ON FUZZY $\varphi\psi$-CONTINUOUS MULTIFUNCTION

M. ALIMOHAMMADY AND M. ROOHI

Received 13 November 2004; Accepted 3 November 2005

Here, we would study and characterize fuzzy $\varphi\psi$-continuity for fuzzy multifunctions which extend fuzzy $\varphi\psi$-continuity of fuzzy functions. Moreover, we obtain some results in fuzzy multifunctions.

1. Introduction and preliminaries

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory. Recently many authors, for example, Albrycht and Matłoka [1] and Beg [3] have studied fuzzy multifunctions and have characterized some property of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied some type of fuzzy continuity for fuzzy functions and fuzzy multifunctions [1–5], [8–12]. In [3] fuzzy $\varphi\psi$-continuous functions have been studied. But this brand of fuzzy continuity has not considered for fuzzy multifunctions which we attempt to study and characterize.

The *fuzzy set* in (on) a universe X is a function with domain X and values in $I = [0,1]$. The class of all fuzzy sets on X will be denoted by I^X and symbols $A,B,\ldots$ are used for fuzzy sets on X. 01_X is called *empty fuzzy set*, where 1_X is the characteristic function on X. For any fuzzy set A in X, the function value $A(x)$ is called the *grade of membership* of x in A. We write $x \in A$ if $A(x) > 0$. For any fuzzy set A, the fuzzy set $1 - A(x)$ is called the *complement* of A which is denoted by A^c. Let A and B be fuzzy sets in X, we write $A \leq B$ if $A(x) \leq B(x)$ for all x in X. For any family $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ of fuzzy sets in X, $\bigvee_{\alpha\in\mathcal{A}} A_\alpha$ and $\bigwedge_{\alpha\in\mathcal{A}} A_\alpha$ are defined by $\sup_\alpha A_\alpha(x)$ and $\inf_\alpha A_\alpha(x)$, respectively. A family τ of fuzzy sets in X is called a *fuzzy topology* for X if (i) $\alpha 1_X \in \tau$ for each $\alpha \in I$; (ii) $A \wedge B \in \tau$ where $A,B \in \tau$ and (iii) $\bigvee_{\alpha\in\mathcal{A}} A_\alpha \in \tau$ whenever $A_\alpha \in \tau$ for all α in $\mathcal{A}$. The pair (X,τ) is called a *fuzzy topological space* [6]. Every member of τ is called *fuzzy open set* and its complements

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 85165, Pages 1–7
DOI 10.1155/JAMSA/2006/85165

are called *fuzzy closed sets* [6]. In a fuzzy topological space X the *interior* and the *closure* of a fuzzy set A (simply int(A) and cl(B), resp.) are defined by

$$\begin{aligned} \operatorname{int}(A) &= \bigvee\{U : A \leq U,\ U \text{ is a fuzzy open set}\}, \\ \operatorname{cl}(A) &= \bigwedge\{F : A \leq F,\ F \text{ is a fuzzy closed set}\}. \end{aligned} \tag{1.1}$$

A *neighborhood of a fuzzy set* A in a fuzzy topological space X is any fuzzy set B for which there is a fuzzy open set V satisfying $A \leq V \leq B$. Any fuzzy open set V that satisfies $A \leq V$ is called a *fuzzy open neighborhood* of A [10]. A fuzzy set A is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is $\epsilon (0 \leq \epsilon \leq 1)$, we denote this fuzzy point by x_ϵ [11]. For any fuzzy point x_ϵ and any fuzzy set A we write $x_\epsilon \in A$ if and only if $\epsilon \leq A(x)$. Let f be a function from X to Y. A fuzzy function $f : X \to Y$ is defined by

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(\{y\})} A(x) & f^{-1}(\{y\}) \neq \varnothing, \\ 0 & f^{-1}(\{y\}) = \varnothing, \end{cases} \tag{1.2}$$

for all y in Y, where A is an arbitrary fuzzy set in X [12]. A fuzzy function $f : X \to Y$ is called *fuzzy continuous* if for each $x_\epsilon \in X$ and each fuzzy neighborhood B of $f(x_\epsilon)$ there exists a fuzzy neighborhood A of x_ϵ such that $f(A) \leq B$ [11]. A *fuzzy multifunction* $f : X \to\to Y$ assigns to each x in X a fuzzy set $f(x)$ of Y [2]. If A is a fuzzy set in X, then the fuzzy set $f(A)$ in Y is defined by

$$f(A)(y) = \bigvee_{x \in X} \big(f(x)(y) \wedge A(x)\big). \tag{1.3}$$

For more details about fuzzy multifunctions and their properties, the reader is referred to [1, 2, 10]. Throughout this paper, (X,τ) and (Y,ν) are fuzzy topological spaces. The symbol $f : X \to\to Y$ is used for a fuzzy multifunction from X to Y, while $f : X \to Y$ for a fuzzy function from X to Y.

Main results

Definition 1.1. (i) A fuzzy function φ on X is called a *fuzzy operation* on X, if $\varphi(01_X) = 01_X$ and $\operatorname{int}(A) \leq \varphi(A)$, where A is any nonempty fuzzy set in X. φ is called a *monotonous fuzzy operation*, if $\varphi(A) \leq \varphi(B)$, whenever $A, B \in I^X$ and $A \leq B$ [5].

(ii) $f : X \to Y$ is called a *fuzzy $\varphi\psi$-continuous function at $x_\epsilon \in X$* if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(B)$, where φ and ψ are fuzzy operation on X and Y, respectively. $f : X \to Y$ is said to be a *fuzzy $\varphi\psi$-continuous function* if it is a fuzzy $\varphi\psi$-continuous function at each $x_\epsilon \in X$.

Definition 1.2. (i) $f : X \to\to Y$ is called a *fuzzy $\varphi\psi$-continuous multifunction at $x_\epsilon \in X$* if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(B)$, where φ and ψ are fuzzy operation on X and Y, respectively. $f : X \to\to Y$ is said to be a *fuzzy $\varphi\psi$-continuous multifunction* if it is a fuzzy $\varphi\psi$-continuous multifunction at each $x_\epsilon \in X$.

(ii) $f : X \to\to Y$ is called a *single valued fuzzy multifunction* if f at each x is a fuzzy point $1_{\{y_x\}}$, where $y_x \in Y$. In this case it would induce a fuzzy function $\widetilde{f} : X \to Y$ by $\widetilde{f}(x) = y_x$. Therefore, $f(x) = 1_{\{\widetilde{f}(x)\}}$.

PROPOSITION 1.3. *Suppose $f : X \to\to Y$ be a single valued fuzzy multifunction. Then for any fuzzy set A in X; $f(A) = \widetilde{f}(A)$. Therefore, f is a $\varphi\psi$-continuous multifunction if and only if $\widetilde{f}$ is a fuzzy $\varphi\psi$-continuous function.*

Proof. The equivalence for any fuzzy set A of X can be derived from the following fact:

$$f(\varphi(A))(y) = \bigvee_{z\in X} \big(f(z)(y) \wedge \varphi(A)(z)\big) = \bigvee_{z\in X} \big(1_{\{\widetilde{f}(z)\}}(y) \wedge \varphi(A)(z)\big). \tag{1.4}$$

On the other hand,

$$\widetilde{f}(\varphi(A))(y) = \begin{cases} \bigvee_{h\in \widetilde{f}^{-1}(\{y\})} \varphi(A)(h) & \widetilde{f}^{-1}(\{y\}) \neq \varnothing, \\ 0 & \widetilde{f}^{-1}(\{y\}) = \varnothing. \end{cases} \tag{1.5}$$

Therefore, $f(\varphi(A)) = \widetilde{f}(\varphi(A))$. Now replacing identity function as a fuzzy operation on X instead of φ completes the proof. □

From the above result this brand of continuity for fuzzy multifunctions is in fact a generalization of $\varphi\psi$-continuity introduced in [3]. Next, we would like to present a result showing the relation between fuzzy $\varphi\psi$-continuity and fuzzy $\varphi\psi$-continuity in respect of nets. First we note to the following results.

PROPOSITION 1.4. *Suppose $f : X \to\to Y$ be a fuzzy multifunction and $A, B \in I^X$ such that $A \leq B$. Then $f(A) \leq f(B)$.*

Proof.

$$f(A)(y) = \bigvee_{z\in X} \big(f(z)(y) \wedge A(z)\big) \leq \bigvee_{z\in X} \big(f(z)(y) \wedge B(z)\big) = f(B)(y). \tag{1.6}$$

□

LEMMA 1.5. *Let $f : X \to\to Y$ be a fuzzy multifunction and let x_ϵ be a fuzzy point in X. Then $f(x_\epsilon) = f(x) \wedge \epsilon$.*

Proof. It is straightforward. □

We say that a net $(x^\alpha_{\epsilon_\alpha})_{\alpha\in\mathcal{A}}$ of fuzzy points in a fuzzy topological space X is *φ-convergent* to a fuzzy point x_ϵ (we will denote it by $x^\alpha_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$) if for any neighborhood set A of x_ϵ, there is an $\alpha_0 \in \mathcal{A}$ in which $x^\alpha_{\epsilon_\alpha} \in \varphi(A)$ for all $\alpha \geq \alpha_0$.

LEMMA 1.6. *Consider a fuzzy set A and a convergent net (A_α) of fuzzy sets which $A_\alpha \to A$ in X. Then $A_\alpha \xrightarrow{\varphi} A$.*

Proof. From the assumption given any fuzzy open neighborhood B of A, there is $\alpha_0 \in \mathcal{A}$ such that for all $\alpha \geq \alpha_0$,

$$A_\alpha \leq B = \operatorname{int}(B) \leq \varphi(B). \tag{1.7}$$

□

A fuzzy multifunction $f : X \to\to Y$ is called *net-fuzzy $\varphi\psi$-continuous* if for each net of fuzzy points $x^\alpha_{\epsilon_\alpha}$ and x_ϵ in X, $f(x^\alpha_{\epsilon_\alpha}) \xrightarrow{\psi} f(x_\epsilon)$, where $(x^\alpha_{\epsilon_\alpha}) \xrightarrow{\varphi} x_\epsilon$.

Theorem 1.7. *Let X be a fuzzy topological space. For any fuzzy multifunction $f : X \to\to Y$ the following are equivalent:*

(i) *f is a fuzzy $\varphi\psi$-continuous;*

(ii) *f is a net-fuzzy $\varphi\psi$-continuous.*

Proof. (i)⇒(ii).

For any fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that

$$f(\varphi(A)) \leq \psi(B). \tag{1.8}$$

From the assumption, there is $\alpha_0 \in A$ for which

$$x^\alpha_{\epsilon_\alpha} \leq \varphi(A) \quad (\forall \alpha \geq \alpha_0). \tag{1.9}$$

According to Proposition 1.4, $f(x^\alpha_{\epsilon_\alpha}) \leq f(\varphi(A)) \leq \psi(B)$.

(ii)⇒(i).

On the contrary, there is a fuzzy point x_ϵ in X, a fuzzy open neighborhood B of $f(x_\epsilon)$ such that, there is not a fuzzy neighborhood A of x_ϵ satisfying in $f(\varphi(A)) \leq \psi(B)$. This means that there is $z_A \epsilon Y$ with the following property:

$$f(\varphi(A))(z_A) > \psi(B)(z_A). \tag{1.10}$$

Therefore,

$$\bigvee_{x \in X} (f(x)(z_A) \wedge \varphi(A)(x)) > \psi(B)(z_A). \tag{1.11}$$

Then $f(x_A)(z_A) \wedge \varphi(A)(z_A) > \psi(B)(z_A)$ for a suitable x_A of X. We conclude that

$$f(x_A)(z_A) > \psi(B)(z_A). \tag{1.12}$$

Consider $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a system of fuzzy neighborhoods at x_ϵ. The following order makes $\mathcal{A}$ as a directed set and so it makes $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a net:

$$\alpha \leq \beta \Longleftrightarrow A_\beta \leq A_\alpha. \tag{1.13}$$

Applying (1.12) for A_α instead of A, there is $x^\alpha_{\epsilon_\alpha}$ in A_α for which $f(x^\alpha_{\epsilon_\alpha}) > \psi(B)$. From the choice of $x^\alpha_{\epsilon_\alpha}$ in A_α, $x^\alpha_{\epsilon_\alpha} \to x_\epsilon$. Lemma 1.6 implies that $x^\alpha_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$. Since $f(x^\alpha_{\epsilon_\alpha})(z_\alpha) = f(x_\alpha)(z_\alpha) \wedge \epsilon_\alpha \leq f(x_\alpha)(z_\alpha)$ so from (1.12), $f(x^\alpha_{\epsilon_\alpha})$ is not ψ-convergent to $f(x_\epsilon)$, which completes the proof. □

In the following result we show continuity of the composition of two fuzzy multifunction. Suppose $f : X \to\to Y$ and $g : Y \to\to Z$. We define *composition* $gof : X \to\to Z$ by $(gof)(x) = g(f(x)) = \bigcup_{t\in f(x)} g(t)$.

Corollary 1.8. *Suppose $f : X \to\to Y$ be a fuzzy $\varphi\psi$-continuous single valued multifunction and suppose $g : Y \to\to Z$ be $\psi\eta$-fuzzy continuous multifunction. Then, $gof : X \to\to Z$ is $\varphi\eta$-fuzzy continuous multifunction.*

Proof. Assume that $x^{\alpha}_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$ in X. Since f is fuzzy $\varphi\psi$-continuous multifunction, so

$$f(x^{\alpha}_{\epsilon_\alpha}) \xrightarrow{\psi} f(x_\epsilon). \tag{1.14}$$

Assume that g is $\psi\eta$-fuzzy continuous multifunction and f is fuzzy $\varphi\psi$-continuous single valued multifunction, so

$$g(f(x^{\alpha}_{\epsilon_\alpha})) \xrightarrow{\eta} g(f(x_\epsilon)). \tag{1.15}$$

Therefore,

$$(gof)(x^{\alpha}_{\epsilon_\alpha}) \xrightarrow{\eta} (gof)(x_\epsilon). \tag{1.16}$$

Theorem 1.7 completes the proof. □

Definition 1.9. Let X_0 be a subset of X, let $i : X_0 \to X$ be the inclusion map, and let $f : X \to\to Y$ be a fuzzy multifunction. Say that foi is the *restriction* of f to X_0.

Lemma 1.10. *Assuming φ is a fuzzy operation on X and $X_0 \subseteq X$. Then $\widetilde{\varphi}(A) = \varphi(\widetilde{A})$ defines a fuzzy operation on X_0, where $\widetilde{A}$ is the extension of A by zero to X.*

Proof. It is easy to see that $\widetilde{\varphi}$ is a well-defined map and $\widetilde{\varphi}(01_X) = 01_X$. φ is a fuzzy operation, so $\text{int}(\widetilde{A}) \le \varphi(\widetilde{A})$. But,

$$\begin{aligned} \text{int}(A) &= \bigvee\{\acute{U} : \acute{U} \le A,\ \acute{U} \text{ is a fuzzy open set}\} \\ &= \bigvee\{Uoi : U \le A,\ U \text{ is a fuzzy open set}\}. \end{aligned} \tag{1.17}$$

For $x_0 \in X_0$, $\text{int}(A)(x_0) = \text{int}(\widetilde{A})(x_0)$. This shows that

$$\text{int}(A) \le \varphi(\widetilde{A}) \le \widetilde{\varphi}(A). \tag{1.18}$$

□

The following result shows the fuzzy continuity of the restriction of fuzzy multifunction.

Theorem 1.11. *Suppose $f : X \to\to Y$ be a fuzzy $\varphi\psi$-continuous multifunction and $X_0 \subseteq X$. Then foi is a $\widetilde{\varphi}\psi$-fuzzy continuous multifunction, where φ is a monotonous fuzzy operation.*

Proof. For any fuzzy point x_ϵ in X_0, $j(x_\epsilon)$ is a fuzzy point in X. It shows that for any fuzzy open neighborhood B of $f(i(x_\epsilon))$, there is a fuzzy open neighborhood A of $i(x_\epsilon)$

for which $f(\varphi(A)) \leq \psi(B)$. But Aoi is a fuzzy open neighborhood of x_ϵ in X_0, only we must show that $foi(\widetilde{\varphi}(Aoi)) \leq \psi(B)$. To see this,

$$\begin{aligned} foi(\widetilde{\varphi}(Aoi))(y) &= \bigvee_{z\in X_0} ((foi)(z)(y) \wedge \widetilde{\varphi}(Aoi)(z)) \\ &= \bigvee_{z\in X_0} (f(z)(y) \wedge \varphi\widetilde{(Aoi)}oi(z)) \\ &\leq \bigvee_{z\in X_0} (f(z)(y) \wedge \varphi(A)(z)) \\ &= f(\varphi(A))(y) \\ &\leq \psi(B)(y). \end{aligned} \tag{1.19}$$

□

Proposition 1.12. *Suppose (X,τ) and (Y,η) be fuzzy topological spaces, φ and ψ are fuzzy operations on X and Y, respectively, where φ is a monotonous fuzzy operation. Let $f : X \to\to Y$ be any fuzzy multifunction and let $\mathcal{B}$ be a base for η. Then f is fuzzy $\varphi\psi$-continuous multifunction if and only if f is fuzzy $\varphi\psi$-continuous multifunction with respect to $\mathcal{B}$.*

Proof. $(\Rightarrow)$ is straightforward.

For $(\Leftarrow)$ consider any fuzzy point x_ϵ in X and any fuzzy open neighborhood B of $f(x_\epsilon)$. $C \in \mathcal{B}$ exists such that $f(x_\epsilon) \leq C$ and $C \leq B$. From the assumption there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \leq \psi(C)$. But ψ is monotonous so $f(\varphi(A)) \leq \psi(B)$. □

References

[1] J. Albrycht and M. Matłoka, *On fuzzy multivalued functions. I. Introduction and general properties*, Fuzzy Sets and Systems **12** (1984), no. 1, 61–69.

[2] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, Journal of Mathematical Analysis and Applications **82** (1981), no. 1, 14–32.

[3] I. Beg, *Continuity of fuzzy multifunctions*, Journal of Applied Mathematics and Stochastic Analysis **12** (1999), no. 1, 17–22.

[4] E. Ekici, *On some types of continuous fuzzy functions*, Applied Mathematics E-Notes **4** (2004), 21–25.

[5] S. Ganguly and S. Saha, *A note on δ-continuity and δ-connected sets in fuzzy set theory*, Simon Stevin. A Quarterly Journal of Pure and Applied Mathematics **62** (1988), no. 2, 127–141.

[6] E. E. Kerre, A. A. Nouh, and A. Kandil, *Operations on the class of all fuzzy sets on a universe endowed with a fuzzy topology*, Proceedings IFSA, Brussels, 1991, pp. 109–113.

[7] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, Journal of Mathematical Analysis and Applications **56** (1976), no. 3, 621–633.

[8] M. N. Mukherjee and B. Ghosh, *Some stronger forms of fuzzy continuous mappings on fuzzy topological spaces*, Fuzzy Sets and Systems **38** (1990), no. 3, 375–387.

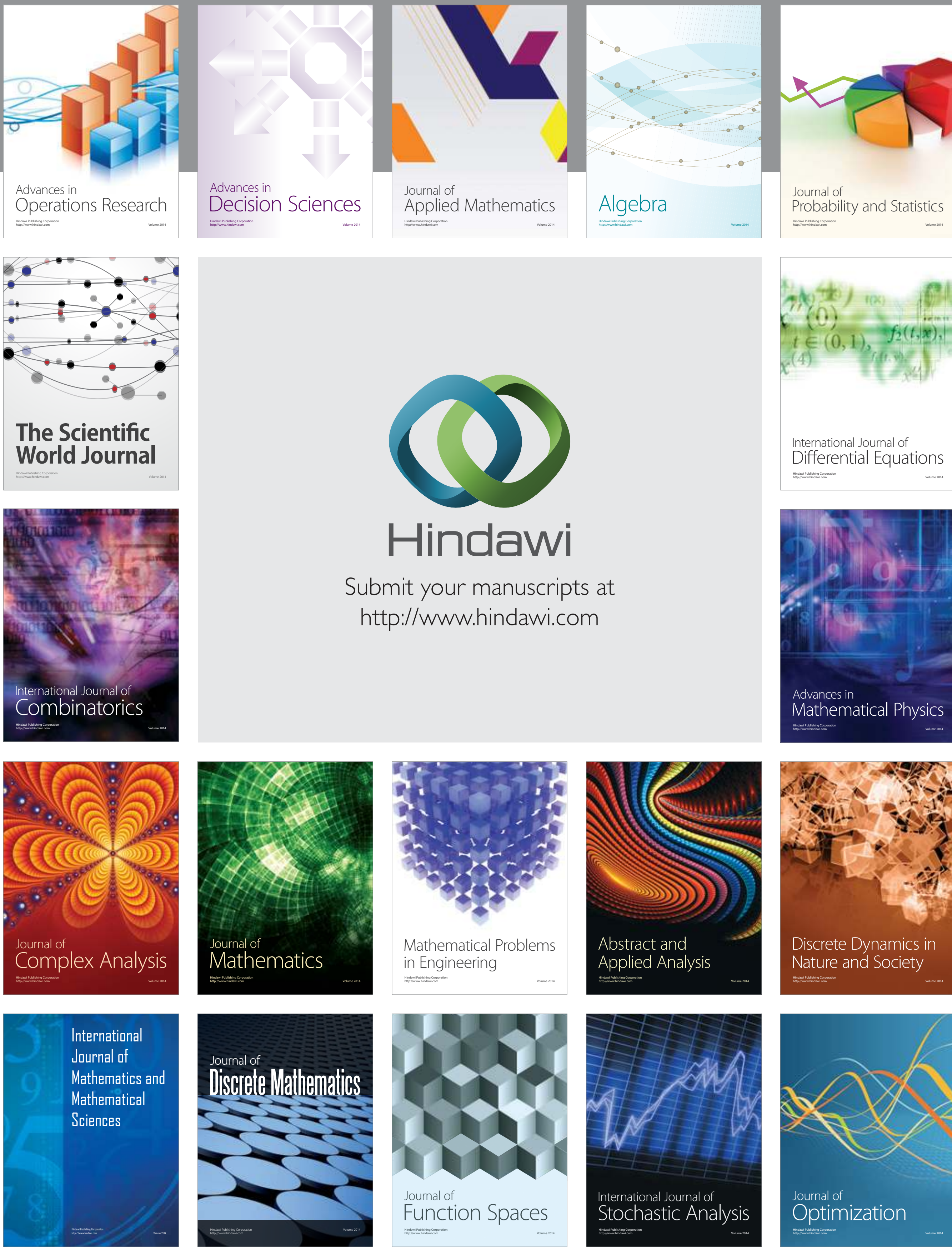
[9] M. N. Mukherjee and S. P. Sinha, *On some near-fuzzy continuous functions between fuzzy topological spaces*, Fuzzy Sets and Systems **34** (1990), no. 2, 245–254.

[10] N. S. Papageorgiou, *Fuzzy topology and fuzzy multifunctions*, Journal of Mathematical Analysis and Applications **109** (1985), no. 2, 397–425.

[11] P.-M. Pu and Y. M. Liu, *Fuzzy topology. II. Product and quotient spaces*, Journal of Mathematical Analysis and Applications **77** (1980), no. 1, 20–37.
[12] S. Saha, *Fuzzy δ-continuous mappings*, Journal of Mathematical Analysis and Applications **126** (1987), no. 1, 130–142.

M. Alimohammady: Mazandaran University, 47416-1467 Babolsar, Iran
E-mail address: m.alimohammady@gmail.com

M. Roohi: Islamic Azad University, Sari Branch, Sari, Iran
E-mail address: mehdi.roohi@gmail.com


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization