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On Fuzzy Ideals in Γ-Semigroups

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Abstract

In this paper the notion of fuzzy ideals of a Γ-semigroup has been introduced and some of their properties have been investigated. Characterization of a regular Γ-semigroup in terms of fuzzy ideals has been obtained. Operator semigroups of a Γ-semigroup has been made to work by obtaining various relationships between fuzzy ideals of a Γ-semigroup and that of its operator semigroups. These results are then used to revisit some results of Γ-semigroups.

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1 Introduction

Γ-semigroup was introduced by Sen and Saha[8] as a generalization of semigroup and ternary semigroup. Many results of semigroups could be extended to Γ-semigroups directly and via operator semigroups[1] of a Γ-semigroup. Many authors have studied semigroups in terms of fuzzy sets[9]. Kuroki [5,6] is the main contributor to this study. Motivated by Kuroki [5], Mustafa et all[7] we have initiated the study of Γ-semigroups in terms of fuzzy sets. The purpose of this paper is as stated in the abstract.
In §3 we introduce the notion of fuzzy ideals of a Γ-semigroup and obtain some of their properties.

In §4 we define composition of fuzzy ideals of a Γ-semigroup and characterize regular Γ-semigroups in terms of fuzzy ideals.
In §5, in order to make operator semigroups of Γ-semigroup to work in the context of fuzzy sets as it worked in the study of Γ-semigroups[1,2,3], we obtain various relationships between fuzzy ideals of a Γ-semigroup and that of its operator semigroups. Here, among other results we obtain an inclusion preserving bijection between the set of all fuzzy ideals of a Γ-semigroup and that of its operator semigroups. This bijection is then used to give new proofs of its ideal analogue obtained in [2].

2 Preliminaries

A *fuzzy subset*[9] of a non-empty set X is a function $\mu : X \rightarrow [0, 1]$.

Let S and Γ be two non-empty sets. S is called a Γ-*semigroup*[2] if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Let S be a Γ-semigroup. By a *left*(*right*) *ideal*[2] of S we mean a non-empty subset A of S such that $S\Gamma A \subseteq A(A\Gamma S \subseteq A)$. By a *two-sided ideal or simply an ideal*[2], we mean a non-empty subset of S which is both a left and a right ideal of S.

A Γ-semigroup S is called *regular*[2] if, for each element x in S , there exist $\beta \in \Gamma$ such that $x = x\beta x$.

3 Fuzzy Ideals in Γ-Semigroup

A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy left ideal*[7] of S if $\mu(x\gamma y) \geq \mu(y)\ \forall x, y \in S, \forall \gamma \in \Gamma$.

A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy right ideal*[7] of S if $\mu(x\gamma y) \geq \mu(x)\ \forall x, y \in S, \forall \gamma \in \Gamma$.

A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy ideal*[7] of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.

Example: Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-semigroup if $a\gamma b$ and $\alpha a\beta$denote the usual multiplication of integers a, γ, b and α, a, β respectively where $a, b \in S$ and $\alpha.\beta, \gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows

$$\mu(x) = \begin{cases} 1, \text{if } x = 0 \\ 0.1, \text{if } x = -1, -2 \\ 0.2, \text{if } x < -2 \end{cases}$$. Then μ is a fuzzy ideal of S.

We omit the proof of the following as they can be deduced easily.

Theorem 3.1 *Let I be a non-empty subset of a Γ-semigroup S and χ_I be the characteristic function of I, then I is a left ideal(right ideal, ideal) of S if and only if χ_I is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

Proposition 3.2 *Let I be a left ideal(right ideal, ideal) of a Γ-semigroup S and $\alpha \leq \beta \neq 0$ be any two elements in $[0,1]$, then the fuzzy subset μ of S, defined by $\mu(x) = \begin{cases} \beta, \text{if } x \in I \\ \alpha, \text{otherwise} \end{cases}$ is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

Theorem 3.3 *Let S be a Γ-semigroup and μ be a non-empty fuzzy subset of S, then μ is a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of S if and only if μ_t's are left ideals(resp. right ideals, ideals) of S for all $t \in Im(\mu)$ where $\mu_t = \{x \in S : \mu(x) \geq t\}$.*

Definition 3.4 *Let S be a Γ-semigroup and μ be a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of a Γ-semigroup S. Then the ideals μ_t's are called the level left ideals(resp. level right ideals, level ideals)of μ where $t \in Im(\mu)$.*

Theorem 3.5 *Let μ be a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of a Γ-semigroup S and $t_1 > t_2$. Then $\mu_{t_1} \subseteq \mu_{t_2}$ where $t_1, t_2 \in Im(\mu)$, equality occurs if and only if there is no $x \in S$ such that $t_1 \leq \mu(x) < t_2$.*

Proof. The first part of the theorem follows easily. Now let μ be a fuzzy left ideal of S such that $\mu_{t_1} = \mu_{t_2}$. If possible let there exists $x \in S$ such that $t_1 \leq \mu(x) < t_2$. Then $x \in \mu_{t_1}$ but $x \notin \mu_{t_2}$, which is a contradiction. So there exist no $x \in S$ such that $t_1 \leq \mu(x) \leq t_2$.

Conversely, suppose μ is a fuzzy left ideal of S such that there does not exist $x \in S$ with $t_1 \leq \mu(x) < t_2$. Since $t_1 < t_2$, then $\mu_{t_2} \subseteq \mu_{t_1}$ $(cf.$ Definition 3.4). If possible let $\mu_{t_1} \neq \mu_{t_2}$. Then there is some $y \in S$ such that $y \in \mu_{t_1}$ but $y \notin \mu_{t_2}$, i.e, $\mu(y) \geq t_1$ but $\mu(y) < t_2$, i.e, $t_1 \leq \mu(y) < t_2$, which contradicts our assumption. Hence $\mu_{t_1} = \mu_{t_2}$. By applying similar argument we can prove the theorem when μ is a fuzzy right ideal and a fuzzy ideal. ■

Theorem 3.6 *Let $f : R \to S$ be a homomorphism of Γ-semigroups (i) If λ is a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of S, then $f^{-1}(\lambda)$ is also a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of R(where $f^{-1}(\lambda)(r\gamma s) := \lambda(f(r\gamma s))$ for all $r, s \in R$ and $\gamma \in \Gamma$), provided $f^{-1}(\lambda)$ is non-empty. (ii) If f is a surjective homomorphism and μ is a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of R then, $f(\mu)$ is also a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S(where $(f(\mu)(x)) := \sup_{f(y)=x} \mu(y), x \in S, y \in R$).*

Proof. (i) Let λ be a fuzzy left ideal of S. Let $r, s \in R$ and $\gamma \in \Gamma$. Then $(f^{-1}(\lambda))(r\gamma s) = \lambda(f(r\gamma s)) = \lambda(f(r)\gamma f(s)) \geq \lambda(f(s)) = (f^{-1}(\lambda))(s)$. Thus $(f^{-1}(\lambda))$ is a fuzzy left ideal of R.

(ii) Let μ be a fuzzy left ideal of R. Since $(f(\mu))(x') = \sup_{f(x)=x'} \mu(x)$ for $x' \in S$, so $f(\mu)$ is non-empty. Let $x', y' \in S$ and$\gamma \in \Gamma$. Then $(f(\mu))(x'\gamma y') = \sup_{f(z)=x'\gamma y'} \mu(z) \geq \sup_{\substack{f(x)=x' \\ f(y)=y'}} \mu(x\gamma y) \geq \sup_{f(y)=y'} \mu(y) = (f(\mu))(y')$.

Thus $f(\mu)$ is a fuzzy left ideal of S. In a similar way we can prove the theorem when λ, μ are fuzzy right ideals and fuzzy ideals. ■

4 Composition of Fuzzy Ideals

Definition 4.1 *Let S be a Γ-semigroup and $\mu_1, \mu_2 \in FLI(S)[FRI(S), FI(S)]$*[1]. *Then the product $\mu_1 \circ \mu_2$ of μ_1 and μ_2 is defined as*

$$(\mu_1 \circ \mu_2)(x) = \begin{cases} \sup_{x=u\gamma v} [\min\{\mu_1(u), \mu_2(v)\} : u, v \in S; \gamma \in \Gamma] \\ 0, \text{ if for any } u, v \in S \text{ and for any } \gamma \in \Gamma, x \neq u\gamma v \end{cases}$$

Theorem 4.2 *In a Γ-semigroup S the following are equivalent. $(i)\mu$ is a fuzzy left(right) ideal of S. $(ii)\chi \circ \mu \subseteq \mu(\mu \circ \chi \subseteq \mu)$, where χ is the characteristic function of S.*

Proof. Let μ be a fuzzy left ideal of S. Let $a \in S$. Suppose there exist $u, v \in S$ and $\delta \in \Gamma$ such that $a = u\delta v$. Then, since μ is a fuzzy left ideal of S, we have $(\chi \circ \mu)(a) = \sup_{a=x\gamma y} [\min\{\chi(x), \mu(y)\}] = \sup_{a=x\gamma y} [\min\{1, \mu(y)\}] = \sup_{a=x\gamma y} \mu(y)$. Now since μ is a fuzzy left ideal, $\mu(x\gamma y) \geq \mu(y)$ for all $x, y \in S$ and for all $\gamma \in \Gamma$. So in particular, $\mu(y) \leq \mu(a)$ for all $a = x\gamma y$. Hence $\sup_{a=x\gamma y} \mu(y) \leq \mu(a)$. Thus $\mu(a) \geq (\chi \circ \mu)(a)$. If there do not exist $x, y \in S, \gamma \in \Gamma$ such that $a = x\gamma y$ then $(\chi \circ \mu)(a) = 0 \leq \mu(a)$. Hence $\chi \circ \mu \subseteq \mu$. By a similar argument we can show that $\mu \circ \chi \subseteq \mu$ when μ is a fuzzy right ideal.

Conversely, let $\chi \circ \mu \subseteq \mu$. Let $x, y \in S$, $\gamma \in \Gamma$ and $a := x\gamma y$. Then $\mu(x\gamma y) = \mu(a) \geq (\chi \circ \mu)(a)$. Now $(\chi \circ \mu)(a) = \sup_{a=u\alpha v} [\min\{\chi(u), \mu(v)\}] \geq \min\{\chi(x), \mu(y)\} = \min\{1, \mu(y)\} = \mu(y)$. Hence $\mu(x\gamma y) \geq \mu(y)$. Hence μ is a fuzzy left ideal of S. By a similar argument we can show that if $\mu \circ \chi \subseteq \mu$ then μ is a fuzzy right ideal of S.

■

Using the above theorem we can deduce the following theorem.

[1] $FLI(S)$, $FRI(S)$, $FI(S)$ denote respectively the set of all fuzzy left ideals, fuzzy right ideals, fuzzy ideals of a Γ-semigroup S.

Theorem 4.3 *In a Γ-semigroup S the following are equivalent.* $(i)\mu$ *is a fuzzy two-sided ideal of* S. $(ii)\chi \circ \mu \subseteq \mu$ *and* $\mu \circ \chi \subseteq \mu$, *where* χ *is the characteristic function of* S.

Proposition 4.4 *Let* μ_1 *be a fuzzy right ideal and* μ_2 *be a fuzzy left ideal of a* Γ*-semigroup* S. *Then* $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$.

Proof. Let μ_1 be a fuzzy right ideal and μ_2 be a fuzzy left ideal of a Γ-semigroup S. Let $x \in S$. Suppose there exist $u_1, v_1 \in S$ and $\gamma_1 \in \Gamma$ such that $x = u_1\gamma_1 v_1$.

Then $(\mu_1 \circ \mu_2)(x) = \sup\limits_{x=u\gamma v} \min\{\mu_1(u), \mu_2(v)\} \leq \sup\limits_{x=u\gamma v} \min\{\mu_1(u\gamma v), \mu_2(u\gamma v)\}$ $= \min\{\mu_1(x), \mu_2)(x) = (\mu_1 \cap \mu_2)(x)$. Suppose there do not exist $u, v \in S$ such that $x = u\gamma v$. Then $(\mu_1 \circ \mu_2)(x) = 0 \leq (\mu_1 \cap \mu_2)(x)$. Thus $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. ■

From the above proposition and the definition of $\mu_1 \cap \mu_2$ the following proposition follows easily.

Proposition 4.5 *Let* $\mu_1, \mu_2 \in FI(S)$. *Then* $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2$.

Proposition 4.6 *Let* S *be a regular* Γ*-semigroup and* μ_1, μ_2 *be two fuzzy subsets of* S. *Then* $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$.

Proof. Let $c \in S$. Since S is regular, then there exists an element $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c = c\gamma_1 x\gamma_2 c = c\gamma c$ where $\gamma := \gamma_1 x\gamma_2 \in \Gamma$. Now $(\mu_1 \circ \mu_2)(c) = \sup\limits_{c=u\alpha v} \{\min\{\mu_1(u), \mu_2(v)\} \geq \min\{\mu_1(c), \mu_2(c)\} = (\mu_1 \cap \mu_2)(c)$. Therefore $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. ■

Theorem 4.7 *In a* Γ*-semigroup* S *the following are equivalent.* $(i)S$ *is regular.* $(ii)\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$ *for every fuzzy right ideal* μ_1 *and every fuzzy left ideal* μ_2 *of* S.

Proof. Let S be a regular Γ-semigroup. Then by Proposition 4.6, $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. Again by Proposition 4.5, $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. Hence $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$.

Conversely, let S be a Γ-semigroup and for every fuzzy right ideal μ_1 and every fuzzy left ideal μ_2 of S, $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$. Let L and R be respectively a left ideal and a right ideal of S and $x \in R \cap L$. Then $x \in R$ and $x \in L$. Hence $\chi_L(x) = \chi_R(x) = 1$(where $\chi_L(x)$ and $\chi_R(x)$ are respectively the characteristic functions of L and R). Thus $(\chi_R \cap \chi_L)(x) = \min\{\chi_R(x), \chi_L(x)\} = 1$. Now by Theorem 3.1, χ_L and χ_R are respectively a fuzzy left ideal and a fuzzy right ideal of S. Hence by hypothesis, $\chi_R \circ \chi_L = \chi_R \cap \chi_L$. Hence $(\chi_R \circ \chi_L)(x) = 1$, i.e., $\sup\limits_{x=y\gamma z} [\min\{\chi_R(y), \chi_L(z)\} : y, z \in S; \gamma \in \Gamma] = 1$. This implies that there exist some $r, s \in S$ and $\gamma_1 \in \Gamma$ such that $x = r\gamma_1 s$ and $\chi_R(r) = 1 = \chi_L(s)$. Hence $r \in R$ and $s \in L$. Hence $x \in R\Gamma L$. Thus $R \cap L \subseteq R\Gamma L$. Also $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. Consequently, the Γ-semigroup S is regular(cf. Theorem 1[2]). ■

5 Corresponding Fuzzy Ideals

Definition 5.1 [2] *Let S be a Γ-semigroup. Let us define a relation ρ on $S\times\Gamma$ as follows : $(x,\alpha)\rho(y,\beta)$ if and only if $x\alpha s = y\beta s$ for all $s\in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma\in\Gamma$. Then ρ is an equivalence relation. Let $[x,\alpha]$ denote the equivalence class containing (x,α). Let $L=\{[x,\alpha]: x\in S, \alpha\in\Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x,\alpha][y,\beta]=[x\alpha y,\beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of Γ-semigroup S is defined where the multiplication is defined by $[\alpha,a][\beta,b]=[\alpha a\beta,b]$.*

Definition 5.2 *For a fuzzy subset μ of R we define a fuzzy subset μ^* of S by $\mu^*(a)=\inf\limits_{\gamma\in\Gamma}\mu([\gamma,a])$, where $a\in S$. For a fuzzy subset σ of S we define a fuzzy subset $\sigma^{*'}$ of R by $\sigma^{*'}([\alpha,a])=\inf\limits_{s\in S}\sigma(s\alpha a)$, where $[\alpha,a]\in R$. For a fuzzy subset δ of L, we define a fuzzy subset δ^+ of S by $\delta^+(a)=\inf\limits_{\gamma\in\Gamma}\delta([a,\gamma])$ where $a\in S$. For a fuzzy subset η of S we define a fuzzy subset $\eta^{+'}$ of L by $\eta^{+'}([a,\alpha])=\inf\limits_{s\in S}\eta(a\alpha s)$, where $[a,\alpha]\in L$.*

Now we recall the following propositions from[2] which were proved therein for one sided ideals. But the results can be proved to be true for two sided ideals.

Proposition 5.3 [2] *Let S be a Γ-semigroup with unities and L be its left operator semigroup. If A is a (right) ideal of L then A^+ is a(right) ideal of S.*

Proposition 5.4 [2] *Let S be a Γ-semigroup with unities and L be its left operator semigroup. If B is a (right)ideal of S then $B^{+'}$ is a(right)ideal of L.*

Proposition 5.5 [2] *Let S be a Γ-semigroup with unities and R be its right operator semigroup. If A is a (left)ideal of R then A^* is a(left)ideal of S.*

Proposition 5.6 [2] *Let S be a Γ-semigroup with unities and R be its right operator semigroup. If B is a (left)ideal of S then $B^{*'}$ is a(left)ideal of R.*

For convenience of the readers, we may note that for a Γ-semigroup S and its left, right operator semigroups L,R respectively four mappings namely $()^+$, $()^{+'}$, $()^+$, $()^{*'}$ occur. They are defined as follows: For $I\subseteq R, I^*=\{s\in S,[\alpha,s]\in I\forall\alpha\in\Gamma\}$; for $P\subseteq S, P^{*'}=\{[\alpha,x]\in R: s\alpha x\in P\forall s\in S\}$; for $J\subseteq L, J^+=\{s\in S,[s,\alpha]\in J\forall\alpha\in\Gamma\}$; for $Q\subseteq S, Q^{+'}=\{[x,\alpha]\in L: x\alpha s\in Q\forall s\in S\}$.

Proposition 5.7 *If μ is a fuzzy subset of R(the right operator semigroup of the Γ-semigroup S), then $(\mu_t)^* = (\mu^*)_t$ for all $t \in [0,1]$ such that the sets are non-empty.*

Proof. Let $s \in S$. Then $s \in (\mu_t)^* \Leftrightarrow [\gamma, s] \in \mu_t \ \forall \gamma \in \Gamma \Leftrightarrow \mu([\gamma, s]) \geq t \ \forall \gamma \in \Gamma \Leftrightarrow \inf\limits_{\gamma \in \Gamma} \mu([\gamma, s]) \geq t \Leftrightarrow \mu^*(s) \geq t \Leftrightarrow s \in (\mu^*)_t$. Thus $(\mu_t)^* = (\mu^*)_t$.
■

Proposition 5.8 *Let σ is a fuzzy subset of a Γ-semigroup S. Then $(\sigma_t)^{*'} = (\sigma^{*'})_t$ for all $t \in [0,1]$ such that the sets under consideration are non-empty.*

Proof. Let $[\alpha, x] \in R$ and t is as mentioned in the statement. Then $[\alpha, x] \in (\sigma_t)^{*'} \Leftrightarrow s\alpha x \subseteq \sigma_t \ \forall s \in S \Leftrightarrow \sigma(s\alpha x) \geq t \ \forall s \in S \Leftrightarrow \inf\limits_{s \in S} \sigma(s\alpha x) \geq t \Leftrightarrow \sigma^{*'}([\alpha, x]) \geq t \Leftrightarrow [\alpha, x] \in (\sigma^{*'})_t$. Thus $(\sigma_t)^{*'} = (\sigma^{*'})_t$.
■

In what follows S denotes a Γ-semigroup with unities[1], L,R its left and right operator semigroups respectively.

Proposition 5.9 *If $\mu \in FI(R)(FLI(R))$, then $\mu^* \in FI(S)$(respectively FLI (S)]).*

Proof. Suppose $\mu \in FI(R)$. Then μ_t is an ideal of R $\forall t \in Im(\mu)$. Hence $(\mu_t)^*$ is an ideal of $S \forall t \in Im(\mu)$(cf. Proposition 5.5). Now since μ is fuzzy ideal of R, μ is a non-empty fuzzy subset of R. Hence for some $[\alpha, s] \in R, \mu([\alpha, s]) > 0$. Then $\mu_t \neq \phi$ where $t := \mu([\alpha, s])$. So by the same argument applied above $(\mu_t)^* \neq \phi$. Let $u \in (\mu_t)^*$. Then $[\beta, u] \in \mu_t$ for all $\beta \in \Gamma$. Hence $\mu([\beta, u]) \geq t$. This implies that $\inf\limits_{\beta \in \Gamma} \mu([\beta, u]) \geq t$, i.e., $\mu^*(u) \geq t$. Hence $u \in (\mu^*)_t$. Hence$(\mu^*)_t \neq \phi$. Consequently, $(\mu_t)^* = (\mu^*)_t$(cf. Proposition 5.7). It follows that $(\mu^*)_t$ is an ideal of S for all $t \in Im(\mu)$. Hence μ^* is a fuzzy ideal of S(cf. Theorem 3.3). The proof for fuzzy left ideal follows similarly.
■

In a similar fashion by using Propositions 5.6, 5.8 and Theorem 3.3 we deduce the following proposition.

Proposition 5.10 *If $\sigma \in FI(S)(\ FLI(S))$, then $\sigma^{*'} \in FI(R)$ (respectively $FLI(R)$).*

We can also deduce the following left operator analogues of the above propositions.

Proposition 5.11 *If $\delta \in FI(L)[\ FRI(L)]$, then $\delta^+ \in FI(S)$ [respectively $FRI(S)$].*

Proposition 5.12 *If* $\eta \in FI(S)(FRI(S))$, *then* $\eta^{+'} \in FI(L)$ (*respectively* $FRI(L)$).

Theorem 5.13 *Let S be a Γ-semigroup with unities and L be its left operator semigroup. Then there exist an inclusion preserving bijection $\sigma \mapsto \sigma^{+'}$ between the set of all fuzzy ideals (fuzzy right ideals) of S and set of all fuzzy ideals (resp. fuzzy right ideals) of L, where σ is a fuzzy ideal (resp. fuzzy right ideal) of S.*

Proof. Let $\sigma \in FI(S)(FRI(S))$ and $x \in S$. Then $(\sigma^{+'})^{+}(x) = \inf_{\gamma\in\Gamma}\sigma^{+'}([x,\gamma]) = \inf_{\gamma\in\Gamma}[\inf_{s\in S}\sigma(x\gamma s)] \geq \sigma(x)$. Hence $\sigma \subseteq (\sigma^{+'})^{+}$. Let $[\gamma, f]$ be the right unity of S. Then $x\gamma f = x$ for all $x \in S$. Now $\sigma(x) = \sigma(x\gamma f) \geq \inf_{\alpha\in\Gamma}[\inf_{s\in S}\sigma(x\alpha s)] = \inf_{\alpha\in\Gamma}\sigma^{+'}([x,\alpha]) = (\sigma^{+'})^{+}(x)$. So $\sigma \supseteq (\sigma^{+'})^{+}$. Hence $(\sigma^{+'})^{+} = \sigma$. Now let $\mu \in FI(L)(FRI(S)$. Then $(\mu^{+})^{+'}([x,\alpha]) = \inf_{s\in S}\mu^{+}(x\alpha s) = \inf_{s\in S}[\inf_{\gamma\in\Gamma}\mu([x\alpha s,\gamma]) = \inf_{s\in S}[\inf_{\gamma\in\Gamma}\mu([x,\alpha][s,\gamma]) \geq \mu([x,\alpha])$. So $\mu \subseteq (\mu^{+})^{+'}$. Let $[e,\delta]$ be the left unity of L. Then $\mu([x,\alpha]) = \mu([x,\alpha][e,\delta]) \geq \inf_{s\in S}[\inf_{\gamma\in\Gamma}\mu([x,\alpha][s,\gamma]) = (\mu^{+})^{+'}([x,\alpha])$. So $\mu \supseteq (\mu^{+})^{+'}$ and hence $\mu = (\mu^{+})^{+'}$. Thus the correspondence $\sigma \mapsto \sigma^{+'}$ is a bijection. Now let $\sigma_1, \sigma_2 \in FI(S)(FRI(S))$ be such that $\sigma_1 \subseteq \sigma_2$. Then for all $[x,\alpha] \in L$, $\sigma_1^{+'}([x,\alpha]) = \inf_{s\in S}\sigma_1(x\alpha s) \leq \inf_{s\in S}\sigma_2(x\alpha s) = \sigma_2^{+'}([x,\alpha])$. Thus $\sigma_1^{+'} \subseteq \sigma_2^{+'}$. Similarly we can show that if $\mu_1 \subseteq \mu_2$ where $\mu_1, \mu_2 \in FI(L)(FRI(L))$ then $\mu_1^{+} \subseteq \mu_2^{+}$. Hence $\sigma \mapsto \sigma^{+'}$ is an inclusion preserving bijection. The rest of the proof follows from Propositions 5.11 and 5.12.
■

In a similar way by using Propositions 5.9 and 5.10 we can deduce the following theorem.

Theorem 5.14 *Let S be a Γ-semigroup with unities and R be its right operator semigroup. Then there exist an inclusion preserving bijection $\sigma \mapsto \sigma^{*'}$ between the set of all fuzzy ideals (fuzzy left ideals) of S and set of all fuzzy ideals (resp. fuzzy left ideals) of R, where σ is a fuzzy ideal (resp. fuzzy left ideal) of S.*

Now to apply the above theorem for giving a new proof of Theorem 4.6 ([2]) and its two sided ideal analogue we obtain the following lemmas.

Lemma 5.15 *Let I be a (left) ideal of the right operator semigroup R of a Γ-semigroup S. Then $(\lambda_I)^* = \lambda_{I^*}$ where λ_I denotes the characteristic function of I.*

Proof. Suppose $s \in I^*$. Then $[\beta, s] \in I$ for all $\beta \in \Gamma$. This means $\inf_{\beta\in\Gamma}(\lambda_I([\beta, s])) = 1$. Also $\lambda_{I^*}(s) = 1$. Now suppose $s \notin I^*$. Then there exists $\delta \in \Gamma$ such that $[\delta, s] \notin I$. Hence $\lambda_I([\delta, s]) = 0$ and so $\inf_{\beta\in\Gamma}(\lambda_I([\beta, s]) = 0$. Hence $(\lambda_I)^*(s) = 0$. Again $(\lambda_{I^*})(s) = 0$. Thus $(\lambda_I)^* = \lambda_I^*$.
■

By applying similar argument as above we deduce the following lemma.

Lemma 5.16 *Let I be a(right) ideal of a Γ-semigroup S and R be the right operator semigroup of S. Then $(\lambda_I)^{*'} = \lambda_{I^{*\prime}}$.*

Remark 5.17 *By drawing an analogy we can deduce results similar to the above lemmas for left operator semigroup L of the Γ-semigroup S.*

Theorem 5.18 [2] *Let S be a Γ-semigroup with unities. Then there exists an inclusion preserving bijection between the set of all ideals (left ideals) of S and that of its right operator semigroup R via the mapping $I \to I^{*'}$.*

Proof. Let us denote the mapping $I \to I^{*'}$ by ϕ. This is actually a mapping follows from Proposition 5.6. Now let $\phi(I_1) = \phi(I_2)$. Then $I_1^{*'} = I_2^{*'}$. This implies that $\lambda_{I_1^{*'}} = \lambda_{I_2^{*'}}$(where λ_I is the characteristic function I). Hence by Lemma 5.16, $(\lambda_{I_1})^{*'} = (\lambda_{I_2})^{*'}$. This together with Theorem 5.14 gives $\lambda_{I_1} = \lambda_{I_2}$ whence $I_1 = I_2$. Consequently ϕ is one-one. Let I be a (left) ideal of R. Then its characteristic function λ_I is a fuzzy (left) ideal of R. Hence by Theorem 5.14, $((\lambda_I)^*)^{*'} = \lambda_I$. This implies that $\lambda_{(I^*)^{*'}} = \lambda_I$ (*cf.* Lemmas 5.15 and 5.16). Hence $(I^*)^{*'} = I$, i.e., $\phi(I^*) = I$. Now since I^* is a (left) ideal of S (*cf.* Proposition 5.5), it follows that ϕ is onto. Let I_1, I_2 be two (left) ideals of S with $I_1 \subseteq I_2$. Then $\lambda_{I_1} \subseteq \lambda_{I_2}$. Hence by Theorem 5.14, we see that $(\lambda_{I_1})^{*'} \subseteq (\lambda_{I_2})^{*'}$ i.e., $\lambda_{I_1^{*'}} \subseteq \lambda_{I_2^{*'}}$ (*cf.* Lemma 5.16) which gives $I_1^{*'} \subseteq I_2^{*'}$.
■

Remark 5.19 *Now by using a similar argument as above and with the help of lemmas dual to Lemmas* 5.15, 5.16(*cf. Remark* 5.17) *and Theorem* 5.15 *we can deduce that the mapping $()^{+'}$ is an inclusion preserving bijection(with $()^+$ as the inverse) between the set of all ideals(right ideals) of S and that of its left operator semigroup L.*

Concluding Remark: *Results of fuzzy semigroups can be extended to the general setting of fuzzy Γ-semigroups via operator semigroups as various relationships between fuzzy subsets of a Γ-semigroup and that of its operator semigroups are found to be true and many results of Γ-semigroup can be rediscovered with the help of fuzzification which is illustrated in Theorem* 5.18 *and Remark* 5.19 .

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