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On G-arc-regular dihedrants and regular dihedral maps

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Abstract A graph Γ is said to be *G-arc-regular* if a subgroup $G \le \mathsf{Aut}(\Gamma)$ acts regularly on the arcs of Γ. In this paper connected G-arc-regular graphs are classified in the case when G contains a regular dihedral subgroup D_{2n} of order $2n$ whose cyclic subgroup $C_n \le D_{2n}$ of index 2 is core-free in G. As an application, all regular Cayley maps over dihedral groups D_{2n}, n odd, are classified.

Keywords G-arc-regular graph · Cayley graph · Cayley map · Dihedral group

1 Introduction

In this paper all groups and graphs are finite. It is also assumed that the graphs are simple and undirected. For a graph Γ let $V(\Gamma)$, $A(\Gamma)$ and $\mathsf{Aut}(\Gamma)$ denote the vertex set, the arc set (or dart set) and the automorphism group of Γ, respectively. A *k-arc* of Γ is a sequence of $k+1$ vertices $v_1, v_2, \dots, v_{k+1}$ in $V(\Gamma)$, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. We say that Γ is *G-k-arc-transitive* and *G-k-arc-regular*, respectively, if $G \le \mathsf{Aut}(\Gamma)$ acts transitively and regularly on the set of k-arcs of Γ. In particular, $\mathsf{Aut}(\Gamma)$-k-arc-transitive and $\mathsf{Aut}(\Gamma)$-k-arc-regular are usually referred to as *k-arc-transitive* and *k-regular*, respectively. Also, we will sometimes omit the prefix k in

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the special case $k = 1$ and talk about G-arc-transitive and G-arc-regular graphs. Let K be a group with identity element 1, and let S be a subset of K such that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $\mathsf{Cay}(K, S)$ over K relative to S is the graph with vertex set K, and the ordered pair $(x, y) \in K \times K$ is an arc if and only if $xy^{-1} \in S$.

The question as to which Cayley graphs $\mathsf{Cay}(K, S)$ are s-arc-transitive for some $s > 0$ has been studied extensively. For instance, 2-arc-transitive Cayley graphs over Abelian groups have been classified in [15], and those over dihedral groups in [3, 16, 17]. As for *arc-transitivity*, a complete classification is known only for Cayley graphs over cyclic groups, see [8, 14] (it can also be found implicitly in [18]).

In this paper we consider the class of connected *G-arc-regular* Cayley graphs over dihedral groups, *dihedrants* in short. In particular, 1*-regular* dihedrants have been studied in a number of papers: 1-regular dihedrants of valency 4 or 6 in [10, 25, 26], whereas those of prime valency in [7], and moreover a construction of a 1-regular dihedrant of valency $2k \geq 4$ is given in [13].

The paper is organized as follows. In Sect. 2 we study those connected G-arc-regular dihedrants, for which G contains a regular dihedral subgroup D_{2n} of order $2n$, and for which the cyclic subgroup $C_n \leq D_{2n}$ of order n is core-free in G (in other words, C_n contains no non-trivial normal subgroup of G). Throughout this paper we will denote the class of these dihedrants by $\mathcal{F}$. The study of the class $\mathcal{F}$ was initiated by the authors in [9], where the following result was proved (see [9, Theorem 1.1]): if Γ is in class $\mathcal{F}$, then Γ is isomorphic to a lexicographical product $(K_{n_1} \otimes \cdots \otimes K_{n_\ell})[K_m^c]$, where the numbers n_i are pairwise coprime. It turns out, however, that not all of the above products are actually contained in $\mathcal{F}$. As the first main result of this paper we give a complete classification of arc-regular dihedrants in class $\mathcal{F}$ (see Theorem 2.8).

In Sect. 3 we apply our classification theorem to Cayley maps over dihedral groups (*dihedral maps* for short). By a *map* with an underlying graph Γ we mean a triple $\mathcal{M} = (\Gamma; R, T)$, where R is a permutation of the arc set $A(\Gamma)$ whose orbits coincide with the sets of arcs initiating in the same vertex, and T is an involution of $A(\Gamma)$ whose orbits coincide with sets of arcs with the same underlying edge. The permutations R and T are called, respectively, *rotation* and *dart-reversing involution* of $\mathcal{M}$. It is well-known that the group $\mathsf{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ acts semi-regularly on $A(\Gamma)$. We say that $\mathcal{M}$ is *regular* if $\mathsf{Aut}(\mathcal{M})$ acts transitively, and hence regularly, on $A(\Gamma)$. (In this paper we consider only orientable embeddings, that is, embeddings of graphs into orientable surfaces, and we consider only orientation preserving automorphisms of the studied maps.) Let S be an inverse closed generating set of a group K satisfying $1 \notin S$ and $S = S^{-1}$, and let p be a cyclic permutation of S. Then the *Cayley map* $\mathsf{CM}(K, S, p)$ is defined as the map $\mathcal{M} = (\Gamma; R, T)$, where the underlying graph Γ coincides with $\mathsf{Cay}(K, S)$, and the rotation R is defined as $(x, sx)^R = (x, p(s)x)$ for any $x \in K$ and $s \in S$. (For more information on regular maps and Cayley maps we refer the reader to the survey papers [20] and [22], respectively.) The class of cyclic groups is the only class of finite groups for which all regular Cayley maps have been classified [1]. (The solution for the special case of cyclic groups of prime order is of an earlier date, see [6].) Apart from cyclic groups only partial classifications are known including the classification of regular balanced Cayley maps over dihedral and generalized quaternion groups, see [24], and the classification of regular t-balanced Cayley maps over dihedral, dicyclic and semi-dihedral groups, see [11, 12, 21]. The second main result in this paper is a complete classification of all regular Cayley maps over dihedral groups D_{2n} of order $2n$ where n is odd (see Theorem 3.2).

2 G-arc-regular dihedrants with trivial cyclic core

We start by fixing relevant notation and terminology. Let G be a group acting on a set Ω. For a subset $\Delta \subseteq \Omega$, denote by G_Δ the elementwise stabilizer of Δ in G, and by $G_{\{\Delta\}}$ the setwise stabilizer of Δ in G. If Δ is a G-invariant subset, then g^Δ denotes the permutation of Δ induced by $g \in G$, and we let $G^\Delta = \{g^\Delta \mid g \in G\}$. Suppose that G acts transitively on Ω, and let $\mathcal{B}$ be an imprimitivity system of G. Then $G_\mathcal{B} = \{g \in G \mid \forall B \in \mathcal{B} : B^g = B\}$ is the kernel of G acting on $\mathcal{B}$. We say that $\mathcal{B}$ is *normal* if $G_\mathcal{B}$ acts transitively on each block $B \in \mathcal{B}$. A block B of G is *minimal* if it does not contain any non-trivial, proper block of G, and an imprimitivity system $\mathcal{B}$ is *minimal* if it is generated by a minimal block B, that is, $\mathcal{B} = \{B^g \mid g \in G\}$.

Suppose next that, in addition, G contains a regular dihedral subgroup D of order $|D| = 2n$ with $n > 2$, and let $C \le D$ denote the cyclic subgroup of index 2 in D. If B is a block of G, then clearly $D_{\{B\}}$ is a subgroup of D of order $|B|$. We shall say that B is of *cyclic* type, *dihedral* type, respectively, if $D_{\{B\}} \le C$, and $D_{\{B\}} \not\le C$.

Let Γ be a graph and let $G \le \mathsf{Aut}(\Gamma)$ be a subgroup acting transitively on $V(\Gamma)$. For an imprimitivity system $\mathcal{B}$ of G, the *quotient graph* $\Gamma/\mathcal{B}$ of Γ with respect to $\mathcal{B}$ is the graph with vertex set $\mathcal{B}$, and $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ is an arc if and only if Γ contains an arc (x_1, x_2) with $x_i \in B_i$. We denote by $G^\mathcal{B}$ the permutation group of $\mathcal{B}$ induced by G acting on $\mathcal{B}$.

For two graphs Γ_1 and Γ_2, their *lexicographical product* $\Gamma_1[\Gamma_2]$ of Γ_1 with Γ_2 is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, and (u_1, u_2) is adjacent to (v_1, v_2) if and only if u_1 is adjacent to v_1 in Γ_1, or $u_1 = v_1$ and u_2 is adjacent to v_2 in Γ_2. The complement of a graph Γ is denoted by Γ^c, and the complete graph with n vertices is denoted by K_n.

For the rest of the paper, D_{2n} denotes the dihedral group of order $2n$ with the presentation

$$D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1\rangle, \quad \text{and we set} \quad C_n = \langle rs\rangle.$$

Note that, if $n \ne 2$, then C_n is the unique cyclic subgroup of D_{2n} of order n. For $d \in D_{2n}$ we denote by d_* the *right translation* of D_{2n} acting according to the rule

$$x^{d_*} = xd$$

for any $x \in D_{2n}$ and we write $H_* = \{x_* \mid x \in H\}$ for any subgroup H of D_{2n}. Our goal in this section is to determine the class $\mathcal{F}$ of connected G-arc-regular Cayley graphs Γ over D_{2n} such that G satisfies the following two properties:

$$(D_{2n})_* \le G, \quad \text{and} \quad \mathrm{core}_G\big((C_n)_*\big) = 1. \tag{1}$$

Recall that for a group A and its subgroup $B \le A$, the *core* of B in A is the largest normal subgroup of A contained in B, and denoted by $\mathrm{core}_A(B)$. Notice that in the case $n = 2$ the second part of (1) should be satisfied by *any* cyclic subgroup of order 2.

In Example 2.1 below an outline of a construction of a graph in $\mathcal{F}$ with $2n$ vertices for any n of the form $n = 2m$, m is odd, and $m \geq 3$, is given (for details see [9, Sect. 3]).

Example 2.1 Let $n = 2m$, where $m \geq 3$ is an odd number, and let D_{2m} be the dihedral group given by the above presentation. Let $C_m \leq D_{2m}$ be the unique cyclic subgroup of order m. Define the group $\mathbb{D}_n = (D_{2m} \times D_{2m}) \rtimes \langle\sigma\rangle$, where σ is an automorphism of $D_{2m} \times D_{2m}$ interchanging the coordinates, that is, $(x, y)^\sigma = (y, x)$ for all $(x, y) \in D_{2m} \times D_{2m}$. Notice that $\mathbb{D}_n$ is isomorphic to the wreath product $D_{2m} \wr \mathbb{Z}_2$. The elements of $\mathbb{D}_n$ are written as triples (a, b, σ^i), $a, b \in D_{2m}$ and $i \in \mathbb{Z}_2$, and the product of two triples is defined as follows:

$$\bigl(a, b, \sigma^i\bigr)\bigl(c, d, \sigma^j\bigr) = \begin{cases} (ac, bd, \sigma^{i+j}) & \text{if } i = 0, \\ (ad, bc, \sigma^{i+j}) & \text{if } i = 1. \end{cases}$$

The subset $D = \{(a, b, 1) \mid a \in D_{2m}, b \in \langle r\rangle\}$ is a subgroup of $\mathbb{D}_n$, and in fact, $D \cong D_{2m} \times \mathbb{Z}_2 \cong D_{2n}$. It transpires that the unique cyclic subgroup of D of order n is given as $C = \{(a, b, 1) \mid a \in C_m, b \in \langle r\rangle\}$. It follows from $C \cap C^\sigma = 1$ that $\mathrm{core}_{\mathbb{D}_n}(C) = 1$. Let $x \in \mathbb{Z}_m^*$, $x^2 \equiv 1 \pmod m$. The subset $A = \{(c^x, c, \sigma^i) \mid c \in C_m, i \in an\mathbb{Z}_2\}$ is a subgroup of $\mathbb{D}_n$ of order n. Clearly, $A \cap D = 1$, and therefore, $\mathbb{D}_n$ is factorized as $\mathbb{D}_n = AD$. A direct check shows that $\mathrm{core}_{\mathbb{D}_n}(A) = 1$.

Consider the action of $\mathbb{D}_n$ on the set $\mathbb{D}_n/A$ of right A-cosets. Since $\mathrm{core}_{\mathbb{D}_n}(A) = 1$, this results in a faithful permutation representation of $\mathbb{D}_n$. The subgroup D induces a regular dihedral subgroup of order $2n$, hence it is possible to identify $\mathbb{D}_n/A$ with D, by the bijection $Ax \leftrightarrow x, x \in D$. Thus we have the coset graph $\mathsf{Cos}(\mathbb{D}_n, A, AgA)$ is isomorphic to $\mathsf{Cay}(D, S)$, where $AgA = AS$. Observe that $\mathsf{Cay}(D, S)$ is then $\mathbb{D}_n$-arc-transitive with $\mathbb{D}_n$ satisfying (1).

Let us choose the double coset $A(1, r, 1)A$. Then $A(1, r, 1)A = AE(1, r, 1)$, where $E \leq D$ is the subgroup $E = \{(r^i c, r^i, 1) \mid c \in C_m, i \in \mathbb{Z}_2\} \cong D_n$. This means that

$$\mathsf{Cos}\bigl(\mathbb{D}_n, A, A(1, r, 1)A\bigr) \cong \mathsf{Cay}\bigl(D, E(1, r, 1)\bigr) \cong K_{n,n},$$

and we conclude that $\mathsf{Cay}(D, E(1, r, 1))$ is also $\mathbb{D}_n$-arc-regular, and therefore it belongs to $\mathcal{F}$.

Herzog and Kaplan proved that the core $\mathrm{core}_G(C)$ of a cyclic subgroup C in a group G is non-trivial if $[G : C] \leq |C|$ (see [4, Theorem A]). The following similar result about dihedral subgroups was obtained by the authors (see [9, Theorem 4.3]).

Proposition 2.2 *Let G be a finite group, $D \leq G$ a dihedral subgroup of order $2n$ with $n > 4$, and C the cyclic subgroup of D order n. If $[G : D] \leq n$ and $\mathrm{core}_G(C) = 1$, then $[G : D] = n$, and $G \cong \mathbb{D}_n$.*

Remark Notice that the formulation of Theorem 4.3 given in [9] contains a mistake. The result given above is formulated in a correct form. The complete list of corrections related to [9] is contained in the last section of the paper.

We start by enumerating all graphs of small order contained in $\mathcal{F}$.

Proposition 2.3 *Let $\Gamma = \mathsf{Cay}(D_{2n}, S)$ be a G-arc-regular graph from class $\mathcal{F}$ with $n \le 4$. Then*

(i) $n = 1$, $\Gamma \cong K_2$, $G \cong S_2$,
(ii) $n = 2$, $\Gamma \cong K_4$, $G \cong A_4$,
(iii) $n = 3$, $\Gamma \cong K_{2,2,2}$, $G \cong S_4$,
(iv) $n = 4$, $\Gamma \cong Q_3$, $G \cong S_4$.[1]

Proof If $n \le 2$ then one can easily find that one of the cases (i)–(iii) holds. If $n = 3, 4$ then $(C_n)_*$ is a characteristic subgroup of $(D_{2n})_*$ implying that $(D_{2n})_*$ is not normal in G. Thus $|S| = [G : (D_{2n})_*] \ge 3$.

Case $n = 3$. In this case $3 \le |S| \le 5$. If $|S| = 5$, then G should be a Frobenius group of order 30 which does not exist. If $|S| = 3$, then a Sylow 3-subgroup P of G is Abelian and normal in G. This yields $(C_3)_* \trianglelefteq G$ contrary to (1). If $|S| = 4$, then $|G| = 24$ and since a Sylow 3-subgroup of G is not normal, we conclude that $G \cong S_4$ and $\Gamma \cong K_{2,2,2}$ (case (iii)).

Case $n = 4$. If $|S| = 7$ then G is a Frobenius group of order 56 with elementary Abelian kernel of order 8 contrary to the inclusion $(D_8)_* \le G$. Also the case of $|S| = 5$ is impossible because in this case a point stabilizer is a Sylow 5-subgroup of order 5 which is normal in G. Thus $|S| = 3, 4, 6$. If $|S| = 3$, then $|G| = 24$ and the point stabilizer of G is a Sylow 3-subgroup of order 3 which is not normal. Then there are four Sylow 3-subgroups. The action of G on the set of its Sylow 3-subgroups yields a homomorphism into S_4 the image of which is either A_4 or S_4. In the first case the kernel of the homomorphism has order 2, and, therefore, is contained in $(C_4)_*$ contrary to (1). In the second case $G \cong S_4$ and $\Gamma \cong Q_3$ (case (iv)).

If $|S| = 4$, then $|G|$ has order 32 and acts transitively on four right cosets of the subgroup $(D_8)_*$. This yields a homomorphism of G into a Sylow 2-subgroup of S_4. Comparing orders we conclude that the kernel of the above homomorphism has order at least 4. Since the kernel is nothing but the core of $(D_8)_*$ in G, we obtain $|\mathrm{core}_G((D_8)_*)| \ge 4$. Since G is a 2-group, the subgroup $\mathrm{core}_G((D_8)_*)$ intersects the center of G non-trivially. Together with $\mathrm{core}_G((D_8)_*) \cap \mathbf{Z}(G) \le \mathbf{Z}((D_8)_*)$ and $|\mathbf{Z}((D_8)_*)| = 2$ we obtain $\mathrm{core}_G((D_8)_*) \cap \mathbf{Z}(G) = \mathbf{Z}((D_8)_*)$ implying $\mathbf{Z}((D_8)_*) \trianglelefteq G$. But now the inclusion $\mathbf{Z}((D_8)_*) \le (C_4)_*$ contradicts (1).

If $|S| = 6$, then $(D_8)_*$ is normal in a Sylow 2-subgroup P which contains $(D_8)_*$. Also $|P_e| = 2$ and P_e normalizes $(D_8)_*$. Let g be the unique involution contained in P_e (recall that it normalizes $(D_8)_*$). Since G_e acts regularly on S, the involution g has no fixed point inside S, and, therefore, it fixes two points of D_8. That means g centralizes two elements of $(D_8)_*$ implying that $C_{(D_8)_*}(g) = \{1, z_*\}$ where $z \in (C_4)_*$ is the unique non-trivial element of the center of D_8. Thus G_e has orbits $\{1\}, \{z\}, D_8 \setminus \{1, z\}$. Hence the cosets of the subgroup $\{1, z\}$ form a G-invariant imprimitivity system $\mathcal{A}$. The kernel K of G-action on the blocks of the $\mathcal{A}$ contains z_*. Moreover, z_* is a unique fixed-point-free element of K. Therefore z_* is in the center of G implying $z_* \in \mathrm{core}_G((C_4)_*)$ contrary to (1). □

[1]Here Q_3 is a three dimensional cube.

Let $\Gamma = \mathsf{Cay}(D_{2n}, S)$ be a G-arc-regular graph from class $\mathcal{F}$ of valency at most n. If $n \leq 4$ then Proposition 2.3 gives a complete list of graphs. If $n > 4$ then by Proposition 2.2 $|S| = n$ and $G \cong \mathbb{D}_n$. In the second case $\Gamma \cong \mathsf{Cos}(\mathbb{D}_n, A, AgA)$, where $A \leq \mathbb{D}_n$ is a core-free subgroup satisfying $A \cap D = 1$ and $\mathbb{D}_n = AD$. It was shown that under such conditions the coset graph $\mathsf{Cos}(\mathbb{D}_n, A, AgA)$ is isomorphic to $K_{n,n}$ (see [9, Proposition 3.3]), and thus our final goal of finding Γ is accomplished in the case when the valency of Γ is less than or equal to n.

In the rest of the section we assume that the valency of Γ is greater than n. We shall first prove two preparatory lemmas. The first one contains some useful properties that hold for arbitrary connected G-arc-regular graphs. The proof is straightforward and we omit it.

Lemma 2.4 *Let Γ be an undirected, connected and G-arc-regular graph. Then the following statements hold.*

(i) *For each arc (x, y) of Γ, there exists a unique involution $t_{xy} \in G$ which inverts (x, y) (that is, $x^{t_{xy}} = y$ and $y^{t_{xy}} = x$).*
(ii) *The set of all involutions $T = \{t_{xy} \mid (x, y) \in A(\Gamma)\}$ form a single conjugacy class of G.*
(iii) *For each $t \in T$, the centralizer $C_G(t)$ acts regularly on the set of all arcs inverted by t.*
(iv) *For each arc (x, y) of Γ, $G = \langle G_x, t_{xy}\rangle$.*

Following [2] we shall call the involutions t_{xy} as in the above lemma the *arc-inverting involutions* of Γ in G.

Lemma 2.5 *Let $\Gamma = \mathsf{Cay}(D_{2n}, S)$ be a connected G-arc-regular graph, and T be the set of arc-inverting involutions of Γ in G. Then the following hold.*

(i) *For each $s \in S \setminus C_n$, the permutation s_* is in T, and*

$$\left|C_G(s_*)\right| = \left|C_{D_{2n}}(s)\right| \cdot \left|S \cap s^{D_{2n}}\right|,$$

where $s^{D_{2n}}$ is the conjugacy class of s in D_{2n}.
(ii) *If $|S \cap C_n| = |S|/2$ and $|S| > n$, then $|T| = 2n$.*

Proof To simplify the notation we set $D = D_{2n}$ and $C = C_n$.

We start by proving (i). Since $s \in S$ and $s^2 = 1$, the pair $(1, s)$ is an arc of Γ and $(1, s)^{s_*} = (s, 1)$. Thus s_* is an arc-inverting involution. Let us count the number of arcs inverted by s_*. An arc of $\mathsf{Cay}(D, S)$ is of the form $(x, u^{-1}x)$ with $x \in D, u \in S$. Thus the number of arcs inverted by s_* is equal to

$$\begin{aligned}\left|\left\{(x, u) \in D \times S \mid \left(x, u^{-1}x\right) = \left(u^{-1}xs, xs\right)\right\}\right| &= \left|\left\{x \in D \mid xsx^{-1} \in S\right\}\right| \\ &= \left|C_D(s)\right| \cdot \left|S \cap s^D\right|.\end{aligned}$$

The statement then follows by Lemma 2.4(iii).

To prove (ii), note that the assumption $|S| > n$ implies that $|S \cap C| = |S|/2 > n/2$. If n is odd, then $s^D = D \setminus C$ for each $s \in D \setminus C$. In this case $S \cap s^D = S \setminus C$, $|C_D(s)| =$

2 and we find that $C_G(s_*)$ has order $2 \cdot |S \setminus C| = |S|$. Since G acts regularly on the set of arcs of Γ, we have $|G| = |S| \cdot |D|$, and therefore $|T| = |(s_*)^G| = |D| = 2n$.

If n is even, then $D \setminus C$ is a union of two D-conjugacy classes of cardinality $n/2$, say R_1 and R_2. Since $|S \setminus C| > n/2$, the set $S \setminus C$ intersects both classes R_1 and R_2 non-trivially. By part (i), for each $s \in S \cap R_i$ we have $|C_G(s_*)| = 4 \cdot |S \cap R_i|$. But the elements from $(S \setminus C)_*$ are arc-inverting involutions. Therefore, they are conjugate in G, implying that $|C_G(s_*)|$ is constant for each $s \in S \setminus C$. Therefore $4 \cdot |S \cap R_1| = 4 \cdot |S \cap R_2|$, and, consequently, $|S \cap R_1| = |S|/4$. Hence $|C_G(s_*)| = |S|$, implying $|T| = |(s_*)^G| = 2n$. □

Note that it follows from the above proof of Lemma 2.5 that if n is even and $|S| > n$ then the two $(D_{2n})_*$-conjugacy classes $(R_1)_*$ and $(R_2)_*$ are fused into one, and we also have the inclusion $(D_{2n})_* \setminus (C_n)_* \subset T$.

Our approach is to analyze the minimal imprimitivity systems of the groups in question. We make use of the following description which can be deduced from the proof of [9, Theorem 1.1].

Proposition 2.6 *Let $\Gamma = \mathsf{Cay}(D_{2n}, S)$ be a connected G-arc-regular graph such that G satisfies* (1) *and $n < |S| < 2n - 1$.*[2] *Then one of the following holds.*

(i) *G has a block system $\mathcal{B}$ with block size q, q is an odd prime, $\Gamma \cong \overline{\Gamma}[K_q^c]$, and $G_{\mathcal{B}} \cong \mathbb{Z}_q^2$.*
(ii) *G has a unique minimal block system $\mathcal{B}$ of dihedral type, $\mathcal{B}$ has block size* 2, *$\Gamma \cong \overline{\Gamma}[K_2^c]$, and $G_{\mathcal{B}} \cong \mathbb{Z}_2^\ell$, $\ell \le 2$.*
(iii) *G has a unique minimal block system $\mathcal{B}$ of dihedral type, $\mathcal{B}$ has block size* 4, *$\Gamma \cong \overline{\Gamma}[K_4^c]$, and $G_{\mathcal{B}} \cong \mathbb{Z}_2^2$.*

As the next step we settle the case (ii) of Proposition 2.6.

Theorem 2.7 *With notation as in Proposition* 2.6, *suppose that case* (ii) *holds. Then $\Gamma \cong K_{2,2,2}$, and $G \cong S_4$.*

Proof We set first some notation. Again, we write D for D_{2n} and C for C_n. We set $S_0 = S \cap C$ and $N = G_{\mathcal{B}}$ for the kernel of G acting on $\mathcal{B}$. It can be assumed without loss of generality that $\mathcal{B}$ consists of the cosets $\{1, r\}g$, $g \in D$. By Proposition 2.6 (ii), $N \cong \mathbb{Z}_2^\ell$, $\ell \le 2$ and $\Gamma \cong (\Gamma/\mathcal{B})[K_2^c]$. The latter isomorphism implies $S = \{1, r\}S_0$. In particular, $|S_0| = |S|/2$. Finally, let T be the set of all arc-inverting involutions of Γ in G. Notice that by Lemma 2.5(ii), $|T| = 2n$.

The rest of the proof is divided into several steps.

(a) n is odd.

Since $\mathcal{B}$ is a block system of G, each involution $t \in T$ fixes the same number of blocks in $\mathcal{B}$. Recall that we have the inclusion $D_* \setminus C_* \subseteq T$. Let c be a generating element of C. Now the permutations $r_*, (rc)_* \in D_* \setminus C_*$ are conjugate in G, and thus

[2] Notice that $n > 1$ in this case.

fix the same number of blocks in $\mathcal{B}$. If n is even, then r_* fixes two blocks $\{1,r\}$ and $\{1,r\}c^{n/2}$, while $(rc)_*$ fixes none, a contradiction.

Notice that since n is odd, each $s_* \in D_* \setminus C_*$ fixes a unique block of $\mathcal{B}$.

(b) If $r_* \in H \le G$, then $C_H(r_*) = C_{H_1}(r_*)\langle r_*\rangle$, and consequently, $|C_H(r_*)| = 2|C_{H_1}(r_*)|$.

Since $\{1,r\}$ is a unique block of $\mathcal{B}$ fixed setwise by r_*, each $g \in C_H(r_*)$ fixes $\{1,r\}$ too. Therefore $C_H(r_*) \le H_{\{\{1,r\}\}} = H_1\langle r_*\rangle$. Now the claim follows from $\langle r_*\rangle \le C_H(r_*)$.

Notice that for $H = G$ we have $|C_{G_1}(r_*)| = |G_1|/2$ because $|C_G(r_*)| = |S| = |G_1|$ (see Lemma 2.5 (i)).

(c) If $D_* \le H \le G$, then either $r_*^H = r_*^{D_*}$ or $r_*^H = T$.

Clearly, $r_*^{D_*} \subseteq r_*^H \subseteq r_*^G = T$. Since $C_{G_1}(r_*)$ has index 2 in G_1, the index of the subgroup $H_1 \cap C_{G_1}(r_*) = C_{H_1}(r_*)$ in H_1 is at most 2. Now we can write

$$\left|r_*^H\right| = \frac{|H|}{2|C_{H_1}(r_*)|} = \frac{|H_1||D|}{|H_1|}\frac{|H_1|}{2|C_{H_1}(r_*)|} \in \{2n, n\}.$$

(d) $\langle T\rangle = G$.

If $\mathrm{core}_{\langle T\rangle}(C)$ is trivial, then by Proposition 2.2 either $n \le 4$ or $[\langle T\rangle : D_*] \ge n$. Using Proposition 2.3 and oddness of n we conclude that $[\langle T\rangle : D_*] \ge n$ in any case. This implies $[G : \langle T\rangle] = [G : D_*]/[\langle T\rangle : D_*] \le (2n-1)/n$, hence $[G : \langle T\rangle] = 1$.

If $\mathrm{core}_{\langle T\rangle}(C)$ is non-trivial, then the orbits of $\mathrm{core}_{\langle T\rangle}(C)$ form a normal imprimitivity system of $\langle T\rangle$, say $\mathcal{L}$, with odd block size. Since $\langle T\rangle$ is normal in G, $\mathcal{L}^g$, $g \in G$ is also an imprimitivity system of $\langle T\rangle$ with the same block size. But $D_* \le \langle T\rangle$, hence $\langle T\rangle$ admits a unique imprimitivity system with odd block size of a given order. Therefore $\mathcal{L}^g = \mathcal{L}$, implying that $\mathcal{L}$ is an imprimitivity system for G too. But this contradicts the assumption that $\mathcal{B}$ is a unique minimal imprimitivity system.

(e) For each $g \in G$ the subsets $D_* \setminus C_*$ and $(D_* \setminus C_*)^g$ either coincide or intersect in at most one element.

Assume toward a contradiction that the subsets $D_* \setminus C_*$, $(D_* \setminus C_*)^g$ are distinct and contain two common elements, say u_*, v_*. Then the subgroup $\langle u_* v_*\rangle$ is normal in D_* and D_*^g and, therefore, is normal in $K = \langle D_* \cup (D^g)_*\rangle$. The subsets $(D_* \setminus C_*)$ and $(D_* \setminus C_*)^g$ are conjugacy classes in the groups D_* and D_*^g respectively. Since they have non-trivial intersection, they are contained in a unique conjugacy class of K, namely r_*^K. It follows from $(D_* \setminus C_*) \ne (D_* \setminus C_*)^g$ that $|r_*^K| > n$. By (c) $r_*^K = T$. Together with (d) we obtain $K = G$. Thus $\langle u_* v_*\rangle$ is normal in G, which contradicts $\mathrm{core}_G(C_*) = 1$.

(f) $n = 3$, $\Gamma \cong K_{2,2,2}$, and $G \cong S_4$.

Consider the subsets $(D_* \setminus C_*)^g \subseteq T$, $g \in G$. Since G acts transitively on T, the union of these sets is the whole T. There are two possibilities: the number of distinct sets of the form $(D_* \setminus C_*)^g$, $g \in G$ is two or at least three. If there are only two subsets conjugate to $D_* \setminus C_*$, then these are blocks of the action of G on T (by conjugation).

In this case the setwise stabilizer H of $D_* \setminus C_*$ has index 2 in G, and hence it is normal in G. Since $D_* \setminus C_*$ is a conjugacy class of H, the subgroup $\langle D_* \setminus C_* \rangle = D_*$ is normal in H. This implies that $C_* \trianglelefteq H$. Pick an arbitrary $g \in G \setminus H$. Then C_*^g is normal in H too. The intersection $C_* \cap C_*^g$ is normal in G, and hence it is trivial. This implies that $C_*^g \cap D_* = 1$ and, therefore, $|H| \geq 2n^2$ implying $|G| \geq 4n^2$, a contradiction.

Thus we may assume that there are at least three subsets conjugate to $D_* \setminus C_*$. By (e) each two of these subsets intersect in at most one element. Therefore their union contains at least $3n-3$ elements. Since the union is contained in T and $|T| = 2n$, we conclude that $n \leq 3$. Thus $n = 3$, and (f) easily follows. □

We accomplish our goal by showing that the list of desired G-arc-regular graphs in $\mathcal{F}$ is completely exhausted by the graphs obtained till now.

Theorem 2.8 *Let $\Gamma = \mathsf{Cay}(D_{2n}, S)$ be a connected G-arc-regular graph such that $(D_{2n})_* \leq G$, and $(C_n)_*$ is core-free in G. Then one of the following holds.*

(i) $n = 1$, $\Gamma \cong K_2$, *and* $G \cong S_2$,
(ii) $n = 2$, $\Gamma \cong K_4$, *and* $G \cong A_4$,
(iii) $n = 3$, $\Gamma \cong K_{2,2,2}$, *and* $G \cong S_4$,
(iv) $n = 4$, $\Gamma \cong Q_3$, *and* $G \cong S_4$,
(v) $n = 2m$, *m is an odd number,* $\Gamma \cong K_{n,n}$, *and* $G \cong \mathbb{D}_n$.

Proof If $n \leq 4$, then Proposition 2.3 yields the result. Thus in what follows we assume that $n > 4$. If the valency of Γ is less than or equal to n, then (v) follows by Proposition 2.2 and the paragraph after Proposition 2.3.

We complete the proof by showing that no graph Γ under the assumption has valency greater than n for $n > 4$. Towards a contradiction let us choose Γ as a counter example which is also minimal relative to its order $2n$. If the group G is primitive, then Wielandt's Theorem [27, Satz 2] gives the result that it is 2-transitive. Thus G is a 2-transitive Frobenius group, and G contains a regular, normal, elementary Abelian Sylow p-subgroup. This implies $n = 2$, a contradiction. Thus G is imprimitive, and Proposition 2.6 is applicable. Fix an arbitrary non-trivial imprimitivity system $\mathcal{B}$ of G. Till the end of the proof we write N for $G_{\mathcal{B}}$, D for D_{2n} and C for C_{2n}.

If case (ii) of Proposition 2.6 occurs, then Theorem 2.7 yields the result. Thus it remains to resolve cases (i) and (iii) of Proposition 2.6.

Assume for the moment that case (iii) of Proposition 2.6 holds. Then n is even, and we let $n = 2m$ and $c = rs$. Recall that $N = G_{\mathcal{B}} \cong \mathbb{Z}_2^2$. We may assume that $\mathcal{B}$ consists of the cosets $\{1, r, c^m, rc^m\}c^i$, $c^i \in C$. Clearly, $(c^m)_*$ is in N. Let T be the set of arc-inverting involutions of Γ in G.

We show that $[t_*, N] = 1$ for some $t \in D \setminus C$. As $c^m \in \mathbf{Z}(D)$, the group D_* acts on N as a group of order at most 2. Thus if $[t_*, N] \neq 1$ for all $t \in D \setminus C$, then $C_{D_*}(N) = C_*$. Pick an element $t \in (D \setminus C) \cap S$, that is, $t = rc^i$ for some $c^i \in C$. Let x be an element in N such that $x^B = (1, r)(c^m, rc^m)$ where B is the block $\{1, r, c^m, rc^m\}$. Since $(c^i)_*$ commutes with x, we have $(c^i)^x = 1^{(c^i)_* x} = 1^{x (c^i)_*} = rc^i$, and $(rc^i)^x = c^i$ Similarly, x switches c^{m+i} and rc^{m+i}, and so we get $x^{Bc^i} = (c^i, rc^i)(c^{m+i}, rc^{m+i})$. From these $1^{t_* x} = (rc^i)^x = c^i$ and $(c^i)^{t_* x} = r^x = 1$. Thus

t_*x inverts the arc $(1, c^i)$. The element $t = rc^i$ was chosen from S, hence $(1, rc^i)$ is an arc of Γ. Since $\Gamma \cong \overline{\Gamma}[K_4^c]$, 1 is adjacent with any element in $\{1, r, c^m, rc^m\}c^i$, in particular, the arc $(1, c^i)$ is in $A(\Gamma)$, hence t_*x is in T, and thus $(t_*x)^2 = 1$. We conclude $[t_*, N] = [t_*, \langle (c^m)_*, x\rangle] = 1$, as required. Observe that $T = t_*^G$ together with $N \trianglelefteq G$ implies $[T, N] = 1$.

We show next that $T = (D_* \setminus C_*)N$. Pick an $s \in S$ and denote the block in $\mathcal{B}$ containing s by B'. Since $\Gamma \cong \overline{\Gamma}[K_4^c]$, there exists $t \in B' \cap (D \setminus C) \cap S$. Also, there exists $y \in N$ such that $t^y = s$ and $s^y = t$. As $[t_*, y] = 1$, we find $1^{t_*y} = s$, and $s^{t_*y} = s^{yt_*} = 1$. Thus t_*y is in T inverting the arc $(1, s)$. We conclude that $(D_* \setminus C_*)N$ contains $t_{1,s}$ for all $s \in S$. Let (z, sz) be any arc of Γ, $z \in D$, $s \in S$. Then $t_{z,zs} = t_{1,s}^{z_*}$, and by the previous observation, $t_{z,zs} \in (D_* \setminus C_*)N$, implying that $T = (D_* \setminus C_*)N$.

Thus we get $D_* \le \langle T\rangle \le D_*N$. It is clear that $\langle T\rangle \trianglelefteq G$. Since $[D_*N : D_*] = 2$ and $T \not\subseteq D_*$, we find $\langle T\rangle = D_*N$. In particular, $D_*N \trianglelefteq G$. Using $[T, N] = 1$ we obtain $[D_*, N] = 1$. Hence $(D_*N)^2 = (D_*)^2N^2 = (C_*)^2$ is normal in G. But the group $(C_*)^2$ has order $m > 1$, contradicting $\mathrm{core}_G(C_*) = 1$.

We are left with case (i) of Proposition 2.6. In this case $N \cong \mathbb{Z}_q^2$, where q is an odd prime. Let $\overline{G} = G/N$, i.e., $\overline{G}$ is the permutation group induced by the action of G on $\mathcal{B}$, and let $\overline{\Gamma}$ be the corresponding quotient graph $\Gamma/\mathcal{B}$. We determine $\overline{\Gamma}$ and $\overline{G}$.

Since $\Gamma = \overline{\Gamma}[K_q^c]$, the graph $\overline{\Gamma}$ is $\overline{G}$-arc-regular. Clearly, the valency of $\overline{\Gamma}$ is greater than $n/q = |\overline{D}|/2$, and $\overline{D_*} \le \overline{G}$. Then $N = \langle Q_*^G\rangle$, where $Q \le C$, $|Q| = q$. Using this fact it is proved (see [9, Proposition 4.1]) that

$$\left|\mathrm{core}_{\overline{G}}(\overline{C_*})\right| = \left|\mathrm{core}_{G/N}(C_*N/N)\right| \le \left|\mathrm{core}_G(C_*)\right| = 1.$$

We conclude that $\overline{G}$ satisfies (1), and thus $\overline{\Gamma}$ is in $\mathcal{F}$. By minimality of Γ it follows that $\overline{\Gamma}$ must be one of the graphs in (ii)–(iii), and $\overline{G}$ is isomorphic to either A_4 or S_4.

Consider the action of $\overline{G}$ on $N \cong \mathbb{Z}_q^2$. Since $C_* \cap N = Q_*$ and any $r \in D_* \setminus C_*$ inverts the elements of Q_*, the group $\overline{D_*}$ acts on N non-trivially. If the kernel of the $\overline{G}$-action on N is non-trivial, then it should contain the Klein 4-group $K \cong \mathbb{Z}_2^2$ (it is a unique minimal normal subgroup of $\overline{G}$). In this case $\overline{G} \cong S_4$, $\overline{D_*} \cong S_3$ and $\overline{G} = K\overline{D_*}$ implying $G = C_G(N)D_*$, hence $Q_* \trianglelefteq G$, contrary to $\mathrm{core}_G(C_*) = 1$. Thus we may assume that $\overline{G}$ acts on N faithfully, that is, it is embedded into $GL(\mathbb{Z}_q^2)$. A faithful action of K on N contains a non-identical element in $\mathbf{Z}(GL(\mathbb{Z}_q^2))$ represented by the matrix $-I$. In particular, $\mathbf{Z}(\overline{G}) \ne 1$. But $\overline{G} \cong A_4$, or $\overline{G} \cong S_4$, leading to a contradiction. This completes the proof of the theorem. □

3 Regular dihedral maps of order $2n$, n is odd

We start with notation and terminology. Given two maps $\mathcal{M}_1 = (\Gamma_1; R_1, T_1)$ and $\mathcal{M}_2 = (\Gamma_2; R_2, T_2)$, an *isomorphism* φ from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a bijection $\varphi\colon A(\Gamma_1) \to A(\Gamma_2)$ such that $R_1\varphi = \varphi R_2$ and $T_1\varphi = \varphi T_2$. Thus φ maps the orbits of R_1 onto the orbits of R_2, and thus induces naturally a bijection from $V(\Gamma_1)$ to $V(\Gamma_2)$. This bijection is clearly a graph isomorphism from Γ_1 to Γ_2, and will also be denoted by

φ. In case $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ the bijection φ is called an *automorphism* of $\mathcal{M}$. Two Cayley maps $\mathsf{CM}(K_1, S_1, p_1)$ and $\mathsf{CM}(K_2, S_2, p_2)$ are said to be *equivalent* if there exists a group isomorphism $\varphi\colon G_1 \to G_2$ which maps S_1 to S_2 such that $p_1\varphi = \varphi p_2$. Equivalent Cayley maps are isomorphic as maps, however, the converse is not true, isomorphic Cayley maps may not be equivalent as Cayley maps.

A Cayley map $\mathcal{M} = \mathsf{CM}(K, S, p)$ is called *t-balanced* if $p(s)^{-1} = p^t(s^{-1})$ for all $s \in S$. In particular, if $t = 1$, then we say that $\mathcal{M}$ is *balanced*. It is known (see [23]) that a regular Cayley map $\mathcal{M}$ is balanced if and only if there exists a group automorphism σ of K whose restriction to S is equal to p. In this case the group K_* of right translations of K is normal in $\mathsf{Aut}(\mathcal{M})$ and $\mathsf{Aut}(\mathcal{M}) \cong K_* \rtimes \langle\sigma\rangle$. More generally, a Cayley map $\mathcal{M} = \mathsf{CM}(K, S, p)$ is regular if and only if there exists a *skew-morphism* ψ of K whose restriction to S is equal to p (see [6, Theorem 1]). By a skew-morphism of K we mean a permutation ψ of K such that there exists a function $\pi : K \to \{0, 1, \dots, m-1\}$, where m is the order of ψ, such that

- $1^\psi = 1$, and
- $(xy)^\psi = x^{\psi^{\pi(y)}} y^\psi$ for all $x, y \in K$.

The function π is called the *power function* of ψ. We remark that our definition of a skew-morphism differs from the one given in [6], which is due to the fact that in our presentation we adopt the convention to write permutations acting on the right. Nevertheless, the theory of skew-morphisms developed in [6] can be repeated, in particular, we have the property that a permutation ψ of K such that $1^\psi = 1$ is a skew-morphism of K if and only if the product $K_*\langle\psi\rangle$ is a subgroup in $\mathsf{Sym}(K)$.

Our main goal in this section is to classify all Cayley maps over dihedral groups D_{2n} such that n is odd. As in the previous section the group D_{2n} is given by the presentation $D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1\rangle$, and $C_n = \langle c\rangle$, where $c = rs$. The automorphism group of D_{2n} is given as

$$\mathsf{Aut}(D_{2n}) = \bigl\{\sigma_{i,j} \mid \sigma_{i,j}(c) = c^i, \sigma_{i,j}(r) = rc^j, i, j \in \{0, 1, \dots, n-1\}, \gcd(i, n) = 1\bigr\}.$$

Let $\ell \in \{0, 1, \dots, n-1\}$, $\gcd(\ell, n) = 1$, and let k be the smallest positive integer such that $1 + \ell + \dots + \ell^{k-1} \equiv 0 \pmod n$. (The number k is in fact equal to the size of the orbit containing 0 for the permutation $x \mapsto \ell x + 1$ of $\mathbb{Z}_n$.) Let

$$S = \bigl\{r, rc, rc^{1+\ell}, \dots, rc^{1+\ell+\dots+\ell^{k-2}}\bigr\}, \quad \text{and}$$
$$p = \bigl(r, rc, rc^{1+\ell}, \dots, rc^{1+\ell+\dots+\ell^{k-2}}\bigr).$$

Consider the automorphism $\sigma_{\ell,1} \in \mathsf{Aut}(D_{2n})$. The restriction of $\sigma_{\ell,1}$ to S is p, hence $\mathsf{CM}(D_{2n}, S, p)$ is a balanced regular Cayley map. It was proved in [24] that any regular balanced Cayley map over D_{2n} can be described as a Cayley map $\mathsf{CM}(D_{2n}, S, p)$ for some ℓ with $\gcd(\ell, n) = 1$. Moreover, two such dihedral maps arising from some ℓ_1 and ℓ_2 such that $\gcd(\ell_1, n) = 1$ and $\gcd(\ell_2, n) = 1$ are isomorphic if and only if $\ell_1 = \ell_2$, therefore, the number of non-isomorphic regular balanced Cayley maps over D_{2n} is equal to $\varphi(n)$, where φ is Euler's totient function.

In the rest of the section we turn to regular non-balanced dihedral maps.

Example 3.1 (Non-balanced, regular Cayley maps $\mathsf{CM}(n, \ell)$ over D_{2n}) Let $\mathfrak{T}$ be the set of all pairs (n, ℓ) of positive integers satisfying the following conditions:

- n is an odd number, $n \equiv 0 \pmod 3$, and
- ℓ is an element in $\mathbb{Z}_n^*$ of odd order m.

For each pair (n, ℓ) in $\mathfrak{T}$ we define a Cayley map $\mathsf{CM}(D_{2n}, S, p)$, which we also denote by $\mathsf{CM}(n, \ell)$, as follows:

$$S = \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \dots, m-1\}\}, \quad \text{and}$$
$$p = (c, rc^{-\ell}, rc^{\ell^2}, c^{-\ell^3}, \dots, c^{\ell^{4m-4}}, rc^{-\ell^{4m-3}}, rc^{\ell^{4m-2}}, c^{-\ell^{4m-1}}).$$

We claim that $\mathsf{CM}(n, \ell)$ is regular and non-balanced. Let $\sigma = \sigma_{\ell,0}$ in $\mathsf{Aut}(D_{2n})$. Then $S^\sigma = S$, thus we find that $\sigma \in \mathsf{Aut}(\mathsf{Cay}(D_{2n}, S))$. For $i \in \{0, 1, 2\}$ define the permutation $\mu_i \in \mathsf{Sym}(D_{2n})$ as

$$(r^x c^y)^{\mu_i} = \begin{cases} r^x c^y & \text{if } y \equiv i \pmod 3, \\ r^{x+1} c^y & \text{otherwise.} \end{cases} \tag{2}$$

Then $N = \{\mathrm{id}, \mu_0, \mu_1, \mu_2\}$ is a subgroup of $\mathsf{Sym}(D_{2n})$, $N \cong \mathbb{Z}_2^2$. It is easily seen that $\mu_i \in \mathsf{Aut}(\mathsf{Cay}(D_{2n}, S))$ for all $i \in \{1, 2, 3\}$. The group N is normalized by all $d_* \in D_*$, and it is centralized by σ. Moreover, $N \cap (D_* \rtimes \langle\sigma\rangle) = 1$, and we can form the semidirect product $G = N \rtimes (D_* \rtimes \langle\sigma\rangle)$. This is a subgroup $G \le \mathsf{Aut}(\mathsf{Cay}(D_{2n}, S))$, and acts transitively on the set of arcs of $\mathsf{Cay}(D_{2n}, S)$. The stabilizer

$$G_1 = \langle \mu_1 r_*, \sigma \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_m \cong \mathbb{Z}_{4m},$$

hence the element $\psi = \mu_1 r_* \sigma$ is a generator of G_1. Then $G = (D_{2n})_* \langle\psi\rangle$, and so ψ is a skew-morphism of D_{2n}. The restriction of ψ on S is equal to p, hence $\mathsf{CM}(n, \ell)$ is regular. It is non-balanced too. Otherwise $C_* \trianglelefteq G$, implying that $S \cap \langle c \rangle = \emptyset$, which is not the case.

The main result of this section is the following theorem.

Theorem 3.2 *Let $\mathcal{M}$ be a regular, non-balanced Cayley map over a dihedral group D_{2n} such that n is odd. Then*

(i) *$\mathcal{M}$ is isomorphic to a map $\mathsf{CM}(n, \ell)$ for some $(n, \ell) \in \mathcal{T}$.*
(ii) *For any two pairs $(n, \ell_1), (n, \ell_2) \in \mathcal{T}$, the maps $\mathsf{CM}(n, \ell_1)$ and $\mathsf{CM}(n, \ell_2)$ are isomorphic if and only if $\ell_1 = \ell_2$.*

The following corollary is immediate. For a positive integer x denote $x_{\mathbf{2}}$ the largest 2-power that divides x, that is, $x_{\mathbf{2}} = 2^e$ such that $2^e \mid x$ and $2^{e+1} \nmid x$.

Corollary 3.3 *The number of non-isomorphic regular Cayley maps over the dihedral group D_{2n} with odd n, is equal to*

$$\begin{cases} \varphi(n) & \text{if } n \not\equiv 0 \pmod 3, \\ \varphi(n)(1 + \frac{1}{\varphi(n)_2}) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where φ denotes Euler's totient function.

Proof By the above discussion the number of non-isomorphic balanced regular Cayley maps over D_{2n} is equal to $\varphi(n)$. If there also exist non-balanced ones, then n must be divisible by 3, and in this case the number of non-isomorphic non-balanced regular Cayley maps over D_{2n} is the same as the number of elements ℓ in $\mathbb{Z}_n^*$ that have odd order. Let n have prime decomposition $n = p_1^{e_1} \cdots p_k^{e_k}$. Since n is odd, the group $\mathbb{Z}_n^*$ is Abelian and it can be written as $\mathbb{Z}_n^* = C_{p_1^{e_1-1}(p_1-1)} \cdots C_{p_k^{e_k-1}(p_k-1)}$ for pairwise trivially intersecting cyclic subgroups $C_{p_i^{e_i-1}(p_i-1)} \leq \mathbb{Z}_n^*$. Therefore, any $\ell \in \mathbb{Z}_n^*$ is written uniquely in the form $\ell = \ell_1 \cdots \ell_k$, and $\ell_i \in C_{p_i^{e_i-1}(p_i-1)}$ for all $i \in \{1, \dots, k\}$. The order of ℓ is odd if and only if the order of ℓ_i is odd for all $i \in \{1, \dots, k\}$. The number of elements of the cyclic subgroup $C_{p_i^{e_i-1}(p_i-1)}$ of odd order is

$$\sum_{d \mid p_i^{e_i-1}(p_i-1)/(p_i^{e_i-1}(p_i-1))_{\mathbf{2}}} \varphi(d) = \frac{p_i^{e_i-1}(p_i-1)}{(p_i^{e_i-1}(p_i-1))_{\mathbf{2}}}.$$

Thus the required number is

$$\varphi(n) + \prod_{i=1}^{k} \frac{p_i^{e_i-1}(p_i-1)}{(p_i^{e_i-1}(p_i-1))_{\mathbf{2}}} = \varphi(n) + \varphi(n)/\varphi(n)_{\mathbf{2}}.$$

□

Theorem 3.2 is proved after a series of preparatory steps. We start with a property that holds for arbitrary G-arc-regular graphs whose stabilizer subgroups are Hamiltonian. Recall that an imprimitivity system $\mathcal{B}$ of G is normal if the kernel $G_{\mathcal{B}}$ acts transitively on every block in $\mathcal{B}$.

Proposition 3.4 *Let Γ be a G-arc-regular undirected graph, and $\mathcal{B}$ be a normal non-trivial imprimitivity system of G. If the stabilizer G_v is a Hamiltonian group for $v \in V(\Gamma)$, then the quotient graph $\Gamma/\mathcal{B}$ is $G^{\mathcal{B}}$-arc-regular.*

Proof Let $(v_1, v_2) \in A(\Gamma)$ be an arbitrary arc of Γ and let t be the unique involution inverting this arc. By Lemma 2.4(iv), $\langle G_{v_i}, t\rangle = G$. The orbits $V_i = v_i^{G_{\mathcal{B}}}$, $i = 1, 2$, are blocks of $\mathcal{B}$ which are connected by at least one arc in Γ. Therefore, (V_1, V_2) is an arc of the quotient graph $\Gamma/\mathcal{B}$. Its stabilizer in $G^{\mathcal{B}}$ coincides with $G_{\{V_1,V_2\}}/G_{\mathcal{B}}$, where $G_{\mathcal{B}}$ is the kernel of G acting on $\mathcal{B}$, and $G_{\{V_1,V_2\}} = G_{\{V_1\}} \cap G_{\{V_2\}}$, $G_{\{V_i\}}$ is the setwise stabilizer of V_i in G ($i = 1, 2$).

Since $G_{\mathcal{B}}$ acts transitively on V_i, we obtain $G_{\{V_i\}} = G_{v_i} G_{\mathcal{B}}$. Since $v_i^t = v_{3-i}$, $i = 1, 2$, we obtain $G_{\{V_i\}} = G^t_{\{V_{3-i}\}}$ implying $G^t_{\{V_1,V_2\}} = G_{\{V_1,V_2\}}$. Since $G_{\mathcal{B}} \leq G_{\{V_1,V_2\}} \leq G_{v_i} G_{\mathcal{B}}$, we can write $G_{\{V_1,V_2\}} = (G_{\{V_1,V_2\}} \cap G_{v_i}) G_{\mathcal{B}}$. Since G_{v_i} is Hamiltonian, the intersection $(G_{\{V_1,V_2\}} \cap G_{v_i})$ is normalized by G_{v_i}, Therefore the group $(G_{\{V_1,V_2\}} \cap G_{v_i}) G_{\mathcal{B}}$ is also normalized by G_{v_i}. Thus the subgroup $G_{\{V_1,V_2\}} = (G_{\{V_1,V_2\}} \cap G_{v_i}) G_{\mathcal{B}}$ is normalized by G_{v_i} and t. Since $\langle G_{v_i}, t\rangle = G$, we conclude that $G_{\{V_1,V_2\}}$ is normal in G. Therefore $G_{\{V_1,V_2\}} \leq G_{\mathcal{B}}$ implying $G_{\{V_1,V_2\}} = G_{\mathcal{B}}$. Hence the arc stabilizer in the quotient graph is trivial. □

Let $\mathcal{M} = (\Gamma; R, T)$ be a regular map with underlying graph Γ. Suppose that $\mathcal{B}$ is a non-trivial normal imprimitivity system of $\mathsf{Aut}(\Gamma)$. Observe that $\mathsf{Aut}(\Gamma)_v$ is a cyclic group, hence Proposition 3.4 is applicable. Let $\sim$ be the relation on the arc set $A(\Gamma)$ defined by $(u_1, u_2) \sim (v_1, v_2)$ if and only if u_i, v_i are in the same block of $\mathcal{B}$ for both $i = 1, 2$. In other words, the classes of $\sim$ correspond to the arcs of the quotient graph $\Gamma/\mathcal{B}$. Let $(u_1, u_2) \sim (v_1, v_2)$, and let u_i, v_i be in the blocks $B_i \in \mathcal{B}$ $(i = 1, 2)$. Since $\mathcal{M}$ is regular, $(u_1, v_1)^g = (u_2, v_2)$ for some $g \in \mathsf{Aut}(\mathcal{M})$. Then $g \in \mathsf{Aut}(\mathcal{M})_{\{B_1, B_2\}}$, and Proposition 3.4 gives the result that $g \in \mathsf{Aut}(\mathcal{M})_{\mathcal{B}}$. By definition $Rg = gR$, and we find $(u_2, v_2)^R = (u_1, v_1)^{gR} = (u_1, v_1)^{Rg} = ((u_1, v_1)^R)^g$, so that $(u_1, v_1)^R \sim (u_2, v_2)^R$. The relation $\sim$ is R-invariant. Denote $R^{\mathcal{B}}$ the permutation of the arc set $A(\Gamma/\mathcal{B})$ induced by the action of R. The *quotient map* $\mathcal{M}/\mathcal{B}$ is defined as

$$\mathcal{M}/\mathcal{B} = \big(\Gamma/\mathcal{B}; R^{\mathcal{B}}, T^{\mathcal{B}}\big),$$

where $T^{\mathcal{B}}$ is the dart-reversing involution of $\mathcal{M}/\mathcal{B}$ switching the arcs of $\Gamma/\mathcal{B}$. The following statement is a consequence of Proposition 3.4.

Corollary 3.5 *Let $\mathcal{M} = (\Gamma; R, T)$ be a regular map, and $\mathcal{B}$ be a non-trivial, normal imprimitivity system of* $\mathsf{Aut}(\mathcal{M})$. *Then the quotient map $\mathcal{M}/\mathcal{B}$ is regular, and* $\mathsf{Aut}(\mathcal{M}/\mathcal{B}) = \mathsf{Aut}(\mathcal{M})^{\mathcal{B}}$.

In the next two lemmas we make some observations about the subset S such that $\mathsf{CM}(D_{2n}, S, p)$ is a regular map and n is odd.

Lemma 3.6 *Let $\mathcal{M} = \mathsf{CM}(D_{2n}, S, p)$ be a regular Cayley map such that n is odd. Then $\mathcal{M}$ is balanced if and only if $S \cap C_n = \emptyset$.*

Proof We set $D = D_{2n}$, $C = C_{2n}$, and $G = \mathsf{Aut}(\mathcal{M})$. If D_* is normal in G (that is, $\mathcal{M}$ is balanced), then S is an orbit of some subgroup of $\mathsf{Aut}(D)$, and, therefore, all elements of S have the same order. Since $S \setminus C \neq \emptyset$, we conclude $S \subseteq D \setminus C$.

Assume now that $S \subseteq D \setminus C$. By the proof of Lemma 2.5 $|C_G(s_*)| = 2|S|$ for any $s \in S$. Therefore $|(s_*)^G| = n$. But $|(s_*)^{D_*}| = n$. Therefore $(s_*)^G = (s_*)^{D_*}$ implying that $D_* = \langle (s_*)^{D_*} \rangle \trianglelefteq G$. □

Lemma 3.7 *Let $\mathcal{M} = \mathsf{CM}(D_{2n}, S, p)$ be a regular, non-balanced Cayley map such that n is odd. Then $|S \cap C_n| = |S|/2$.*

Proof We set $D = D_{2n}$, $C = C_n$, $G = \mathsf{Aut}(\mathcal{M})$, and further that $S_0 = S \cap C$. We prove the lemma by induction on n. If $n = 3$, then it follows directly that $S = D \setminus \{1, r\}$ for some $r \in D \setminus C$. Let $n > 3$. Let $\mathcal{B}$ be the (unique) maximal imprimitivity system of G of cyclic type. Let $B \in \mathcal{B}$ be a block containing 1 (that is, $B \le C$). Then the blocks of $\mathcal{B}$ are orbits of B_* implying that $\mathcal{B}$ is normal. Assume that $\mathcal{B}$ is trivial. Then $\mathrm{core}_G(C) = 1$ and $n = 3$ by Theorem 2.8, which is a contradiction. Thus $\mathcal{B}$ is non-trivial.

Apply the induction hypothesis to the quotient map $\mathcal{M}/\mathcal{B}$. This is a regular Cayley map over the dihedral group D/B. This results in $|(SB/B)_0|/|(SB/B)| = 1/2$. Since $(SB/B) \cap C/B = (SB \cap C)/B$, we can write $|(SB/B)_0| = |(SB)_0|/|B|$. But $SB \cap C = (S \cap C)B = S_0B$. Since $|sB \cap S|$ does not depend on a choice of $s \in S$, we obtain $|S_0B| = |S_0||B|/|sB \cap S|$ implying $|(SB/B)_0| = |S_0|/|sB \cap S|$. Analogously, $|SB/B| = |S|/|sB \cap S|$. Taking the ratio we obtain $|S_0|/|S| = |(SB/B)_0|/|SB/B| = 1/2$. □

A crucial observation in our proof of Theorem 3.2 is that if $\mathcal{M}$ is a regular, non-balanced dihedral map over D_{2n} such that n is odd, then $\mathsf{Aut}(\mathcal{M})$ always admits an imprimitivity system with block size 2. This fact we prove in Lemma 3.9, but before that discuss some of its consequences. Recall that if $x \in \mathsf{Aut}(\mathcal{M})$ (acting on D_{2n}), then we write $\mathsf{Fix}(x) = \{d \in D_{2n} \mid d^x = d\}$.

Lemma 3.8 *Let $\mathcal{M} = \mathsf{CM}(D_{2n}, S, p)$ be a regular, non-balanced Cayley map such that n is odd, and suppose that $\mathcal{B}$ is an imprimitivity system of $\mathsf{Aut}(\mathcal{M})$ with block size 2. Then*

(i) *The kernel $\mathsf{Aut}(\mathcal{M})_{\mathcal{B}}$ is isomorphic to $\mathbb{Z}_2^2$.*
(ii) *$|\mathsf{Fix}(x)| = 2n/3$ for all $x \in \mathsf{Aut}(\mathcal{M})_{\mathcal{B}}$ with $x \neq 1$. In particular, $n \equiv 0 \pmod 3$.*
(iii) *The non-trivial elements in $\mathsf{Aut}(\mathcal{M})_{\mathcal{B}}$ form a conjugacy class of G.*

Proof We set $D = D_{2n}$, $C = C_n$, $G = \mathsf{Aut}(\mathcal{M})$, and $N = G_{\mathcal{B}}$. The graph $\mathsf{Cay}(D, S)$ is G-arc-regular, thus $N \cong \mathbb{Z}_2^{\ell}$, $\ell \le 2$. We may assume that $\mathcal{B}$ consists of the right cosets $\{1, r\}d$, $d \in D$.

(i) Consider the action of G on the blocks of $\mathcal{B}$. We may choose $\mathcal{B}$ so that $B = \{1, r\}$ is a block in $\mathcal{B}$. Let y denote a generating element of the stabilizer G_1. Then $G_{\{B\}} = G_1\langle r_*\rangle = \langle y\rangle\langle r_*\rangle$ and $\langle y\rangle$ is normal in $G_{\{B\}}$ as it is an index 2 subgroup. Since r_* fixes setwise a unique block of $\mathcal{B}$, namely B, each element commuting with r_* fixes B too. Therefore $C_G(r_*) \le G_{\{B\}}$. It follows from $|C_G(r_*)| = |S|$, $|G_{\{B\}}| = 2|S|$ and $G_{\{B\}} = \langle y\rangle\langle r_*\rangle$ that $C_{\langle y\rangle}(r_*) = \langle y^2\rangle$. In particular, $G_{\{B\}}$ is non-Abelian.

Let G^+ be the intersection of G with the alternating group $\mathsf{Alt}(D)$ in $\mathsf{Sym}(D)$. Since G contains odd permutations (those belonging to $D_* \setminus C_*$), we conclude $[G : G^+] = 2$. Since $C_* \le \mathsf{Alt}(D)$, the group G^+ either has two orbits on D (the two orbits of C_*) or acts transitively. In the first case the orbits of C_* form an imprimitivity system of G contrary to $S \cap C \neq \emptyset$. Hence G^+ acts transitively on D which, in turn, implies that $[G_1 : G_1^+] = 2$, or, equivalently, $|G^+| = |S|n$. It follows from $[G_1 : G_1^+] = 2$ that y is an odd permutation. Since y^2 and r_*y are even permutations, $\langle y^2, r_*y\rangle \le G_{\{B\}}^+$. By the fact that $r_*y \notin \langle y\rangle$, we get that $|\langle y^2, r_*y\rangle| \ge |S| = |G_{\{B\}}^+|$, and thus $G_{\{B\}}^+ = \langle y^2, r_*y\rangle$. The element r_* centralizes y^2, hence the group $\langle y^2, r_*\rangle = G_{\{B\}}^+$ is Abelian, and G^+ is a product of two Abelian groups: $G_{\{B\}}^+$ and C_*. Thus by Ito's Theorem $[G^+, G^+]$ is Abelian.

Assume first that $[G^+, G^+]$ has odd order. Then its orbits on D form an imprimitivity system, say $\mathcal{C}$, with blocks of odd cardinality. The quotient $G^+/[G^+, G^+]$ is an Abelian group acting transitively on $\mathcal{C}$. Hence $(G^+)^{\mathcal{C}}$ is a regular Abelian group of order $2k$ for some odd k. The subgroup K of $(G^+)^{\mathcal{C}}$ of order k has two orbits on $\mathcal{C}$ of size k. K is characteristic in $(G^+)^{\mathcal{C}}$, and hence it is normal in $G^{\mathcal{C}}$. Hence its preimage is normal in G and has two orbits on D of cardinality n. But this contradicts $S \cap C \neq \emptyset$.

Thus we may assume that a Sylow 2-subgroup P of $[G^+, G^+]$ is non-trivial. Since $[G^+, G^+]$ is characteristic in G^+, it is normal in G. This implies that P is normal in

G, and so $P \le N$. We conclude that $N \ne 1$, implying that N is transitive on B, and thus $G_{\{B\}} = G_1 N$. If $N \cong \mathbb{Z}_2$, then N is in the center of G. But then $G_{\{B\}}$ is Abelian, a contradiction. Thus $N \cong \mathbb{Z}_2^2$, and (i) is proved.

(ii) Since $N \trianglelefteq G$, the subset $\mathsf{Fix}(N_1) = \bigcap_{x \in N_1} \mathsf{Fix}(x)$ is a block of G (see [9, Proposition 5.2]). Obviously, $B \subseteq \mathsf{Fix}(N_1)$. Hence $|\mathsf{Fix}(N_1)|$ is even, implying that the number of the blocks in the imprimitivity system $\mathsf{Fix}(N_1)^G$ is odd. Since there are three subgroups of N of index 2, the number of blocks in $\mathsf{Fix}(N_1)^G$ is at most 3. But $\mathsf{Fix}(N_1)^G$ contains more than one block, because N is not semi-regular on D, and (ii) follows.

(iii) This follows directly from (ii). □

Lemma 3.9 *Let $\mathcal{M} = \mathsf{CM}(D_{2n}, S, p)$ be a regular, non-balanced Cayley map such that n is odd. Then there exists an imprimitivity system of $\mathsf{Aut}(\mathcal{M})$ with block size* 2.

Proof Towards a contradiction let us choose $\mathcal{M}$ as a counter example which is minimal relative to the order $2n$ of the corresponding dihedral group D_{2n}. We set $D = D_{2n}$, $C = C_n$, and $G = \mathsf{Aut}(\mathcal{M})$. Then $\mathbf{O}_2(G) = 1$.

Let K be a minimal normal subgroup of G. By Huppert–Ito Theorem (see [5]) the group $G = D_* G_1$ is solvable. We obtain $K \cong \mathbb{Z}_q^\ell$ for some odd prime q. The orbits of K form an imprimitivity system $\mathcal{K}$, the block size of which is a power of q. Since the block size of $\mathcal{K}$ is odd, $\mathcal{K}$ consists of the right cosets of a q-subgroup $Q \le C$. By [19, Proposition 3.2] we conclude that $|Q| = q$. Clearly, $Q_* \le K$. By the regularity of G on the set of arcs we obtain either $K = Q_*$, or $K \cong \mathbb{Z}_q^2$. Factoring out by the imprimitivity system $\mathcal{K}$ we obtain the dihedral map $\mathcal{M}/\mathcal{K}$ over the group D/Q. The group $\mathsf{Aut}(\mathcal{M}/\mathcal{K}) \cong G/L$, where $L = G_{\mathcal{K}}$. By the minimality of $\mathcal{M}$ the permutation group $\mathsf{Aut}(\mathcal{M}/\mathcal{K})$ has an imprimitivity system with block size two. By Lemma 3.8, the group G/L contains an elementary Abelian normal subgroup of order 4, say $M/L \cong \mathbb{Z}_2^2$. Moreover, since the involutions of M/L form a single conjugacy class of G/L, the subgroup M/L is a minimal normal subgroup of G/L.

Since $Q_* \le L$ and Q_* acts transitively on each block of $\mathcal{K}$, we obtain $L = L_1 Q_*$. By the regularity of G on the set of arcs we conclude that $|L_1| \le q$.

Consider first the case that $|L_1| = q$. In this case $L \cong \mathbb{Z}_q^2$, and, therefore, $K = L$. Since $M/L \cong \mathbb{Z}_2^2$, we may write $M = L \,.\, N$, where $N \cong \mathbb{Z}_2^2$. It follows from $\mathbf{O}_2(G) = 1$ that $\mathbf{O}_2(M) = 1$. Therefore the kernel of the action of N on L is trivial, or equivalently N acts faithfully on L. In any faithful action of $\mathbb{Z}_2^2$ on $\mathbb{Z}_q^2$ two involutions of $\mathbb{Z}_2^2$ centralize q elements while the third one centralizes only the identity. Let us denote the first two by a, b, and the third one by c. Thus $N = \{1, a, b, c\}$, and $|C_L(a)| = |C_L(b)| = q$, $|C_L(c)| = 1$. Then $|C_M(a)| = |C_M(b)| = 4q$, $|C_M(c)| = 4$, and, consequently $|a^M| = q$, $|b^M| = q$, $|c^M| = q^2$. By Sylow's Theorems each involution of M is M-conjugate to one of a, b, c. Therefore a^M, b^M, c^M are the only conjugacy classes of involutions contained in M. Since M is normal in G, G permutes these conjugacy classes. But c^M is the only one among the three which has cardinality q^2. Therefore $(c^M)^G = c^M$, or equivalently, $c^G = c^M$. Therefore $\langle c^G \rangle = \langle c^M \rangle = L\langle c \rangle \trianglelefteq G$. But in this case $\langle c \rangle L/L$ is a normal subgroup of G/L contrary to minimality of M/L in G/L.

Consider now the remaining case $|L_1| < q$. In this case $K = Q_*$ implying that $Q_* \trianglelefteq G$. It follows from $|M| = 4|L| = 4q|L_1|$ that Q_* is a unique Sylow q-subgroup in M. Since Q_* and M are normal in G, the subgroup $C_M(Q_*)$ is normal in G as well. Since $C_L(Q_*)$ is a q-group, we conclude that $C_L(Q_*) = Q_*$. This implies that $C_M(Q_*) \cap L = Q_*$, and therefore, the quotient group $C_M(Q_*)/Q_* = C_M(Q_*)/(C_M(Q_*) \cap L) \cong C_M(Q_*)L/L$ is embedded into $M/L \cong \mathbb{Z}_2^2$. Thus $C_M(Q_*) \cong \mathbb{Z}_q \times \mathbb{Z}_2^e$ with $e \le 2$. It follows from $C_M(Q_*) \trianglelefteq G$ that the subgroup $\mathbf{O}_2(C_M(Q_*)) \cong \mathbb{Z}_2^e$ is contained in $\mathbf{O}_2(G) = 1$. Therefore $e = 0$ and $C_M(Q_*) = Q_*$. In this case the quotient group M/Q_* is embedded into $\mathsf{Aut}(\mathbb{Z}_q)$, and hence it is cyclic. But $(M/Q_*)/(L/Q_*) \cong M/L \cong \mathbb{Z}_2^2$, a contradiction. $\square$

Proof of Theorem 3.2 Let $\mathcal{M} = \mathsf{CM}(D_{2n}, S, p)$ be a regular, non-balanced Cayley map over the dihedral group D_{2n} such that n is odd. We set $D = D_{2n}$, $C = C_{2n}$ and $G = \mathsf{Aut}(\mathcal{M})$. Let y be a generating element of the stabilizer G_1. Lemma 3.7 gives $|S \cap C| = |S|/2$. Also, $|S \cap C|$ is even because of $(S \cap C)^{-1} = S \cap C$, hence 4 divides $|S|$. Let $|S| = 4m$.

By Lemma 3.9, G has an imprimitivity system $\mathcal{B}$ of block size 2. Up to equivalence of $\mathcal{M}$, we may assume that $\mathcal{B}$ consists of the cosets $\{1, r\}d$, $d \in D$. Let $N = G_{\mathcal{B}}$. By Lemma 3.8(i), $N \cong \mathbb{Z}_2^2$. Observe that Lemma 3.8(ii) implies that $N = \{\mu_0, \mu_1, \mu_2, 1\}$, where the permutations μ_i are defined in (2).

The group D_* acts on $N \setminus \{1\}$ either trivially or as S_2 or S_3. In the first two cases there exists a non-trivial element of N, μ_i say, centralized by D_*. Since D_* is transitive on D, μ_i is fixed-point-free. But this contradicts Lemma 3.8(ii). Hence D_* acts on $N \setminus \{1\}$ as S_3. Since the elements of D_* are fixed-point-free, the intersection $N \cap D_* = 1$. Therefore, $ND_* \cong \mathbb{Z}_2^2 \rtimes D$, and $|ND_*| = 8n$.

Since D_* acts on $N \setminus \{1\}$ as S_3, each element of $D_* \setminus C_*$ centralizes one involution in N. Therefore the centralizer of each element $s_* \in D_* \setminus C_*$ in ND_* has order 4, implying $|(s_*)^{ND_*}| = 2n$. Therefore $T = (s_*)^G = (s_*)^{ND_*}$, and the subgroup $\langle (s_*)^{ND_*} \rangle = ND_*$ is normal in G. Since both N and ND_* are normal in G, the subgroup $C_{ND_*}(N)$ is normal in G too. The group D_* acts on N as $\mathsf{Aut}\, N \cong S_3$. Hence $C_{ND_*}(N) = NC_*^3$ (here C_*^3 is the unique subgroup of C_* of index 3). Thus $NC_*^3 \trianglelefteq G$. Since NC_*^3 is Abelian, the subgroup $(NC_*^3)^2 = C_*^3$ is characteristic in NC_*^3 and, therefore, $C_*^3 \trianglelefteq G$.

Consider now the factor group $\overline{G} = G/C_*^3$. Then $\overline{N} \cong N \cong \mathbb{Z}_2^2$, $\overline{D_*} \cong D_6 \cong S_3$, and $\langle \overline{y} \rangle \cong \langle y \rangle$. Since $\overline{D_*} \cong S_3$ acts on $\overline{N}$ faithfully, we obtain $\overline{ND_*} \cong S_4$. Since $\mathsf{Aut}(S_4) = \mathsf{Inn}(S_4)$, the group $\overline{G}$ is isomorphic to a direct sum $S_4 \times K$, where $K = C_{\overline{G}}(\overline{ND_*})$. It follows from $G = \langle y \rangle D_*$ that $\overline{G} = \langle \overline{y}, \overline{D_*} \rangle$ with $\langle \overline{y} \rangle \cap \overline{D_*} = 1$. Thus $|\langle \overline{y} \rangle \cap \overline{ND_*}| = 4$ and $K \cong \overline{G}/\overline{ND_*} \cong \mathbb{Z}_m$. Thus $\overline{G} \cong S_4 \times \mathbb{Z}_m$. The group $S_4 \times \mathbb{Z}_m$ contains an element of order $4m$ if and only if m is odd. Hence m is odd and $\overline{G} = \overline{ND_*} \times \langle \overline{y}^4 \rangle$.

The subgroup $\overline{C_*}$ has order 3 and is contained in $\overline{ND_*} \cong S_4$. Therefore its normalizer in $\overline{G}$ is $\overline{D_*} \times \langle \overline{y}^4 \rangle$. Since C_*^3 is the kernel of the homomorphism $G \to \overline{G}$ and $C_*^3 < D_*$, we conclude that $N_G(C_*) = D_* \langle y^4 \rangle$ (together with $N_{ND_*}(D_*) = D_*$ we obtain $G = N \rtimes N_G(C_*)$). It is proved that $N_{\mathsf{Sym}(D)}(C_*) = N_{\mathsf{Sym}(D)}(D_*)$ (see [9, Proposition 6.2]). Hence y^4 is an automorphism of D_{2n} of order m such that y^4 centralizes r_*. Therefore we get $y^4 = \sigma_{\ell,0}$ such that $(n, \ell) \in \mathfrak{T}$. It follows that

$y = \mu_1 r_* \sigma_{\ell,0}$, and G is as described in Example 3.1. Now S is an orbit of $\langle y\rangle$ such that S generates D_{2n}. Then $S \cap C$ contains a generator of C, implying that S has a conjugate

$$S^{\sigma_{\ell',0}} = \left\{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \;\middle|\; i \in \{0, \dots, m-1\}\right\},$$

where $\sigma_{\ell',0}$ is in $\mathsf{Aut}(D_{2n})$. Therefore $\mathcal{M}$ is equivalent to the Cayley map $\mathsf{CM}(n,\ell)$, and (i) follows.

For two pairs $(n,\ell_1), (n,\ell_1) \in \mathfrak{T}$ suppose that the two Cayley maps $\mathsf{CM}(n,\ell_1) = \mathsf{CM}(D_{2n}, S, p)$ and $\mathsf{CM}(n,\ell_2) = \mathsf{CM}(D_{2n}, T, q)$ are isomorphic. Let us write $p = (s_1, \dots, s_{4m})$ and $q = (t_1, \dots, t_{4m})$ so that $s_1 = t_1 = c$. The two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same *rotation type*, that is, satisfy the property that $s_i^{-1} = s_i^{p^j}$ if and only if $t_i^{-1} = t_i^{q^j}$ for all $i, j \in \{1, 2, \dots, 4m\}$. It is proved that in this case $\mathcal{M}_1$ and $\mathcal{M}_2$ are not only isomorphic but also equivalent (see [12, Lemma 2.4]). Therefore, there is a group automorphism $\sigma_{i,j} \in \mathsf{Aut}(D_{2n})$ which maps S to T and $p\sigma_{i,j} = \sigma_{i,j} q$. There exists an automorphism $\sigma \in \langle \sigma_{\ell_2,0}\rangle$ such that $\sigma_{i,j}\sigma$ maps c to $c^{\pm 1}$. The restriction of σ on T commutes with q, hence $p(\sigma_{i,j}\sigma) = (\sigma_{ij} q)\sigma = (\sigma_{i,j}\sigma)q$. Put $\sigma' = \sigma_{i,j}\sigma$. Assume that σ' maps c to c^{-1}. Then $(rc^{-\ell_1})^{\sigma'} = c^{p\sigma'} = c^{\sigma' q} = (c^{-1})^q = c^{\ell_2}$, a contradiction. Let σ' map c to c. Then $(rc)^{\sigma'} = c^{p^{2m}\sigma'} = c^{\sigma' q^{2m}} = rc$, and we find that σ' is the identity. Therefore $rc^{-\ell_1} = (rc^{-\ell_1})^{\sigma'} = c^{p\sigma'} = c^{\sigma' q} = rc^{-\ell_2}$, implying $\ell_1 = \ell_2$, and this proves (ii). □

4 Corrigendum to [9, Sect. 4]

As we have already mentioned in Sect. 2 of this paper, Theorem 4.3 of [9] contains a mistake. The purpose of this section is to correct this statement and the related ones in Sect. 4 of the aforementioned paper. The list of corrections is given below.

Lemma 4.2: the condition $\mathrm{core}_G(C) = 1$ should be replaced by $\mathrm{core}_G(D) = 1$, and line 9, p. 417 "$c^{n/2} \in \mathrm{core}_G(C)$" should be replaced by "$c^{n/2} \in \mathrm{core}_G(D)$". The rest of the proof is correct.

Theorem 4.3: the condition $n > 2$ should be replaced by $n > 4$. Add the following argument after the first sentence of the proof of Theorem 4.3 (line -2, p. 417). "Let us show first that $\mathrm{core}_G(D)$ is trivial. The commutator $(\mathrm{core}_G(D))'$ is a normal subgroup of G contained in C, and, therefore, it is trivial. Hence $\mathrm{core}_G(D)$ is Abelian and normal in D. But any Abelian normal subgroup of D is contained in C unless $|D| = 4, 8$. Now the assumption $n > 4$ yields that $\mathrm{core}_G(D)$ is contained in C, and, therefore, is trivial." After that the proof goes in the same way till line 16 on p. 418. In this line the text "Since n is divisible by at least three primes, $n/q > 2$," should be replaced by "Since n is divisible by at least three distinct primes, $n/q > 4$,".

Corollary 4.4 remains correct, but the proof should be replaced by the following one. "If $m \le 4$, then the claim follows from $[G : \mathrm{core}_G(D)] \le m! \le 2m^2$. Thus we may assume $m > 4$. Consider the quotient group G/N where $N = \mathrm{core}_G(D)$. Then $\mathrm{core}_{G/N}(D/N)$ is trivial. If $|D/N| \le 2m$, then we are done. If $|D/N| > 2m$, then by Theorem 4.3 $\mathrm{core}_{G/N}(C/N)$ is non-trivial, which contradicts $\mathrm{core}_{G/N}(D/N) = 1$."

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