqslant C$, whence by 2.41 this inequality is satisfied for every v. It follows that

$$\sup_t{}^* |D(t,v_0)| \leqslant C + \sup_t{}^* |D(t,a)| = C_1,$$

$$\sup{}^* |D(t,v| \leqslant C + C_1,$$

whence $x(v)\in M$ for every v.

3. Applications. Now we prove

Theorem 4. *Let $f(t)$ be a real valued function integrable in $[0,b]$ and of period b. Then the following conditions are equivalent:*

$$\operatorname{var}^* f(t) < \infty,$$

$$\int_0^b |f(t+h)-f(t)|\,dt \leqslant K|h| \quad \textit{for} \quad |h|<b.$$

The smallest constant K in this formula is equal to

$$\operatorname{var}^* f(t) + |f^*(0)-f^*(b-0)|,$$

where $f^*(x)$ denotes the reduced function of $f(x)$ in $[0,b]$ the variation being taken over $[0,b]$ [2]).

To prove this put $D(t,v)=f(t+v)-f(t)$ for $0\leqslant t\leqslant b$, $0\leqslant v\leqslant b$. Then $x=x(v)=D(:,v)\,\epsilon\, L$ and $x(0)=D(:,0)=0\,\epsilon\, M$. The condition (J) is satisfied.

It may easily be verified that if $V^*x(v)<\infty$, then $\operatorname{var}^* f(v)<\infty$, and conversely, the formula

$$V^*x(v)=Vx(v)=\sup_t{}^* \operatorname{var}_v^* D(t,v)$$
$$=\sup_t{}^* \operatorname{var}_v^* [f(t+v)-f(t)] = \sup_t{}^* \operatorname{var}_v^* f(t+v),$$

is valid if $\operatorname{var}^* f(v)<\infty$ in virtue of (c) of Theorem 2, for in this case $D(:,v)\,\epsilon\, M$ if $v\,\epsilon\,[0,b]$.

If t is a point of continuity for $f(t)$, then

$$\operatorname{var}_v^* f(t+v) = \operatorname{var}^* f(v) + |f^*(0)-f^*(b-0)|,$$

whence the same results almost everywhere. Since, moreover,

$$\int_0^b |D(t,v+h)-D(t,v)|\,dt = \int_0^b |f(t+h)-f(t)|\,dt,$$

for $-v\leqslant h\leqslant b-v$, the theorem results from Theorem 2 and Remark 1.

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[2]) G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals I*, Math. Zeitschr. 27 (1928), p. 599, Theorem 24. These authors do not give a precise value of the constant K.

Let $\{\varphi_i(t)\}$ be an orthonormal system in $[a,b]$, complete in L^2 and composed of bounded functions. The sequence $\{\lambda_i\}$ is said to be a *multiplicator of the class* (M,M), if

$$x(t) \sim \sum_{i=1}^{\infty} a_i \varphi_i(t)$$

being the Fourier development of an arbitrary function of M,

$$\sum_{i=1}^{\infty} \lambda_i a_i \varphi_i(t)$$

is a development of a function $y(t)\,\epsilon\, M$. As the function $y(t)$ depends on $x=x(t)$, we may write $y(t)=U(x,t)$. It is known that the operation $U(x,t)$ is linear from M to M [3]).

The series

$$\sum_{i=1}^{\infty}\left(\int_a^u \varphi_i(t)\,dt\right)^2$$

converges uniformly in $[a,b]$. Indeed, for every u this series, being the series of squares of coefficients of the characteristic function of $[a,u]$, converges by the identity of Parseval to the continuous function $u-a$. The series, consisting of positive continuous functions, converges to a continuous function, hence it must converge uniformly by the theorem of Dini.

Suppose now the sequence $\{\lambda_i\}$ to be bounded. Then the series

$$\sum_{i=1}^{\infty} \lambda_i \int_a^t \varphi_i(\tau)\,d\tau \int_a^v \varphi_i(\tau)\,d\tau = D(t,v)$$

must converge uniformly in $[a,a;b,b]$, and its sum $D(t,v)$ is continuous in both variables.

Note further that the series

$$\sum_{i=1}^{\infty}\left(\lambda_i \int_a^v \varphi_i(\tau)\,d\tau\right)^2$$

is uniformly convergent in $[a,b]$. Putting

$$\mu_i(v) = \lambda_i \int_a^v \varphi_i(\tau)\,d\tau,$$

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[3]) See W. Orlicz, *Beiträge zur Theorie der Orthogonalreihen I*, Studia Mathematica 1 (1929), p. 1-39.

the series

$$\sum_{i=1}^{\infty}\mu_i(v)\varphi_i(t)$$

becomes the Fourier series of a function $G(t,v)$ such that $G(:,v)\in L^2$ and the sums

$$\sum_{i=1}^{n}\mu_i(v)\varphi_i(t)$$

converge asymptotically in $[a,a;b,b]$ to $G(t,v)$, which implies the measurability of $G(t,v)$ in $[a,a;b,b]$. We easily observe that for $v\in[a,b]$

$$D(t,v)=\int_a^t G(\tau,v)\,d\tau,$$

(5)

$$D'(t,v)=G(t,v),$$

for almost every t and arbitrary v, and almost everywhere in the interval $[a,a;b,b]$, the functions on both sides being measurable.

Theorem 5. *Let the system $\{\varphi_i(t)\}$ be complete in L^2. Then the sequence $\{\lambda_i\}$ is a multiplicator of the class (M,M) if and only if the following conditions are satisfied:*

(I) *the sequence $\{\lambda_i\}$ is bounded;*

(IIa) $$\sup_t{}^*\operatorname{var}_v{}^* D_t'(t,v)<\infty,$$

or the equivalent inequality

(IIb) $$\int_a^b|D_t'(t,v+h)-D_t'(t,v)|\,dt\leqslant K|h|\quad \textit{for}\quad a-h\leqslant v\leqslant b-h.$$

The equivalence of (IIa) and (IIb) is to be understood as follows: (I) implies the existence of $D_t'(t,v)$ almost everywhere for $v\in[a,b]$. We extend the definition of $D_t'(t,v)$ to a function defined in $[a,a;b,b]$ setting $D_t'(t,v)=0$ for those (t,v) for which the partial derivative does not exist. This function appears in the conditions (IIa) and (IIb).

Proof. Necessity. For every $x=x(t)\in M$

$$\left|\int_a^b U(x,t)\varphi_i(t)\,dt\right|=\left|\lambda_i\int_a^b x(t)\varphi_i(t)\,dt\right|\leqslant\|U\|\|x\|\int_a^b|\varphi_i(t)|\,dt;$$

if, in particular, $x(t)=\operatorname{sign}\varphi_i(t)$, then

$$|\lambda_i|\leqslant\|U\|,$$

and (I) is satisfied.

Since the functions $\mu_i(v)$ are by hypothesis the Fourier coefficients of a bounded function, the completeness of the system $\{\varphi_i(t)\}$ implies $G(:,v)\in M$. Let

$$a=v_1<\ldots<v_n=b$$

be a partition $[a,b]$, and let $z(v)=\varepsilon_i$ for $v_i\leqslant v<v_{i+1}$, $\varepsilon_i=\pm1$, then

(6) $$U(z,t)=\sum_{i=1}^{n-1}\varepsilon_i\{G(t,v_{i+1})-G(t,v_i)\},$$

whence $\|U(z,:)\|\leqslant\|U\|\|z\|=\|U\|$, and the function $y(v)=D_t'(:,v)$ from the interval $[a,b]$ to M has a finite variation. Since the derivative $D_t'(t,v)$ extended as in Theorem 5 is measurable in $[a,a;b,b]$ and fulfils the condition (J), the conditions (IIa) and (IIb) follow from Theorem 3.

Now suppose the condition (I) to be satisfied. If, moreover, the condition (IIb) is satisfied, we can apply the Remark 1. By Theorem 2(b) the inequality $Vx(v)<\infty$ holds, and by Theorem 2(c) and Theorem 1 the condition (IIa) is satisfied. Now suppose the condition (IIa) to be satisfied. Since $D'(t,a)=0$, we can apply the Remark 2 to the function $D_t'(t,v)$, whence in virtue of Theorem 3 the condition (IIb) is satisfied. Thus the conditions (IIa) and (IIb) are equivalent.

If $x(v)=a_i$ for $v_i\leqslant v<v_{i+1}$, then

(7) $$\begin{aligned}|U(x,t)|&=\left|\sum_{i=1}^{n-1}a_i\{G(t,v_{i+1})-G(t,v_i)\}\right|\\&\leqslant\|x\|\,VG(:,v)=\|x\|\,VD_t'(:,v).\end{aligned}$$

Here $V^*D_t'(:,v)=VD_t'(:,v)<\infty$.

Now, $x(t)$ being an arbitrary bounded measurable function, choose a sequence $x_n(t)$ of step-functions converging to $x(t)$ and such that $\|x_n\|\leqslant\|x\|$; then

$$\lambda_i\int_a^b x_n(\tau)\varphi_i(\tau)\,d\tau=\int_a^b U(x_n,\tau)\varphi_i(\tau)\,d\tau\qquad\text{for } i=1,2,\ldots$$

Since (7) implies

$$|U(x_n,t)| \leqslant \|x\| \, VD'_t(:,v),$$

for almost every t there exists a subsequence $\{x_{n_i}\}$ and a function $U(x,\tau)\in M$ such that

$$\int_a^b U(x_{n_i},\tau)\varphi_j(\tau)\,d\tau \to \int_a^b U(x,\tau)\varphi_j(\tau)\,d\tau \qquad \text{for} \quad j=1,2,\ldots$$

Since $x_{n_i}(t)\to x(t)$, we get

$$\lambda_j \int_a^b x_{n_i}(\tau)\varphi_j(\tau)\,d\tau \to \lambda_j \int_a^b x(\tau)\varphi_j(\tau)\,d\tau = \int_a^b U(x,\tau)\varphi_j(\tau)\,d\tau,$$

i. e.

$$\lambda_j \int_a^b x(\tau)\varphi_j(\tau)\,d\tau = \int_a^b U(x,\tau)\varphi_j(\tau)\,d\tau.$$

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ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ МАЗУРА-ОРЛИЧА ИЗ ТЕОРИИ СУММИРОВАНИЯ

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Пусть дана последовательность функций $\{a_n(t)\}$ определённых на некотором множестве чисел T. Этой последовательности можно поставить в соответствие некоторый, так называемый *континуальный*, метод суммирования последовательностей чисел следующим образом.

Пусть t_∞ предельная точка множества T, а $x=\{\xi_k\}$ какая нибудь последовательность чисел. Если ряды

$$\sum_{k=0}^{\infty} a_k(t)\,\xi_k$$

сходятся для любого $t\in T$ отличного от t_∞, причём их суммы

$$A_t(x)=\sum_{k=0}^{\infty} a_k(t)\xi_k$$

стремятся к пределу, $\xi=A(x)=A_{t_\infty}(x)$, $t\to t_\infty$, то будем говорить, что последовательность $x=\{\xi^k\}$ *суммируется континуальным методом* A к этому пределу.

Частным случаем континуальных методов являются методы суммирования, задаваемые бесконечными матрицами, если например в качестве T возьмём множество натуральных чисел.

Континуальный метод A называется *перманентным*, если он суммирует всякую сходящуюся последовательность к тому же пределу, к которому она сходится.

Для того, чтобы континуальный метод был перманентным, необходимо и достаточно выполнение следующих трёх условий, аналогичных условиям, налагаемым на матрицы Тэплица

1⁰ $a_n(t)\to 0$, когда $t\to t_\infty$ $(n=1,2,\ldots)$.