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ON FUZZY BITOPOLOGICAL SPACES IN ŠOSTAK'S SENSE

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Abstract. In this paper, we used the supra fuzzy topology which generated from a fuzzy bitopological space [1] to introduce and study the concepts of continuity (resp. openness, closeness) of mapping, separation axioms and compactness for a fuzzy bitopological spaces. Our definition preserve much of the correspondence between concepts of fuzzy bitopological spaces and the associated fuzzy topological spaces.

1. Introduction and preliminaries

Šostak [16], introduce the fundamental concept of fuzzy topological structure as an extension of both crisp topology and Chang's fuzzy topology [5], in the sense that not only the object were fuzzified, but also the axiomatics. In [17, 18] Šostak gave some rules and showed how such an extension can be realized. Chattopdhyay et al. [6, 7] have redefined the similar concept. In [15, 8] Ramadan gave a similar definition namely "Smooth fuzzy topology" for lattice $L = [0, 1]$, it has been developed in many direction [4, 10-13, 17, 18]. Ghanim et al. [9] introduce the supra fuzzy topology as an extension of supra fuzzy topology in sense of Abd El-Monsef and Ramadan [2]. Abbas [1] generated the supra fuzzy topology from a fuzzy bitopological spaces. In this paper we have used the supra fuzzy topology which created from fuzzy bitopological spaces to introduce and study the concepts of continuity of mapping, separation axioms and compactness of the fuzzy bitopological spaces.

Throughout this paper, let X be a nonempty set $I = [0, 1]$, $I_0 = (0, 1]$ and I^X denote the set of all fuzzy subsets of X. FP (resp. FP^*)

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stand for fuzzy pairwise (resp. fuzzy P^*). For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \not{q} \mu$[14].

DEFINITION 1.1 [9, 16]. A mapping $\tau : I^X \longrightarrow I$ is called supra fuzzy topology on X if it satisfies the following conditions:

(S1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$.

(S2) $\tau(\vee_{i \in J} \mu_i) \geq \wedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called supra fuzzy topological space (briefly, sfts). A supra fuzzy topology τ is called fuzzy topology on X if

(T) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,

and the pair (X, τ) is called fuzzy topological space (briefly, fts).The triple (X, τ_1, τ_2) is called fuzzy bitopological space (briefly, fbts) where, τ_1 and τ_2 are fuzzy topologies on X. Throughout this paper, the indices $i, j \in \{1, 2\}$ and $i \neq j$.

DEFINITION 1.2 [11]. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*) is said to be:

(i)FP-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \tau_i^*(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$

(ii) FP-open if and only if $\tau_i^*(f(\mu)) \geq \tau_i(\mu)$ for each $\mu \in I^X$ and $i = 1, 2$

(iii) FP-closed if and only if $\tau_i^*(\underline{1} - f(\mu)) \geq \tau_i(\underline{1} - \mu)$ for each $\mu \in I^X$ and $i = 1, 2$.

THEOREM 1.1 [3]. *Let (X, T) be an ordinary topological space (resp. supra topological space). Then, the mapping $\omega(T) : I^X \longrightarrow I$ defined by*

$$\omega(T)(\lambda) = \vee\{\alpha \in I : \lambda^{-1}(\alpha, 1] \in T\}$$

for every $\lambda \in I^X$ is fuzzy topology (resp. supra fuzzy topology) on X.

This provides a "goodness of extension" criterion for fuzzy topological properties. Recall that a fuzzy extension of a topological property of (X, T) is said to be good when it is possessed by $\omega(T)$ if and only if the original property is possessed by T.

THEOREM 1.2 [3]. *Let (X, T) and (Y, T^*) be ordinary topological spaces. If a mapping $f : (X, T) \longrightarrow (Y, T^*)$ is continuous, then $f : (X, \omega(T)) \longrightarrow (Y, \omega(T^*))$ is fuzzy continuous.*

DEFINITION 1.3 [1]. A mapping $C : I^X \times I_0 \longrightarrow I^X$ is called supra fuzzy closure operator on X if for $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(C1) $C(\underline{0}, r) = \underline{0}$.
(C2) $\lambda \leq C(\lambda, r)$.
(C3) $C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$.
(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$.
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The pair (X, C) is called supra fuzzy closure space.

THEOREM 1.3 [1]. *Let (X, τ) be a sfts. For each $\lambda \in I^X$, $r \in I_0$ we define a mapping $C_\tau : I^X \times I_0 \longrightarrow I^X$ as follows:*

$$C_\tau(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, \tau(\underline{1} - \mu) \geq r\}.$$

Then, (X, C_τ) is supra fuzzy closure space. The mapping $I_\tau : I^X \times I_0 \longrightarrow I^X$ which defined by: $I_\tau(\lambda, r) = \{\mu : \mu \leq \lambda, \tau(\mu) \geq r\}$ satisfies $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda, r)$.

THEOREM 1.4 [1]. *Let (X, τ_1, τ_2) be a fbts . For each $\lambda \in I^X, r \in I_0$ we define a mapping $C_{12} : I^X \times I_0 \longrightarrow I^X$ as follows:*

$$C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r).$$

Then, (X, C_{12}) is supra fuzzy closure space. The mapping $I_{12} : I^X \times I_0 \longrightarrow I^X$ which defined by $I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r)$ satisfies $I_{12}(\underline{1} - \lambda, r) = \underline{1} - C_{12}(\lambda, r)$.

THEOREM 1.5 [1]. *Let (X, τ_1, τ_2) be a fbts. Let (X, C_{12}) be a supra fuzzy closure space. Define the mapping $\tau_s : I^X \longrightarrow I$ on X by*

$$\tau_s(\lambda) = \vee\{\tau_1(\lambda_1) \wedge \tau_1(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\},$$

where $\vee$ is taken over all families $\{\lambda_1, \lambda_2 : \lambda = \lambda_1 \vee \lambda_2\}$. Then,

(i) *$\tau_s = \tau_{C_{12}}$is the coarsest supra fuzzy topology on X which is finer than τ_1 and τ_2.*
(ii) $C_{12} = C_{\tau_s} = C_{\tau_{C_{12}}}$.

2. FP^*-continuity

We are now going to use the family τ_s which is generated by the two fuzzy topologies τ_1 and τ_2 to introduce another type of fuzzy continuity (resp. openness, closeness)of the fuzzy pairwise mappings.

DEFINITION 2.1. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*). f is called FP^*-continuous (resp. FP^*-open, FP^*-closed) if and only if $f : (X, \tau_s) \to (Y, \tau_s^*)$ is FP-continuous (resp. FP-open, FP-closed).

THEOREM 2.1. *Every FP-continuous (resp. FP-open, FP-closed) mapping is FP^*-continuous (resp. FP^*-open, FP^*-closed).*

PROOF. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a FP-continuous mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*) and (X, τ_s), (Y, τ_s^*) their associated sfts's. Suppose that there exists $\mu \in I^Y$ and $r \in I_0$ such that

$$\tau_s(f^{-1}(\mu)) < r \leq \tau_s^*(\mu).$$

There exists $\mu_1, \mu_2 \in I^Y$ with $\mu = \mu_1 \vee \mu_2$ such that $\tau_s^*(\mu) = \tau_1^*(\mu_1) \wedge \tau_2^*(\mu_2) \geq r$. Then, $\tau_1^*(\mu_1) \geq r$ and $\tau_2^*(\mu_2) \geq r$. From FP-continuity we have

$$\tau_1(f^{-1}(\mu_1)) \geq \tau_1^*(\mu_1) \geq r \quad \text{and} \quad \tau_2(f^{-1}(\mu_2)) \geq \tau_2^*(\mu_2) \geq r.$$

This implies that

$$\tau_1(f^{-1}(\mu_1)) \wedge \tau_2(f^{-1}(\mu_2)) \geq r.$$

Since $f^{-1}(\mu) = f^{-1}(\mu_1) \vee f^{-1}(\mu_2)$, we have $\tau_s(f^{-1}(\mu)) \geq r$. It is a contradiction. Hence,

$$\tau_s(f^{-1}(\mu)) \geq \tau_s^*(\mu), \quad \forall \mu \in I^Y.$$

Thus, f is FP^*-continuous. The other parts can be proved by the same manner. □

EXAMPLE 2.1. Let $X = \{a, b, c\}$ and $Y = \{x, y\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1(a) = 0.3 \qquad \lambda_1(b) = 0.3 \qquad \lambda_1(c) = 0.5$$

$$\lambda_2(a) = 0.5 \qquad \lambda_2(b) = 0.5 \qquad \lambda_2(c) = 0.3$$

$$\mu_1(x) = 0.5 \qquad \mu_1(y) = 0.3$$

$$\mu_2(x) = 0.3 \qquad \mu_2(y) = 0.5$$

We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ and $\tau_1^*, \tau_2^* : I^Y \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.5, & \text{if } \lambda = \lambda_1 \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.3, & \text{if } \lambda = \lambda_2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.3, & \text{if } \mu = \mu_1 \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.2, & \text{if } \mu = \mu_2 \\ 0, & \text{otherwise.} \end{cases}$$

From fbts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce supra fuzzy topologies τ_s and τ_s^* as follows:

$$\tau_s(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.5, & \text{if } \lambda = \lambda_1 \\ 0.3, & \text{if } \lambda = \lambda_2 \\ 0.3, & \text{if } \lambda = \lambda_1 \vee \lambda_2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_s^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.3, & \text{if } \mu = \mu_1 \\ 0.2, & \text{if } \mu = \mu_2 \\ 0.2, & \text{if } \mu = \mu_1 \vee \mu_2 \\ 0, & \text{otherwise.} \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ defined by:

$$f(a) = x, \qquad f(b) = x, \qquad f(c) = y.$$

Then, f is FP^*-continuous but not FP-continuous.

EXAMPLE 2.2. In the above example we define τ_1^* and τ_2^* as follows:

$$\tau_1^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.6, & \text{if } \mu = \mu_1 \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.5, & \text{if } \mu = \mu_2 \\ 0, & \text{otherwise.} \end{cases}$$

From fbts (Y, τ_1^*, τ_2^*) we can induce the supra fuzzy topology τ_s^* as follows:

$$\tau_s^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \underline{1} \\ 0.6, & \text{if } \mu = \mu_1 \\ 0.5, & \text{if } \mu = \mu_2 \\ 0.5, & \text{if } \mu = \mu_1 \vee \mu_2 \\ 0, & \text{otherwise.} \end{cases}$$

Then, the mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ is FP^*-open but not FP-open.

THEOREM 2.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*). Then, the following conditions are equivalent:*

(i) *f is FP^*-continuous.*

(ii) *$\tau_s(\underline{1} - f^{-1}(\mu)) \geq \tau_s^*(\underline{1} - \mu)$ for each $\mu \in I^Y$.*

(iii) *$f(C_{12}(\lambda, r)) \leq C_{12}(f(\lambda), r)$for each $\lambda \in I^X$ and $r \in I_0$.*

(iv) *$C_{12}(f^{-1}(\mu), r) \leq f^{-1}(C_{12}(\mu, r))$for each $\mu \in I^Y$ and $r \in I_0$.*

(v) *$f^{-1}(I_{12}(\mu, r)) \leq I_{12}(f^{-1}(\mu), r)$for each $\mu \in I^Y$ and $r \in I_0$.*

PROOF. (i) $\Rightarrow$ (ii) It is easily proved from Definition 2.1, and the fact $f^{-1}(\underline{1} - \mu) = \underline{1} - f^{-1}(\mu)$.

(ii) $\Rightarrow$ (iii) For each $\lambda \in I^X$ and $r \in I_0$ we have,

$$\begin{aligned} f^{-1}(C_{12}(f(\lambda), r)) &= f^{-1}(C_{\tau_s^*}(f(\lambda), r)) \qquad \text{(from Theorem 1.5.)} \\ &= f^{-1}[\wedge\{\eta \in I^Y : \eta \geq f(\lambda), \tau_s^*(\underline{1} - \eta) \geq r\}] \\ &= \wedge\{f^{-1}(\eta) \in I^X : f^{-1}(\eta) \geq \lambda, \tau_s^*(\underline{1} - \eta) \geq r\} \\ &\geq \wedge\{f^{-1}(\eta) \in I^X : f^{-1}(\eta) \geq \lambda, \tau_s(\underline{1} - f^{-1}(\eta)) \geq r\} \\ &= C_{\tau_s}(\lambda, r) = C_{12}(\lambda, r). \end{aligned}$$

Thus, $f(C_{12}(\lambda, r)) \leq C_{12}(f(\lambda), r)$.

(iii) $\Rightarrow$ (iv) For all $\mu \in I^Y, r \in I_0$, put $\lambda = f^{-1}(\mu)$. From (iii) we have,

$$f(C_{12}(f^{-1}(\mu), r)) \leq C_{12}(f(f^{-1}(\mu)), r) \leq C_{12}(\mu, r).$$

This implies that

$$C_{12}(f^{-1}(\mu), r) \leq f^{-1}(f(C_{12}(f^{-1}(\mu), r))) \leq f^{-1}(C_{12}(\mu, r)).$$

(iv) $\Rightarrow$ (v) For all $\mu \in I^Y, r \in I_0$ we have

$$C_{12}(f^{-1}(\underline{1} - \mu), r) \leq f^{-1}(C_{12}(\underline{1} - \mu, r)).$$

This implies that

$$\underline{1} - f^{-1}(C_{12}(\underline{1} - \mu, r)) \leq \underline{1} - C_{12}(f^{-1}(\underline{1} - \mu), r).$$

Therefore

$$f^{-1}(\underline{1} - C_{12}(\underline{1} - \mu, r)) \leq \underline{1} - C_{12}(f^{-1}(\underline{1} - \mu), r).$$

By Theorem 1.4, we have

$$f^{-1}(I_{12}(\mu, r)) \leq I_{12}(\underline{1} - f^{-1}(\underline{1} - \mu), r) = I_{12}(f^{-1}(\mu), r).$$

(v) $\Rightarrow$ (i) Suppose that there exists $\mu \in I^Y$ and $r \in I_0$ such that

$$\tau_s(f^{-1}(\mu)) < r \leq \tau_s^*(\mu).$$

Then, there exist $\mu_1, \mu_2 \in I^Y$ such that

$$\tau_s^*(\mu) = \tau_1^*(\mu_1) \vee \tau_2^*(\mu_2) \geq r \quad \text{and} \quad \mu = \mu_1 \vee \mu_2.$$

This implies that $\tau_1^*(\mu_1) \geq r$ and $\tau_2^*(\mu_2) \geq r$. Then,

$$I_{\tau_1^*}(\mu_1, r) = \mu_1 \quad \text{and} \quad I_{\tau_2^*}(\mu_2, r) = \mu_2.$$

From Theorem 1.4, we have

$$I_{12}(\mu, r) = I_{\tau_1^*}(\mu_1, r) \vee I_{\tau_2^*}(\mu_2, r) = \mu_1 \vee \mu_2 = \mu.$$

By (v) we have

$$f^{-1}(\mu) = f^{-1}(I_{12}(\mu, r)) \leq I_{12}(f^{-1}(\mu), r).$$

Thus,

$$\begin{aligned} f^{-1}(\mu) = I_{12}(f^{-1}(\mu), r) &= \underline{1} - C_{12}(\underline{1} - f^{-1}(\mu), r) \\ &= \underline{1} - C_{\tau_s}(\underline{1} - f^{-1}(\mu), r) = I_{\tau_s}(f^{-1}(\mu), r). \end{aligned}$$

This implies that

$$\tau_s(f^{-1}(\mu)) \geq r.$$

It is a contradiction. So, $\tau_s(f^{-1}(\mu)) \geq \tau_s^*(\mu)$ for each $\mu \in I^Y$. Hence, $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ is FP^*-continuous.

3. Separation axioms for fuzzy bitopological spaces

DEFINITION 3.3. A fbts (X, τ_1, τ_2) is called:
(i) FPR_0 if and only if $x_t \not q C_{\tau_i}(y_m, r)$ implies that $y_m \not q C_{\tau_j}(x_t, r)$.
(ii) FPR_1 if and only if $x_t \not q C_{\tau_i}(y_m, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda \not q \mu$.
(iii) FPR_2 if and only if $x_t \not q \rho = C_{\tau_i}(\rho, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ such that $x_t \in \lambda$, $\rho \leq \mu$ and $\lambda \not q \mu$.
(iv) FPR_3 if and only if $\eta = C_{\tau_i}(\eta, r) \not q \rho = C_{\tau_j}(\rho, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda \not q \mu$.
(v) FPT_0 if and only if $x_t \not q y_m$ implies that there exists $\lambda \in I^X$ such that for i=1 or 2 $\tau_i(\lambda) \geq r$ and $x_t \in \lambda$, $y_m \not q \lambda$ or $y_m \in \lambda$, $x_t \not q \lambda$.
(vi) FPT_1 if and only if $x_t \not q y_m$ implies that there exists $\lambda \in I^X$ such that for i=1 or 2 $\tau_i(\lambda) \geq r$, $x_t \in \lambda$ and $y_m \not q \lambda$.
(vii) FPT_2 if and only if $x_t \not q y_m$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda \not q \mu$.
(viii) $FPT_{2\frac{1}{2}}$ if and only if $x_t \not q y_m$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $C_{\tau_j}(\lambda, r) \not q C_{\tau_i}(\mu, r)$.
(ix) FPT_3 if and only if it is FPR_2 and FPT_1.
(x) FPT_4 if and only if it is FPR_3 and FPT_1.

Now, by making use of the supra fuzzy topology τ_s generated by the two fuzzy topologies τ_1 and τ_2, we introduce and study weaker forms of the fuzzy pairwise separation axioms FPT_i $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$ and FPR_i $i = 0, 1, 2$.

DEFINITION 3.1. A fbts (X, τ_1, τ_2) is called:
(i) FP^*R_i if and only if its associated sfts (X, τ_s) is FR_i, $i = 0, 1, 2$.
(ii) FP^*T_i if and only if its associated sfts (X, τ_s) is FT_i, $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$.

THEOREM 3.1. *Let (X, τ_1, τ_2) be a fbts. Then, we have*
(i) $FPR_i \Rightarrow FP^*R_i$, $i = 0, 1, 2$.
(ii) $FPT_i \Rightarrow FP^*T_i$, $i = 0, 1, 2, 2\frac{1}{2}, 3$.
(iii) $FPT_i \Leftrightarrow FP^*T_i$, $i = 0, 1$.

PROOF. (i) Let (X, τ_1, τ_2) be FPR_0 and let $x_t \not q C_{\tau_s}(y_m, r)$ from Theorem 1.5, we have $x_t \not q C_{12}(y_m, r)$. Also, by Theorem 1.4, we have

$x_t \not{q}[C_{\tau_1}(y_m, r) \wedge C_{\tau_2}(y_m, r)]$. Then

$x_t \in \underline{1} - [C_{\tau_1}(y_m, r) \wedge C_{\tau_2}(y_m, r)] = [\underline{1} - C_{\tau_1}(y_m, r)] \vee [\underline{1} - C_{\tau_2}(y_m, r)]$.

This implies that $x_t \in \underline{1} - C_{\tau_1}(y_m, r)$ or $x_t \in \underline{1} - C_{\tau_2}(y_m, r)$, therefore $x_t \not{q}C_{\tau_1}(y_m, r)$ or $x_t \not{q}C_{\tau_2}(y_m, r)$. Since (X, τ_1, τ_2) is a FPR_0, we have

$$y_m \not{q}C_{\tau_1}(x_t, r) \text{ or } \quad y_m \not{q}C_{\tau_2}(x_t, r).$$

This implies that

$$y_m \not{q}[C_{\tau_1}(x_t, r) \wedge C_{\tau_2}(x_t, r)] = C_{12}(x_t, r).$$

Then, $y_m \not{q}C_{\tau_s}(x_t, r)$, so (X, τ_1, τ_2) is FP^*R_0. (For $i = 1, 2$ the proof is similar).

(ii)Let (X, τ_1, τ_2) be a $FPT_{2\frac{1}{2}}$ and $x_t \not{q}y_m$. Then there exist $\lambda, \mu \in I^X$ such that $x_t \in \lambda$, $y_m \in \mu$, $\tau_i(\lambda) \geq r$, $\tau_j(\mu) \geq r$ and $C_{\tau_j}(\lambda, r) \not{q}C_{\tau_i}(\mu, r)$. Since, $C_{\tau_s} \leq C_{\tau_i}$ for $i = 1, 2$ we have, $C_{\tau_s}(\lambda, r) \ /qC_{\tau_s}(\mu, r)$. Then (X, τ_1, τ_2) be a $FP^*T_{2\frac{1}{2}}$. (For $i = 0, 1, 2, 3$ the proof is similar).

(iii) Necessity: follows from (ii). Sufficiency: Let (X, τ_1, τ_2) be FP^*T_1 and $x_t \not{q}y_m$. Then, there exists $\lambda \in I^X$ such that $x_t \in \lambda$, $\tau_s(\lambda) \geq r$ and $y_m \not{q}\lambda$. Since $\tau_s(\lambda) \geq r$ then there exist $\lambda_1, \lambda_2 \in I^X$ such that

$$\tau_s(\lambda) = \tau_1(\lambda_1) \wedge \tau_2(\lambda_2), \quad \lambda = \lambda_1 \vee \lambda_2.$$

Then, $\tau_1(\lambda_1) \geq r$ and $\tau_2(\lambda_2) \geq r$. $x_t \in \lambda$ implies that $x_t \in \lambda_1$ or $x_t \in \lambda_2$. Also, $y_m \not{q}\lambda$ implies that $y_m \not{q}\lambda_1$ and $y_m \not{q}\lambda_2$. Thus

$$(x_t \in \lambda_1, \tau_1(\lambda) \geq r, \text{and} y_m \not{q}\lambda_1) \quad \text{or} \quad (x_t \in \lambda_2, \tau_2(\lambda) \geq r, \text{and} y_m \not{q}\lambda_2).$$

Hence, (X, τ_1, τ_2) is FPT_1. (For $i = 0$ the proof is similar). □

LEMMA 3.1. *Let (X, τ_1, τ_2) be a fbts. Then:*
(i) *If (X, τ_1) or (X, τ_2) is FT_i, then (X, τ_1, τ_2) is FP^*T_i $i = 0, 1, 2, 2\frac{1}{2}, 3$.*
(ii) *If (X, τ_1) or (X, τ_2) is FR_i, then (X, τ_1, τ_2) is FP^*R_i $i = 0, 1, 2$.*

PROOF. The proof is easy. □

EXAMPLE 3.1. Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{3}, & \text{if } \lambda \in \{x_\alpha, y_\alpha\}, \alpha \in (0,1) \\ \frac{1}{2}, & \text{if } \lambda \in \{x_\alpha \vee y_\alpha, x_\alpha \vee y_1, x_1 \vee y_\alpha\}, \alpha \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{x_\alpha, y_\alpha, x_\alpha \vee y_\alpha\}, \alpha \in (\frac{1}{2}, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq \frac{1}{3}$ the fbts (X, τ_1, τ_2) is FP^*T_3 since, (X, τ_1) is FT_1 and FR_2(see Lemma 3.1,) but (X, τ_1, τ_2) is not FPT_3.

EXAMPLE 3.2. Let $X = \{x, y, z\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \text{if} \quad \lambda = x_\alpha, \alpha \in (0, \frac{1}{2}] \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{3}, & \text{if} \quad \lambda \in \{x_\alpha, y_\alpha, z_\alpha\}, \alpha \in (0, 1] \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{x_\alpha \vee y_\alpha, x_\alpha \vee z_\alpha, y_\alpha \vee z_\alpha\}, \alpha \in (0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq \frac{1}{3}$ the fts (X, τ_2) is FT_2 and from Lemma 3.1, we have (X, τ_1, τ_2) is FP^*T_2, but it is not FPT_2.

EXAMPLE 3.3. Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{x_1, y_1\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{2}{3}, & \text{if} \quad \lambda \in \{x_{0.5}, \underline{0.5}\} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq \frac{1}{2}$ the fts (X, τ_1) is $FT_{2\frac{1}{2}}$ and from Lemma 3.1, we have (X, τ_1, τ_2) is $FP^*T_{2\frac{1}{2}}$, but it is not $FPT_{2\frac{1}{2}}$.

EXAMPLE 3.4. Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \text{if} \quad \lambda = \underline{\alpha}, \alpha \in (0, 0.3] \cup (0.4, 1) \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{4}, & \text{if} \quad \lambda = \underline{\alpha}, \alpha \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq \frac{1}{4}$ the fts (X, τ_1) is FR_2 and from Lemma 3.1, we have (X, τ_1, τ_2) is FP^*R_2, but it is not FPR_2.

EXAMPLE 3.5. Let $X = \{x, y\}$. We define fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{3}, & \text{if} \quad \lambda \in \{x_1, y_1\} \\ 0, & \text{otherwise} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{2}{3}, & \text{if} \quad \lambda = \underline{0.3} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq \frac{1}{3}$ the fts (X, τ_1) is FR_1 (resp.FR_0) and from Lemma 3.1, we have (X, τ_1, τ_2) is FP^*R_1, (resp. FP^*R_0) but it is not FPR_1, (resp. FPR_0).

LEMMA 3.2. *Let (X, τ_1, τ_2) be a fbts. For $r \in I_0$ we have the following:*

(i) *For all $\lambda \in I^X$ with $\tau_s(\lambda) \geq r$, $\lambda q \mu$ if and only if $\lambda q C_{12}(\mu, r)$, $\mu \in I^X$.*

(ii) *$x_t q C_{12}(\lambda, r)$ if and only if $\lambda q \mu$ for all $\mu \in I^X$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$.*

PROOF. (i) Let $\lambda \in I^X$ with $\tau_s(\lambda) \geq r$, $\lambda q \mu$. Since $\mu \leq C_{12}(\mu, r)$, $\lambda q C_{12}(\mu, r)$. Conversely, let $\lambda \in I^X$ with $\tau_s(\lambda) \geq r$ and suppose that $\lambda \not{q} \mu$, then $\mu \leq \underline{1} - \lambda$ this implies that

$$C_{\tau_s}(\mu, r) \leq C_{\tau_s}(\underline{1} - \lambda, r) = \underline{1} - \lambda.$$

By using Theorem 1.5, we have $C_{12}(\mu, r) \leq \underline{1} - \lambda$. Then, $\lambda \not{q} C_{12}(\mu, r)$. This is a contradiction.

(ii) Let $x_t q C_{12}(\lambda, r)$. Since $x_t \in \mu$, $\mu q C_{12}(\lambda, r)$. By (i) we have $\mu q \lambda$ for all $\mu \in I^X$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$. Conversely, suppose that $x_t \not{q} C_{12}(\lambda, r)$. Then, $x_t \in \underline{1} - C_{12}(\lambda, r)$. Let $\mu = \underline{1} - C_{12}(\lambda, r)$. By Theorem 1.5, $\mu = \underline{1} - C_{\tau_s}(\lambda, r)$, then $\tau_s(\mu) \geq r$. Since, $\lambda \leq C_{12}(\lambda, r)$, then $\mu = \underline{1} - C_{12}(\lambda, r) \leq \underline{1} - \lambda$ this implies that $\lambda \not{q} \mu$ a contradiction.□

THEOREM 3.2. *Let (X, τ_1, τ_2) be a fbts. Then the following statements are equivalent:*

(i) *(X, τ_1, τ_2) is FP^*R_0.*

(ii) $C_{12}(x_t, r) \leq \mu$ *for all* $\mu \in I^X, r \in I_0$ *with* $\tau_s(\mu) \geq r$ *and* $x_t \in \mu$

(iii) *If* $x_t \not{q} \lambda = C_{\tau_s}(\lambda, r)$, *there exits* $\mu \in I^X$ *with* $\tau_s(\mu) \geq r$ *such that* $x_t \not{q}\mu$ *and* $\lambda \leq \mu$, $r \in I_0$.

(iv) *If* $x_t \not{q}\lambda = C_{\tau_s}(\lambda, r)$ *then,* $C_{\tau_s}(x_t, r) \not{q}\lambda = C_{\tau_s}(\lambda, r)$, $\lambda \in I^X, r \in I_0$.

(v) *If* $x_t \not{q} C_{\tau_s}(y_m, r)$ *then,* $C_{\tau_s}(x_t, r) \not{q} C_{\tau_s}(y_m, r)$, $r \in I_0$.

PROOF. (i) $\Rightarrow$ (ii) Let $y_m q C_{12}(x_t, r)$. By Theorem 1.5, we have $y_m q C_{\tau_s}(x_t, r)$. By using (i) we obtain $x_t q C_{\tau_s}(y_m, r)$ i.e. $x_t q C_{12}(y_m, r)$. By using Lemma 3.2(ii), we can found that, $y_m q \mu$ for all $\mu \in I^X$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$. Then, we have $C_{12}(x_t, r) \leq \mu$ for all $\mu \in I^X, r \in I_0$ with $\tau_s(\mu) \geq r$ and $x_t \in \mu$.

(ii) $\Rightarrow$ (i) If $y_m \not{q} C_{\tau_s}(x_t, r)$, we have $y_m \in \underline{1} - C_{\tau_s}(x_t, r)$. By (ii) and the fact $\tau_s(\underline{1} - C_{\tau_s}(x_t, r)) \geq r$ we obtain

$$C_{12}(y_m, r) \leq \underline{1} - C_{\tau_s}(x_t, r) \leq \underline{1} - x_t.$$

Thus

$$x_t \not{q} C_{12}(y_m, r) = C_{\tau_s}(y_m, r).$$

Hence, (X, τ_1, τ_2) is FP^*R_0.

(i) $\Rightarrow$ (iii) Let $x_t \not{q}\lambda = C_{\tau_s}(\lambda, r)$. Since $C_{\tau_s}(y_m, r) \leq C_{\tau_s}(\lambda, r)$ for each $y_m \in \lambda$ we have $x_t \not{q} C_{\tau_s}(y_m, r)$. By (i) we have $y_m \not{q} C_{\tau_s}(x_t, r)$. By Lemma 3.2(ii), for each $y_m \not{q} C_{\tau_s}(x_t, r)$, there exists $\eta_{y_m} \in I^X$ such that $x_t \not{q}\eta_{y_m}$, $\tau_s(\eta_{y_m}) \geq r$, $y_m \in \eta_{y_m}$. Let

$$\mu = \vee_{y_m \in \lambda}\{\eta_{y_m} : x_t \not{q}\eta_{y_m}\}.$$

From Definition 1.1, we have $\tau_s(\mu) \geq r$. Then,

$$x_t \not{q}\mu, \lambda \leq \mu, \tau_s(\mu) \geq r.$$

(iii) $\Rightarrow$ (iv) Let $x_t \not{q}\lambda = C_{\tau_s}(\lambda, r)$. By (iii) there exists $\mu \in I^X$ such that

$$x_t \not{q}\mu, \lambda \leq \mu, \tau_s(\mu) \geq r.$$

Since $x_t \not{q}\mu$, it follows that $x_t \in \underline{1} - \mu$, this implies that

$$C_{\tau_s}(x_t, r) \leq C_{\tau_s}(\underline{1} - \mu, r) = \underline{1} - \mu \leq \underline{1} - \lambda.$$

Hence, $C_{\tau_s}(x_t, r) \not{q}\lambda = C_{\tau_s}(\lambda, r)$.

(iv) ⇒ (v) Let $x_t \not{q} C_{\tau_s}(y_m, r)$. Since, $C_{\tau_s}(C_{\tau_s}(y_m, r), r) = C_{\tau_s}(y_m, r)$ and by using (iv) we have $C_{\tau_s}(x_t, r) \not{q} C_{\tau_s}(y_m, r)$.

(v) ⇒ (i) Let $x_t \not{q} C_{\tau_s}(y_m, r)$. By (v) we have $C_{\tau_s}(x_t, r) \not{q} C_{\tau_s}(y_m, r)$ and since $y_m \leq C_{\tau_s}(y_m, r)$ then $y_m \not{q} C_{\tau_s}(x_t, r)$. Hence, (X, τ_1, τ_2) is FP^*R_0. □

LEMMA 3.3. *Let (X, T_1, T_2) be an ordinary topological space. Let $\omega(T_1)$ and $\omega(T_2)$ be the induced fuzzy topologies of T_1 and T_2 respectively. Also, let $\omega(T_s)$ be the induced supra fuzzy topology of the supra topology T_s and let*

$$(\omega(T))_s(\lambda) = \vee\{\omega(T_1)(\lambda_1) \wedge \omega(T_2)(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2\}.$$

Then,

$$\omega(T_s) \geq (\omega(T))_s$$

PROOF. Suppose that there exists $\lambda \in I^X$ and $r_0 \in I_0$ such that

$$(\omega(\tau))_s(\lambda) \geq r_0 > \omega(T_s)(\lambda).$$

Then, there exist $\lambda_1, \lambda_2 \in I^X$ such that $\lambda = \lambda_1 \vee \lambda_2$ with ,

$$\omega(T_1)(\lambda_1) \geq r_0 \quad \text{and} \quad \omega(T_2)(\lambda_2) \geq r_0.$$

Then

$$\lambda_1^{-1}(r_0, 1] \in T_1 \quad \text{and} \quad \lambda_2^{-1}(r_0, 1] \in T_2.$$

This implies that

$$\lambda^{-1}(r_0, 1] = \lambda_1^{-1}(r_0, 1] \cup \lambda_2^{-1}(r_0, 1] \in T_s.$$

Then, $(\omega(T_s))(\lambda) \geq r_0$ which is a contradiction. Hence,

$$(\omega(T))_s \leq \omega(T_s).$$

□

THEOREM 3.3. *Let (X, T_1, T_2) be an ordinary bitopological space.*
(i) *If $(X, \omega(T_1), \omega(T_2))$ is FP^*T_i, then (X, T_1, T_2) is P^*T_i, $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$.*
(ii) *If $(X, \omega(T_1), \omega(T_2))$ is FP^*R_i, then (X, T_1, T_2) is P^*R_i, $i = 0, 1, 2$.*

PROOF. (i) (For $i = 2$): Let $x \neq y$ and suppose that $x_t \not{q} y_m$. Let $r \in I_0$ such that $r < t, m$. Since $(X, \omega(T_1), \omega(T_2))$ is FP^*T_2 there exist $\lambda, \mu \in I^X$ with $(\omega(T))_s(\lambda) \geq r, (\omega(T))_s(\mu) \geq r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda \not{q} \mu$. By Lemma 3.3, we have

$$\omega(T_s)(\lambda) \geq (\omega(T))_s(\lambda) \geq r.$$

Then, $\lambda^{-1}(r, 1] \in T_s$. Since $x_t \in \lambda$ then, $\lambda(x) \geq t > r$ implies that $x \in \lambda^{-1}(r, 1]$. Similarly $\mu^{-1}(r, 1] \in T_s$ and $y \in \mu^{-1}(r, 1]$. Since, $\lambda \not{q} \mu$ then, $\lambda \leq \underline{1} - \mu$, implies that

$$\lambda^{-1}(r, 1] \subseteq X - \mu^{-1}(r, 1].$$

Then,

$$\lambda^{-1}(r, 1] \cap \mu^{-1}(r, 1] = \phi.$$

Hence, (X, T_1, T_2) is P^*T_2. (for, $i = 0, 1, 2\frac{1}{2}, 3, 4$ the proof is similar).

(ii) The proof is similar to part (i). □

4. FP^*-compactness

DEFINITION 4.1. Let (X, τ) be a fts and $\mu \in I^X$, $r \in I_0$. Then :

(i) The family $\{\eta_j : \tau(\eta_j) \geq r, j \in J\}$ is called τ-cover of μ if and only if for each $x_t \in \mu$ there exists $j_0 \in J$ such that $x_t \in \eta_{j_0}$.

(ii) μ is C-set if and only if every τ-cover of μ have a finite subcover.

(iii)(X, τ) is called F-compact if and only if for every $\lambda \in I^X$ such that $\tau(\underline{1} - \lambda) \geq r$ is C-set .

DEFINITION 4.2. A fbts (X, τ_1, τ_2) is called FP^*-compact if and only if its associated sfts (X, τ_s) is F-compact.

THEOREM 4.1. *Let (X, τ_1, τ_2) be a fbts . If (X, τ_1) or (X, τ_2) is F-compact, then (X, τ_1, τ_2) is FP^*-compact.*

PROOF. Let (X, τ_1) be F-compact. Let $\lambda \in I^X$ such that $\tau_s(\underline{1} - \lambda) \geq r$, $r \in I_0$ and $\{\eta_j : \tau_s(\eta_j) \geq r, j \in J\}$ be a τ_s-cover of λ. Since $\tau_s(\underline{1} - \lambda) \geq r$, then we can write

$$\lambda = \lambda_1 \wedge \lambda_2, \quad \tau_i(\underline{1} - \lambda_i) \geq r, \quad (i = 1, 2).$$

Then, for every $x_t \in \lambda$, there exists $\eta_{j_0} \in I^X$ with $\tau_s(\eta_{j_0}) \geq r$ such that

$$x_t \in \eta_{j_0} = \eta_{j_0}^{(1)} \vee \eta_{j_0}^{(2)}$$

for some $\eta_{j_0}^{(i)} \in I^X$ with $\tau_i(\eta_{j_0}^{(i)}) \geq r$, $(i = 1, 2)$, then $x_t \in \eta_{j_0}^{(1)}$ or $x_t \in \eta_{j_0}^{(2)}$. Now, $\{\eta_{j_0}^{(1)}(i) : \tau_1(\eta_{j_0}^{(1)}(i)) \geq r, i = 1, 2, 3, \dots\}$ is a τ_1-cover of λ_1 or $\{\eta_{j_0}^{(2)}(i) : \tau_2(\eta_{j_0}^{(2)}(i)) \geq r, i = 1, 2, 3, \dots\}$ is a τ_2-cover of λ_2. If (X, τ_1) is F-compact, then λ_1 is C-set. So, there exists a finite τ_1-cover $\{\eta_{j_0}^{(1)}(i) : i = 1, 2, 3, \dots, n\}$ of λ_1, this implies that

$$\lambda \leq \lambda_1 \leq \vee_{i=1}^{n} \eta_{j_0}^{(1)}(i).$$

Hence, λ is C-set consequently, the fbts (X, τ_1, τ_2) is FP^*-compact. Similarly, if (X, τ_2) is F-compact then, (X, τ_1, τ_2) is FP^*-compact. □

THEOREM 4.2. *Let (X, τ_1, τ_2) be a FP^*T_3 and $\mu \in I^X$ is C-set. Then, for every $\lambda \in I^X$ with $\tau_s(\underline{1} - \lambda) \geq r, r \in I_0$ such that $\lambda \not{q} \mu$, there is $\eta, \rho \in I^X$ with $\tau_s(\eta) \geq r$, $\tau_s(\rho) \geq r$ and $\eta \not{q} \rho$.*

PROOF. Since $\lambda \, /q\mu$, then for each $x_t \in \mu$ we have, $x_t \, /q\lambda = C_{\tau_s}(\lambda, r)$, $r \in I_0$. Since (X, τ_1, τ_2) is FP^*T_3, there exists $\eta^*, \rho^* \in I^X$ with $\tau_s(\eta^*) \geq r$, $\tau_s(\rho^*) \geq r$ such that $x_t \in \eta^*$, $\lambda \leq \rho^*$ and $\eta^* \, /q\rho^*$. Then, $\{\eta^* : \tau_s(\eta^*) \geq r, x_t \in \mu\}$ is τ_s-cover of μ. Since, μ is a C-set, then $\mu \leq \vee_{i=1}^{n} \eta^*(i)$. Let $\eta = \vee_{i=1}^{n} \eta^*(i)$ and $\rho = \rho^*(x_t^i)$ for all i. Then $\eta \not{q} \rho$.□

THEOREM 4.3. *Let (X, τ_1, τ_2) be a FP^*T_2, x_t be any fuzzy point of X and $\lambda \in I^X$ is a C-set such that $x_t \not{q} \lambda$. Then, there exist $\eta_1, \eta_2 \in I^X$ with $\tau_s(\eta_1) \geq r$, $\tau_s(\eta_2) \geq r$, $r \in I_0$ such that $x_t \in \eta_1$, $\lambda \leq \eta_2$ and $\eta_1 \not{q} \eta_2$. Moreover, if $\lambda, \mu \in I^X$ are C-sets such that $\lambda \not{q} \mu$, then there exist $\rho_1, \rho_2 \in I^X$ with $\tau_s(\rho_1) \geq r$, $\tau_s(\rho_2) \geq r$, $r \in I_0$ such that $\lambda \leq \rho_1$, $\mu \leq \rho_2$ and $\rho_1 \not{q} \rho_2$.*

PROOF. Since $x_t \not{q} \lambda$, then $x_t \not{q} y_m$ for each $y_m \in \lambda$. Since (X, τ_1, τ_2) is FP^*T_2, there exist $\eta_1, \eta^* \in I^X$ with $\tau_s(\eta_1) \geq r$, $\tau_s(\eta^*) \geq r$, $r \in I_0$ such that $x_t \in \eta_1$, $y_m \in \eta^*$ and $\eta_1 \not{q} \eta^*$. Then, $\{\eta^* : \tau_s(\eta^*) \geq r, y_m \in \lambda\}$ is a τ_s-cover of λ. Since λ is C-set, there exists a finite subcover $\{\eta^*(i) : i = 1, 2, 3 \dots, n\}$ of λ. Let $\eta_2 = \vee_{i=1}^{n} \eta^*(i)$. Then

$$\tau_s(\eta_2) = \tau_s(\vee_{i=1}^{n} \eta^*(i)) \geq \wedge_{i=1}^{n} \tau_s(\eta^*(i)) \geq r.$$

Since $\eta_1 \not{q} \eta^*(i)$, $(i = 1, 2, 3, \dots, n)$, then $\eta_1 \leq \underline{1} - \eta^*(i)$, $(i = 1, 2, 3, \dots, n)$, this implies that

$$\eta_1 \leq \wedge_{i=1}^{n} (\underline{1} - \eta^*(i)) = \underline{1} - \vee_{i=1}^{n} \eta^*(i) = \underline{1} - \eta_2.$$

Then, $\eta_1 \not{q} \eta_2$. For the second part, let $x_t \in \mu$, since $\lambda \not{q} \mu$, then $x_t \not{q} \lambda$, by the first part there exist $\rho^*, \rho_2 \in I^X$ with $\tau_s(\rho^*) \geq r$, $\tau_s(\rho_2) \geq r$, $r \in I_0$

such that $x_t \in \rho^*$, $\lambda \leq \rho_2$ and $\rho^* \not{q} \rho_2$. Then, $\{\rho^* : \tau_s(\rho^*) \geq r, x_t \in \mu\}$ is a τ_s-cover of μ, so there exist a finite subcover $\{\rho^*(i) : i = 1, 2, 3, \ldots, n\}$. Let $\rho_1 = \vee_{i=1}^{n} \rho^*(i)$. Then

$$\tau_s(\rho_1) = \tau_s(\vee_{i=1}^{n} \rho^*(i)) \geq \wedge_{i=1}^{n} \tau_s(\rho^*(i)) \geq r.$$

Since $\rho_2 \not{q} \rho^*(i)$, $(i = 1, 2, 3, \ldots, n)$, then $\rho_2 \not{q} \rho_1$. □

THEOREM 4.4. *Let (X, τ_1, τ_2) be a FP^*T_2. If $\eta \in I^X$ is a C-set, then $C_{\tau_s}(\eta, r) = \eta$, $r \in I_0$.*

PROOF. Let $x_t \in \underline{1} - \eta$. Then, $x_t \not{q} \eta$. Since η is C-set, then be Theorem 4,3, there exist $\mu_{x_t}, \lambda \in I^X$ with $\tau_s(\mu_{x_t}) \geq r$, $\tau_s(\lambda) \geq r$, $r \in I_0$ such that $x_t \in \mu_{x_t}$, $\eta \leq \lambda$ and $\mu_{x_t} \not{q} \lambda$. This implies that

$$x_t \in \mu_{x_t} \leq \underline{1} - \lambda \leq \underline{1} - \eta.$$

Thus

$$\underline{1} - \eta = \vee\{\mu_{x_t} : x_t \in \underline{1} - \eta\}.$$

Then, $\tau_s(\underline{1} - \eta) \geq r$. Hence $C_{\tau_s}(\eta, r) = \eta$.

THEOREM 4.5. *Let (X, τ_1, τ_2) be a FP^*-compact and FP^*T_2. Then, it is FP^*T_4.*

PROOF. Since (X, τ_1, τ_2) is FP^*T_2 it is clear that it is FP^*T_1. Remains we prove that (X, τ_1, τ_2) is FP^*R_3, so let $\lambda_1 = C_{\tau_s}(\lambda_1, r) \not{q} \lambda_2 = C_{\tau_s}(\lambda_2, r)$. Then, $\tau_s(\underline{1} - \lambda_1) \geq r$, $\tau_s(\underline{1} - \lambda_2) \geq r$ and since (X, τ_1, τ_2) is FP^*-compact, then λ_1 and λ_2 are C-sets. Since, $\lambda_1 \not{q} \lambda_2$, then by Theorem 4.3, there exist $\rho_1, \rho_2 \in I^X$ with $\tau_s(\rho_1) \geq r$, $\tau_s(\rho_2) \geq r$, $r \in I_0$ such that $\lambda_1 \leq \rho_1$, $\lambda_2 \leq \rho_2$ and $\rho_1 \not{q} \rho_2$. Thus, (X, τ_1, τ_2) is FP^*R_3. Hence (X, τ_1, τ_2) is FP^*T_4. □

THEOREM 4.6. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a FP^*-continuous mapping from a fbts (X, τ_1, τ_2) to another fbts (X, τ_1^*, τ_2^*). If $\mu \in I^X$ is C-set, then $f(\mu)$ is C-set in Y.*

PROOF. Let $\{\eta_i : i \in J\}$ be a τ_s^*-cover of $f(\mu)$. Then $f(\mu) \leq \vee_{i \in J} \eta_i$, $\tau_s^*(\eta_i) \geq r$, for all i, this implies that

$$\mu \leq f^{-1}(f(\mu)) \leq f^{-1}(\vee_{i \in J} \eta_i) = \vee_{i \in J} f^{-1}(\eta_i).$$

Since f is FP^*-continuous, then

$$\tau_s(f^{-1}(\eta_i)) \geq \tau_s^*(\eta_i) \geq r.$$

Then, $\{f^{-1}(\eta_i) : i \in J\}$ is a τ_s-cover of μ and since μ is C-set, then $\mu \leq \vee_{i=1}^{n} f^{-1}(\eta_i)$ this implies that

$$f(\mu) \leq f(\vee_{i=1}^{n} f^{-1}(\eta_i)) = \vee_{i=1}^{n} f(f^{-1}(\eta_i)) \leq \vee_{i=1}^{n} \eta_i.$$

Hence, $f(\mu)$ is C-set in Y. □

COROLLARY 4.1. *The FP^*-continuous image of an FP^*-compact is FP^*-compact.*

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