ON G-REGULAR LOCAL RINGS

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Abstract. In this paper, we define a G-regular local ring as a commutative, noetherian, local ring over which all totally reflexive modules are free. We study G-regular local rings, and observe that they behave similarly to regular local rings. We extend Eisenbud's matrix factorization theorem and Knörrer's periodicity theorem to G-regular local rings.

Introduction

In the 1960s, Auslander [1] introduced a homological invariant for finitely generated modules over a noetherian ring which is called Gorenstein dimension, or G-dimension for short. After that, he developed the theory of G-dimension with Bridger [2]. G-dimension has been studied deeply from various points of view; details can be found in [2] and [8].

Modules of G-dimension zero are called totally reflexive modules. Any finitely generated projective module is totally reflexive. Over a Gorenstein local ring, the totally reflexive modules are precisely the maximal Cohen-Macaulay modules. Therefore, every singular Gorenstein local ring has a nonfree totally reflexive module.

In the present paper, we will call a commutative noetherian local ring *G-regular* if every totally reflexive module over the ring is free. Regular local rings are trivial examples of G-regular local rings. Avramov and Martsinkovsky [5, Examples 3.5] proved that any Golod local ring that is not a hypersurface (e.g. a Cohen-Macaulay non-Gorenstein local ring with minimal multiplicity [3, Example 5.2.8]) is G-regular. Yoshino [22, Theorem 3.1] gives some sufficient conditions for an artinian local ring of Loewy length three to be G-regular. Takahashi and Watanabe [19, Theorem 1.1] showed that there exist two-dimensional, non-G-regular, non-Gorenstein normal domains. A recent result due to Christensen, Piepmeyer, Striuli and Takahashi [9, Theorem B] says that every non-Gorenstein local ring over which there exist only finitely many isomorphism classes of indecomposable totally reflexive modules is a G-regular ring. The same result in special cases and similar results were earlier shown in [13]–[18].

In this paper we find that G-regular local rings behave similarly to regular local rings. We give two theorems, stated below, as the main results of this paper. The first is a generalization of Eisenbud's matrix factorization theorem [10, Section 6] (cf. [21, Theorem (7.4)]), and the second is a generalization of Knörrer's periodicity theorem [11, Theorem 3.1].

Let S be a G-regular local ring, $f \in S$ an S-regular element, and $R = S/(f)$ the residue ring. We denote by $\mathcal{M}_S(f)$ the quotient category of the category of matrix factorizations of f over S by the matrix factorization $(1, f)$, by $\underline{\mathcal{M}}_S(f)$ the quotient category of $\mathcal{M}^0_S(f)$ by $(f, 1)$, by $\mathcal{G}(R)$ the category of totally reflexive R-modules, and by $\underline{\mathcal{G}}(R)$ the stable category of $\mathcal{G}(R)$.

Theorem A (matrix factorization)**.** *There are equivalences of categories:*

$$\mathcal{M}_S(f) \simeq \mathcal{G}(R),$$
$$\underline{\mathcal{M}}_S(f) \simeq \underline{\mathcal{G}}(R).$$

Theorem B (Knörrer's periodicity)**.** *Let $B = S[[x, y]]/(f + xy)$.*

(1) *There is a fully faithful functor $\Delta : \underline{\mathcal{G}}(R) \to \underline{\mathcal{G}}(B)$.*
(2) *Suppose that $\frac{1}{2}, \sqrt{-1} \in S$ and that R is henselian. Then the functor Δ is an equivalence.*

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1. Basic definitions

In this paper we use commutative noetherian rings and their categories of finitely generated modules. In this section let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $\operatorname{mod} R$ denote the category of finitely generated R-modules. A *subcategory* always means a full subcategory closed under isomorphism.

Definition 1.1. (1) Let $(-)^*$ denote the R-dual functor $\operatorname{Hom}_R(-, R)$. An R-module M is called *totally reflexive* if
  (i) the natural homomorphism $M \to M^{**}$ is an isomorphism, and
  (ii) $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for any $i > 0$.
(2) Let M be a nonzero R-module. If there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

of R-modules such that each X_i is totally reflexive, then we say that M has *G-dimension at most* n. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\operatorname{Gdim}_R M = \infty$. If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has *G-dimension* n, and write $\operatorname{Gdim}_R M = n$. We set $\operatorname{Gdim}_R 0 = -\infty$.

Remark 1.2. An R-module M is totally reflexive if and only if $\operatorname{Gdim}_R M \leq 0$.

Definition 1.3. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if it satisfies the following four conditions.
(1) $\mathcal{X}$ contains R.
(2) $\mathcal{X}$ is closed under direct summands: if M is an R-module in $\mathcal{X}$ and $N \oplus P \cong M$, then N is also in $\mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are in $\mathcal{X}$, then M is also in $\mathcal{X}$.
(4) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N are in $\mathcal{X}$, then L is also in $\mathcal{X}$.

A resolving subcategory is a subcategory such that any two "minimal" resolutions of a module by modules in it have the same length; see [2, (3.12)].

Here we introduce three subcategories of $\operatorname{mod} R$.

Notation 1.4. We denote by $\mathcal{F}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all free R-modules, by $\mathcal{G}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all totally reflexive R-modules, and by $\mathcal{C}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all R-modules M satisfying the inequality $\operatorname{depth}_R M \geq \operatorname{depth} R$.

Let M be an R-module. Take a minimal free resolution

$$F_\bullet = (\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$$

of M. For a nonnegative integer n, we set $\Omega^n_R M = \operatorname{Im} d_n$ and call it the *nth syzygy* of M. Note that the nth syzygy of a given R-module is uniquely determined up to isomorphism.

We will often use the following lemma. The assertion (1) is proved in [8, Theorem (1.4.9)], (2) in [7, Theorem 1.3.3] and [8, Theorem (1.4.8)], (3) in [8, Corollary (1.4.6) and Theorem (2.2.8)], and (4) in [8, Corollary (1.2.9)].

Lemma 1.5. (1) *The following are equivalent:*
  (i) *R is Gorenstein;*
  (ii) $\operatorname{Gdim}_R M < \infty$ *for all R-modules M;*
  (iii) $\operatorname{Gdim}_R k < \infty$.
(2) *Let M be an R-module.*
  (i) *If* $\operatorname{pd}_R M < \infty$, *then* $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$.
  (ii) *If* $\operatorname{Gdim}_R M < \infty$, *then* $\operatorname{Gdim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$.
(3) *Let M be an R-module and $\boldsymbol{x} = x_1, \dots, x_n$ a sequence of elements of R.*
  (i) *If $\boldsymbol{x}$ is an R- and M-sequence, then* $\operatorname{Gdim}_{R/(\boldsymbol{x})} M/\boldsymbol{x}M = \operatorname{Gdim}_R M$.
  (ii) *If $\boldsymbol{x}$ is an R-sequence in* $\operatorname{Ann}_R M$, *then* $\operatorname{Gdim}_{R/(\boldsymbol{x})} M = \operatorname{Gdim}_R M - n$.
(4) *For an R-module M and a nonnegative integer n,* $\operatorname{Gdim}_R \Omega^n M = \sup\{\operatorname{Gdim}_R M - n, 0\}$.

Remark 1.6. The following are basic properties of the subcategories $\mathcal{F}(R)$, $\mathcal{G}(R)$ and $\mathcal{C}(R)$.

(1) All of $\mathcal{F}(R)$, $\mathcal{G}(R)$ and $\mathcal{C}(R)$ are resolving subcategories of $\operatorname{mod} R$.
(2) If R is Cohen-Macaulay, then $\mathcal{C}(R)$ consists of all maximal Cohen-Macaulay R-modules.
(3) $\mathcal{C}(R)$ contains $\mathcal{G}(R)$, and $\mathcal{G}(R)$ contains $\mathcal{F}(R)$.
(4) R is Gorenstein if and only if $\mathcal{C}(R)$ coincides with $\mathcal{G}(R)$.
(5) R is regular if and only if $\mathcal{C}(R)$ coincides with $\mathcal{F}(R)$.

The fact that $\mathcal{G}(R)$ is resolving is shown in [2, (3.11)] and [5, Lemma 2.3]. The first assertion in (3) follows from Lemma 1.5(2). As to (4), if R is Gorenstein, then $\mathcal{C}(R)$ consists of all totally reflexive R-modules by Lemma 1.5(1) and (2). Conversely, suppose that $\mathcal{C}(R)$ coincides with $\mathcal{G}(R)$. Putting $t = \operatorname{depth} R$, we have $\operatorname{depth} \Omega_R^t k = t$. Hence $\Omega_R^t k$ is in $\mathcal{C}(R) = \mathcal{G}(R)$. This implies that the R-module k has G-dimension (at most) t, and thus R is Gorenstein by Lemma 1.5(1). The assertion (5) is shown similarly to (4).

Definition 1.7. We say that a local ring R is *G-regular* if $\mathcal{G}(R)$ coincides with $\mathcal{F}(R)$.

Proposition 1.8. (1) *A local ring is regular if and only if it is G-regular and Gorenstein.*
(2) *A local ring R is G-regular if and only if $\operatorname{Gdim}_R M = \operatorname{pd}_R M$ for any R-module M.*
(3) *A normal local ring R is G-regular if and only if $\operatorname{Gdim}_R R/I = \operatorname{pd}_R R/I$ for any ideal I of R.*

Proof. (1) The assertion immediately follows from Remark 1.6(4) and (5).

(2) This can easily be shown using the definition.

(3) Let M be a totally reflexive R-module. Then M is reflexive, so M is torsionfree. Hence there exists an exact sequence $0 \to R^n \to M \to I \to 0$ such that I is an ideal of R; see [6, Theorem 6 in Chapter VII §4]. We obtain an exact sequence

$$0 \to R^n \to M \to R \to R/I \to 0.$$

It follows by definition that the R-module R/I has G-dimension at most 2. If the equality $\operatorname{Gdim}_R R/I = \operatorname{pd}_R R/I$ holds, then the R-module R/I has finite projective dimension, and so does M. Thus M is free by Lemma 1.5(2). □

2. Matrix factorizations

In this section, we generalize Eisenbud's matrix factorization theorem [10]. Throughout this section, let S be a G-regular local ring with maximal ideal $\mathfrak{n}$, $f \in \mathfrak{n}$ an S-regular element, and $R = S/(f)$ the residue ring. First of all, let us make the definition of a matrix factorization.

Definition 2.1. For a nonnegative integer n, we call a pair (ϕ, ψ) of $n \times n$ matrices over S a *matrix factorization* of f (over S) if $\phi\psi = \psi\phi = fI_n$, where I_n is the identity matrix. When $n = 0$, both ϕ and ψ can be considered as the 0×0 matrix over S which we denote by ζ, and we call the matrix factorization (ζ, ζ) the *zero matrix factorization* of f.

Remark 2.2. If (ϕ, ψ) is a matrix factorization of f, then so are (ψ, ϕ), $({}^t\phi, {}^t\psi)$ and $({}^t\psi, {}^t\phi)$, where ${}^t(-)$ denotes the transpose.

In what follows, we will often identify an $m \times n$ matrix over S with a homomorphism $S^n \to S^m$ of free S-modules. Thus the matrix ζ gives the identity map of the free S-module $S^0 = 0$ of rank zero.

A matrix factorization corresponds to an R-module which has projective dimension at most one as an S-module, as we see next.

Proposition 2.3. (1) *Let (ϕ, ψ) be a matrix factorization of f. Then $M := \operatorname{Coker} \phi$ is an R-module and there is an exact sequence $0 \to S^n \xrightarrow{\phi} S^n \to M \to 0$ in $\operatorname{mod} S$.*
(2) *Let M be an R-module and suppose that there is an exact sequence $0 \to S^n \xrightarrow{\phi} S^m \to M \to 0$ in $\operatorname{mod} S$. Then one has $m = n$, and there is a matrix ψ such that (ϕ, ψ) is a matrix factorization of f.*

Proof. (1) By using the equalities $\phi\psi = \psi\phi = fI_n$, we easily see that $fM = 0$ and that the endomorphism ϕ is injective over S.

(2) The equality $fM = 0$ implies $M_f = 0$. Hence we see that $m = n$. For each $x \in S^n$ we have $fx \in fS^n \subseteq \operatorname{Im} \phi$, and the injectivity of ϕ shows that there uniquely exists $y \in S^n$ such that $fx = \phi(y)$. Defining an endomorphism $\psi : S^n \to S^n$ by $\psi(x) = y$, we have $\phi\psi = f \cdot \mathrm{id}_{S^n}$. We get $\phi(\psi\phi - f \cdot \mathrm{id}_{S^n}) = 0$, and $\psi\phi = f \cdot \mathrm{id}_{S^n}$ by the injectivity of ϕ again. It follows that (ϕ, ψ) is a matrix factorization of f. □

Each matrix factorization of f gives rise to a totally reflexive R-module.

Proposition 2.4. *Let (ϕ, ψ) be a matrix factorization of f, and let n be the (common) size of the matrices ϕ and ψ. Then the sequence*

$$\cdots \xrightarrow{\phi} R^n \xrightarrow{\psi} R^n \xrightarrow{\phi} R^n \xrightarrow{\psi} \cdots \tag{2.4.1}$$

is an exact sequence of R-modules whose R-dual is also exact. Hence $\mathrm{Coker}(S^n \xrightarrow{\phi} S^n) \cong \mathrm{Coker}(R^n \xrightarrow{\phi} R^n)$ *is a totally reflexive R-module.*

Proof. It is obvious that (2.4.1) is a complex of R-modules. We denote by $\overline{x}$ the residue class of an element $x \in S^n$ in R^n. Let $\overline{x}$ be an element of R^n with $\phi(\overline{x}) = \overline{0}$. Then $\phi(x) \in fS^n$, so $\phi(x) = fy$ for some $y \in S^n$, and we have $fx = \psi\phi(x) = f\psi(y)$. Since f is an S-regular element, we get $x = \psi(y)$, and so $\overline{x} = \psi(\overline{y})$. Therefore $\mathrm{Ker}(R^n \xrightarrow{\phi} R^n) = \mathrm{Im}(R^n \xrightarrow{\psi} R^n)$. Similarly we obtain $\mathrm{Ker}(R^n \xrightarrow{\psi} R^n) = \mathrm{Im}(R^n \xrightarrow{\phi} R^n)$. Thus (2.4.1) is an exact sequence. The last statement follows from [8, Theorem (4.1.4)]. $\square$

Matrix factorizations form a category:

Definition 2.5. We define the category $\mathcal{M}^0_S(f)$ by setting

(1) the matrix factorizations of f as the objects of $\mathcal{M}^0_S(f)$, and
(2) a pair (α, β) of matrices making the following diagram commute

$$\begin{array}{ccccc} S^n & \xrightarrow{\psi} & S^n & \xrightarrow{\phi} & S^n \\ \alpha\downarrow & & \beta\downarrow & & \alpha\downarrow \\ S^{n'} & \xrightarrow{\psi'} & S^{n'} & \xrightarrow{\phi'} & S^{n'} \end{array} \tag{2.5.1}$$

as a morphism from an object (ϕ, ψ) to an object (ϕ', ψ').

Remark 2.6. (1) The commutativity of the right square in a diagram of the form (2.5.1) implies the commutativity of the left one. In fact, if $\alpha\phi = \phi'\beta$, then $\phi'(\beta\psi - \psi'\alpha) = \alpha(fI_n) - (fI_{n'})\alpha = 0$, and $\beta\psi = \psi'\alpha$ by the injectivity of ϕ'.
(2) The zero matrix factorization (ζ, ζ) is an object of $\mathcal{M}^0_S(f)$, both terminal and initial, hence zero.
(3) The category $\mathcal{M}^0_S(f)$ is an additive category. Indeed, for two matrix factorizations (ϕ, ψ) and (ϕ', ψ'),

$$(\phi, \psi) \oplus (\phi', \psi') = \left(\left(\begin{smallmatrix} \phi & 0 \\ 0 & \psi \end{smallmatrix}\right), \left(\begin{smallmatrix} \phi' & 0 \\ 0 & \psi' \end{smallmatrix}\right) \right).$$

Definition 2.7. (1) We say that two matrix factorizations $(\phi, \psi), (\phi', \psi')$ are *equivalent*, and denote this situation by $(\phi, \psi) \sim (\phi', \psi')$, if there is an isomorphism $(\phi, \psi) \to (\phi', \psi')$ in $\mathcal{M}^0_S(f)$.
(2) We say that a matrix factorization (ϕ, ψ) is *reduced* if all entries of the matrices ϕ, ψ are in $\mathfrak{n}$.

Remark 2.8. (1) Every matrix factorization equivalent to a reduced one is reduced.
(2) The pairs $(1, f), (f, 1)$ of elements of S are always non-reduced matrix factorizations of f.

Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ a set of objects of $\mathcal{A}$. Then the category $\mathcal{A}/\mathcal{B}$ has $\mathrm{Ob}(\mathcal{A}/\mathcal{B}) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A_1, A_2) = \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)/\mathcal{B}(A_1, A_2)$ for $A_1, A_2 \in \mathrm{Ob}(\mathcal{A}/\mathcal{B})$, where $\mathcal{B}(A_1, A_2)$ is the subgroup consisting of all morphisms from A_1 to A_2 that factor through finite direct sums of objects in $\mathcal{B}$. Note that $\mathcal{A}/\mathcal{B}$ is also an additive category.

Definition 2.9. We define the following additive categories:

$$\begin{aligned} \mathcal{M}_S(f) &= \mathcal{M}^0_S(f)/\{(1, f)\}, \\ \underline{\mathcal{M}}_S(f) &= \mathcal{M}_S(f)/\{(f, 1)\} = \mathcal{M}^0_S(f)/\{(1, f), (f, 1)\}, \\ \underline{\mathcal{G}}(R) &= \mathcal{G}(R)/\{R\}. \end{aligned}$$

Note that $\underline{\mathcal{G}}(R)$ is the stable category of $\mathcal{G}(R)$.

The following theorem is the main result of this section, which is a generalization of Eisenbud's matrix factorization theorem [10, Section 6] (see also [21, Theorem (7.4)]).

Theorem 2.10. *There are equivalences of categories:*

$$\begin{aligned} \mathcal{M}_S(f) &\simeq \mathcal{G}(R), \\ \underline{\mathcal{M}}_S(f) &\simeq \underline{\mathcal{G}}(R). \end{aligned}$$

Proof. For a matrix factorization (ϕ, ψ) of f, the module $F((\phi, \psi)) := \operatorname{Coker} \phi$ is in $\mathcal{G}(R)$ by Proposition 2.4. For a morphism $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$ of matrix factorizations of f, let $F((\alpha, \beta))$ be the induced homomorphism $F((\phi, \psi)) \to F((\phi', \psi'))$. We obtain an additive functor $F : \mathcal{M}_S(f) \to \mathcal{G}(R)$. Compare this with [21, Proposition (7.2) and Theorem (7.4)].

Let M be a totally reflexive R-module. Then we have

$$0 \geq \operatorname{Gdim}_R M = \operatorname{Gdim}_{S/(f)} M = \operatorname{Gdim}_S M - 1 = \operatorname{pd}_S M - 1.$$

Here, the second equality follows from the fact that f is an S-regular element in $\operatorname{Ann}_S M$ and Lemma 1.5(3), and the third follows by Proposition 1.8(2). Hence the S-module M has projective dimension at most one, and there exists an exact sequence $0 \to S^n \xrightarrow{\phi} S^m \to M \to 0$. By Proposition 2.3(2), we have $n = m$ and there is a matrix ψ such that (ϕ, ψ) is a matrix factorization of f. By analogous arguments to the proof of [21, Theorem (7.4)], we obtain an additive functor $G : \mathcal{G}(R) \to \mathcal{M}_S(f)$ with $G(M) = (\phi, \psi)$, and see that $FG = 1_{\mathcal{G}(R)}$ and $GF \cong 1_{\mathcal{M}_S(f)}$. Thus F forms an equivalence between the additive categories $\mathcal{M}_S(f)$ and $\mathcal{G}(R)$. Since $F((f, 1)) = R$, the functor F induces an additive functor $\underline{\mathcal{M}}_S(f) \to \underline{\mathcal{G}}(R)$ of additive categories which is an equivalence. □

The above theorem yields the following corollary; in the case where R is henselian, one can uniquely decompose a given matrix factorization into a direct sum of the form in the corollary. One can prove the corollary similarly to the arguments in [21, Remark (7.5)]. The henselian property of R is used in showing the uniqueness of the direct sum decomposition of R-modules induced from (2.11.1) along the first equivalence in Theorem 2.10.

Corollary 2.11. *Suppose that R is henselian. Then every matrix factorization (ϕ, ψ) of f has a direct sum decomposition unique up to similarity*

(2.11.1)
$$(\phi, \psi) \sim (\phi_0, \psi_0) \oplus (1, f)^{\oplus p} \oplus (f, 1)^{\oplus q},$$

where (ϕ_0, ψ_0) is a reduced matrix factorization and p, q are nonnegative integers.

To prove our next result, we establish a lemma.

Lemma 2.12. *Let (ϕ, ψ) be a matrix factorization of f. Assume that ψ has an entry which is a unit of S. Then (ϕ, ψ) has a direct summand equivalent to $(f, 1)$.*

Proof. By assumption, there is a commutative diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{\psi} & S^n & \xrightarrow{\phi} & S^n \\ \alpha \downarrow \cong & & \beta \downarrow \cong & & \alpha \downarrow \cong \\ S^n & \xrightarrow{\psi'} & S^n & \xrightarrow{\phi'} & S^n \end{array}$$

such that ψ' is a matrix of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right)$ and that the vertical maps are isomorphisms. We can directly check that (ϕ', ψ') is a matrix factorization of f, and $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$ is an isomorphism in $\mathcal{M}_S^0(f)$. Writing $\phi' = \left(\begin{smallmatrix} a & b \\ c & \mu \end{smallmatrix}\right)$ and using the equalities $\phi'\psi' = \psi'\phi' = fI_n$, we see that $a = f$, $b = 0$ and $c = 0$, and that (μ, ν) is a matrix factorization of f. We obtain $(\phi, \psi) \sim (\phi', \psi') = \left(\left(\begin{smallmatrix} f & 0 \\ 0 & \mu \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right)\right) = (f, 1) \oplus (\mu, \nu)$, which proves the lemma. □

Now we can prove the following proposition. (Note that we do not assume that the local ring R is henselian.)

Proposition 2.13. (cf. [10, Corollary 6.3] and [21, Corollary (7.6)]) *The assignment $[(\phi, \psi)] \mapsto [\operatorname{Coker} \phi]$ makes a bijection from the set of equivalence classes of reduced matrix factorizations of f to the set of isomorphism classes of totally reflexive R-modules without free summand.*

Proof. Let (ϕ, ψ) be a reduced matrix factorization of f. Then Proposition 2.4 and similar arguments to the proof of [21, (7.5.1)] show that $\operatorname{Coker} \phi$ is a totally reflexive R-module without free summand. If (ϕ, ψ) is equivalent to another reduced matrix factorization (ϕ', ψ') of f, then the R-module $\operatorname{Coker} \phi$ is isomorphic to $\operatorname{Coker} \phi'$. Thus we obtain a well-defined map

$$\chi : [(\phi, \psi)] \mapsto [\operatorname{Coker} \phi]$$

from the set of equivalence classes of reduced matrix factorizations of f to the set of isomorphism classes of totally reflexive R-modules without free summand.

Let $(\phi, \psi), (\phi', \psi')$ be reduced matrix factorizations such that $\operatorname{Coker} \phi$ is isomorphic to $\operatorname{Coker} \phi'$. Then by Proposition 2.3(2) we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^n & \xrightarrow{\phi} & S^n & \longrightarrow & \operatorname{Coker} \phi & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \alpha & & \downarrow \cong & & \\ 0 & \longrightarrow & S^{n'} & \xrightarrow{\phi'} & S^{n'} & \longrightarrow & \operatorname{Coker} \phi' & \longrightarrow & 0 \end{array}$$

of S-modules with exact rows. Since all entries of ϕ, ϕ' are nonunits of S, the two rows are minimal free resolutions of the S-modules $\operatorname{Coker} \phi, \operatorname{Coker} \phi'$. Hence the vertical maps α, β are isomorphisms (cf. [12, §18, Lemma 8]). According to Remark 2.6(1), we have an isomorphism $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$ in the category $\mathcal{M}_S^0(f)$. Thus the map χ is injective.

Let M be a totally reflexive R-module. Then it is seen from the proof of Theorem 2.10 that there exists an exact sequence $0 \to S^n \xrightarrow{\phi} S^n \to M \to 0$ of S-modules. We can choose ϕ such that all the entries of ϕ are in the maximal ideal $\mathfrak{n}$ of S. Proposition 2.3(2) shows that there is a matrix ψ such that (ϕ, ψ) is a matrix factorization of f. By Lemma 2.12 if M has no free R-summand, all the entries of the matrix ψ must be in $\mathfrak{n}$. Therefore when M is without free R-summand, (ϕ, ψ) is a reduced matrix factorization of f such that $\operatorname{Coker} \phi = M$. Thus, the map χ is surjective. This completes the proof of the proposition. □

We end this section by mentioning extensions of totally reflexive modules:

Remark 2.14. (cf. [21, Remark (7.8)])

(1) Let $h : M \to M'$ be a homomorphism of totally reflexive R-modules. Then the following hold.
   (i) There is a morphism $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$ in $\mathcal{M}_S^0(f)$ which induces h.
   (ii) One has a matrix factorization $\left(\left(\begin{smallmatrix} \psi' & \beta \\ 0 & \phi \end{smallmatrix}\right), \left(\begin{smallmatrix} \phi' & -\alpha \\ 0 & \psi \end{smallmatrix}\right)\right)$ of f.
   (iii) For the exact sequence $0 \to N \to R^{n'} \to M' \to 0$ with $N = \operatorname{Coker} \psi'$, the connecting homomorphism $\operatorname{Hom}_R(M, M') \to \operatorname{Ext}_R^1(M, N)$ sends h to an element corresponding to an exact sequence $0 \to N \to L \to M \to 0$ with $L = \operatorname{Coker} \left(\begin{smallmatrix} \psi' & \beta \\ 0 & \phi \end{smallmatrix}\right)$.
(2) Let M, N be totally reflexive R-modules. Every extension $0 \to N \to L \to M \to 0$ of M by N is obtained in the way shown in (1).

3. Knörrer's periodicity

In this section, we extend the concept of Knörrer's periodicity [11]. Throughout this section, as in the previous section, let S be a G-regular local ring with maximal ideal $\mathfrak{n}$, $f \in \mathfrak{n}$ an S-regular element, and $R = S/(f)$ the residue ring. Set

$$A = S[[x]]/(f + x^2) \quad \text{and} \quad B = S[[x, y]]/(f + xy),$$

where x, y are indeterminates over S.

We can directly check that the following statements hold.

(1) One has $A/(x) \cong R$.
(2) The element x is A-regular.
(3) The ring A is a free S-module with basis $\{1, x\}$.
(4) The element $f + x^2$ is $S[[x]]$-regular.

For a totally reflexive A-module M, we set $\Theta M = M/xM$.

Proposition 3.1. *One has an additive functor $\Theta : \mathcal{G}(A) \to \mathcal{G}(R)$.*

Proof. It is seen from Lemma 1.5(3) that ΘM is a totally reflexive R-module for a totally reflexive A-module M. The proposition follows from this. □

Proposition 3.2. *An A-module is totally reflexive if and only if it is free as an S-module.*

Proof. Let M be a nonzero A-module. We have an equality

$$\operatorname{Gdim}_A M = \operatorname{Gdim}_{S[[x]]} M - 1 \tag{3.2.1}$$

by Lemma 1.5(3).

Suppose that M is free as an S-module. Let $m \in M$ with $xm = 0$. Then $fm = -x^2m = 0$. Since f is S-regular and M is assumed to be S-free, f is M-regular. Therefore $m = 0$. Nakayama's lemma implies that $xM \neq M$. Thus x is an M-regular element. It follows from Lemma 1.5(3) that $\operatorname{Gdim}_{S[[x]]} M = \operatorname{Gdim}_S M/xM$. There is an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, and M is totally reflexive over S since it is free over S. This yields an inequality $\operatorname{Gdim}_S M/xM \leq 1$, so we have $\operatorname{Gdim}_A M \leq 0$ by (3.2.1). Hence M is totally reflexive over A.

On the other hand, suppose that M is a totally reflexive A-module. Then $\operatorname{Gdim}_A M \leq 0$, and $\operatorname{Gdim}_{S[[x]]} M \leq 1$ by (3.2.1). Corollary 4.4 and Proposition 1.8(2) imply that $\operatorname{Gdim}_{S[[x]]} M = \operatorname{pd}_{S[[x]]} M$. Hence $\operatorname{pd}_{S[[x]]} M \leq 1$, and there is an exact sequence

$$0 \to F_1 \to F_0 \to M \to 0$$

such that F_0, F_1 are free $S[[x]]$-modules. Note that F_0, F_1 are flat as S-modules, so we have $\operatorname{Tor}_i^S(M, S/\mathfrak{n}) = 0$ for any $i \geq 2$. The ring A is finitely generated as an S-module, hence so is M. It follows that $\operatorname{pd}_S M \leq 1 < \infty$ (cf. [7, Corollary 1.3.2]). Applying Lemma 1.5(2), we obtain $\operatorname{pd}_S M = \operatorname{depth} S - \operatorname{depth}_S M$. It is obvious that the closed fiber $A/\mathfrak{n}A$ of the flat local homomorphism $S \to A$ is artinian. Therefore we have equalities $\operatorname{depth}_A M = \operatorname{depth}_S M$ and $\operatorname{depth} A = \operatorname{depth} S + \operatorname{depth} A/\mathfrak{n}A = \operatorname{depth} S$. Thus $\operatorname{pd}_S M = \operatorname{depth} A - \operatorname{depth}_A M = \operatorname{Gdim}_A M \leq 0$ by Lemma 1.5(2), and M is S-free. □

As a direct consequence of Proposition 3.2, we have the following result.

Corollary 3.3. *The totally reflexive A-modules are precisely the free S-modules with A-module structure, or equivalently, the free S-modules on which x acts.*

Recall that two square matrices ϕ, ψ over S of the same size are *similar* if there exists an $n \times n$ invertible matrix α over S such that $\phi = \alpha^{-1}\psi\alpha$. For a totally reflexive A-module M, we denote by ϕ_M a representation matrix of the linear map $M \xrightarrow{x} M$ (the multiplication map by the variable x) of free S-modules. Note that ϕ_M is not uniquely determined by M. Instead, we have the following.

Corollary 3.4. *The assignment $[M] \mapsto [\phi_M]$ makes a bijection from the set of isomorphism classes of totally reflexive A-modules to the set of similarity classes of square matrices ϕ over S with $\phi^2 = -fI$, where I is the identity matrix.*

Proof. Let M be a totally reflexive A-module. Then Proposition 3.2 shows that there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow[\rho]{\cong} & S^n \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle \phi_M} \\ M & \xrightarrow[\rho]{\cong} & S^n \end{array}$$

where ρ is an S-isomorphism. We have $\phi_M = \rho x \rho^{-1}$, and hence $\phi_M^2 = (\rho x \rho^{-1})(\rho x \rho^{-1}) = \rho x^2 \rho^{-1} = \rho(-f)\rho^{-1} = -f(\rho\rho^{-1}) = -fI_n$.

Let M and N be totally reflexive A-modules with $[M] = [N]$. Then there exists an A-isomorphism $\lambda : M \to N$, and we have $x\lambda = \lambda x$. There are S-isomorphisms $\rho_M : M \to S^n$ and $\rho_N : N \to S^n$ such that $\phi_M\rho_M = \rho_M x$ and $\phi_N\rho_N = \rho_N x$. Setting $\alpha = \rho_N \lambda \rho_M^{-1}$, we easily see that α is an invertible matrix over S satisfying $\alpha^{-1}\phi_N\alpha = \phi_M$. Therefore $[\phi_M] = [\phi_N]$. Thus, we obtain a well-defined map

$$\chi : [M] \mapsto [\phi_M]$$

from the set of isomorphism classes of totally reflexive A-modules to the set of similarity classes of square matrices ϕ over S with $\phi^2 = -fI$.

Let M, N be totally reflexive A-modules with $[\phi_M] = [\phi_N]$. Then there exists an invertible matrix α over S with $\phi_M = \alpha^{-1}\phi_N\alpha$. As before, there exist S-isomorphisms $\rho_M : M \to S^n$ and $\rho_N : N \to S^n$ such that $\phi_M\rho_M = \rho_M x$ and $\phi_N\rho_N = \rho_N x$. Putting $\lambda = \rho_N^{-1}\alpha\rho_M$, we have $\lambda x = x\lambda$, which means that λ is an A-homomorphism, hence an A-isomorphism. Thus we have $[M] = [N]$, and the map χ is injective.

Let ϕ be an $n \times n$-matrix over S with $\phi^2 = -fI_n$. Then, letting M be the free S-module S^n equipped with the action of x by $xz = \phi(z)$ for $z \in M$, we have $x^2 z = \phi^2(z) = -fz$ and we see that M is an A-module. Proposition 3.2 says that M is a totally reflexive A-module. Since ϕ is a representation matrix of the S-linear map $M \xrightarrow{x} M$, we have $[\phi] = [\phi_M]$. Thus the map χ is surjective. □

Using Corollary 3.4, we can show the following result along the same lines as in the proof of [21, Lemma (12.2)].

Lemma 3.5. *Let M be a totally reflexive A-module. Then the following hold.*

(1) *One has a matrix factorization $(xI - \phi_M, xI + \phi_M)$ of $f + x^2$ over $S[[x]]$, and $M \cong \operatorname{Coker}(xI - \phi_M)$.*
(2) *One has a matrix factorization $(\phi_M, -\phi_M)$ of f over S, and $\Theta M \cong \operatorname{Coker}\phi_M$.*

We have a functor in the opposite direction to that of the functor Θ.

Proposition 3.6. *Taking the first syzygy makes an additive functor $\Omega_A : \mathcal{G}(R) \to \mathcal{G}(A)$.*

Proof. For a totally reflexive R-module M, we have $0 \geq \operatorname{Gdim}_{A/(x)} M = \operatorname{Gdim}_A M - 1$ by Lemma 1.5(3), and $\operatorname{Gdim}_A(\Omega_A M) \leq 0$ by Lemma 1.5(4). Therefore $\Omega_A M$ is a totally reflexive A-module. □

The following lemma is an analogue of the second statement in Lemma 3.5. We can show it similarly to the proof of [21, Lemma (12.3)] by using Proposition 2.13.

Lemma 3.7. (1) *Let (ϕ, ψ) be a matrix factorization of f over S, and put $M = \operatorname{Coker}\phi$. Then $\left(\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}, \begin{pmatrix} \phi & xI \\ -xI & \psi \end{pmatrix}\right)$ is a matrix factorization of $f + x^2$ over $S[[x]]$, and $\Omega_A M \oplus F \cong \operatorname{Coker}\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}$ for some free A-module F.*
(2) *In (1), assume in addition that (ϕ, ψ) is reduced. Then $\left(\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}, \begin{pmatrix} \phi & xI \\ -xI & \psi \end{pmatrix}\right)$ is also reduced, and $\Omega_A M \cong \operatorname{Coker}\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}$.*

Remark 3.8. Let (ϕ, ψ) be a reduced matrix factorization of f over S and set $M = \operatorname{Coker}\phi$. Then one has an equality

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \phi & -xI \\ xI & \psi \end{pmatrix},$$

which shows that $\operatorname{Coker}\begin{pmatrix} \psi & -xI \\ xI & \phi \end{pmatrix}$ is isomorphic to $\operatorname{Coker}\begin{pmatrix} \phi & -xI \\ xI & \psi \end{pmatrix}$. Therefore it follows from Lemma 3.7(2) that the A-module $\Omega_A\Omega_R M$ is isomorphic to $\Omega_A M$.

Applying Lemmas 3.5 and 3.7 and Proposition 2.13, one can prove the following result along the same lines as in the proof of [21, Proposition (12.4)].

Proposition 3.9. (1) *For a totally reflexive R-module M without free summand, one has $\Theta\Omega_A M \cong M \oplus \Omega_R M$.*
(2) *Assume that $\frac{1}{2} \in S$. Then for a totally reflexive A-module N, one has $\Omega_A\Theta N \cong N \oplus \Omega_A N$ up to free summand.*

Corollary 3.10. (cf. [21, Remark (12.7)]) *Suppose that R is henselian. Then the following hold.*

(1) (i) *For any nonfree indecomposable totally reflexive R-module M, there exists a nonfree indecomposable totally reflexive A-module N such that M is isomorphic to a direct summand of ΘN.*
(ii) *Assume that $\frac{1}{2} \in S$. Then for any nonfree indecomposable totally reflexive A-module N, there exists a nonfree indecomposable totally reflexive R-module M such that N is isomorphic to a direct summand of $\Omega_A M$.*
(2) *For an indecomposable totally reflexive R-module M, the A-module $\Omega_A M$ has at most two nonzero direct summands.*

Proof. The assertion (1) follows from Proposition 3.9 and analogous arguments to the proof of [21, Theorem (12.5)]. As to the assertion (2), we may assume that the R-module M is nonfree, hence M has no free summand. Suppose that there is a direct sum decomposition $\Omega_A M \cong X \oplus Y \oplus Z$ of A-modules. Then we have $\Theta X \oplus \Theta Y \oplus \Theta Z \cong \Theta\Omega_A M \cong M \oplus \Omega_R M$ by Proposition 3.9(1). According to [16, Proposition 7.1], $\Omega_R M$ is also indecomposable. By virtue of the Krull-Schmidt theorem, one of the R-modules $\Theta X, \Theta Y, \Theta Z$ is zero; we may assume that $\Theta Z = 0$. Then we have $xZ = Z$, and $Z = 0$ by Nakayama's lemma. This shows the assertion (2). □

For a matrix factorization (ϕ, ψ) of f over S, set

$$\Delta^0(\phi,\psi) = \left(\left(\begin{smallmatrix} \phi & xI \\ yI & -\psi \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi & xI \\ yI & -\phi \end{smallmatrix}\right)\right).$$

Note that this is a matrix factorization of $f + xy$ over $S[[x, y]]$. For a morphism $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$ in the category $\mathcal{M}^0_S(f)$ of matrix factorizations of f over S, let

$$\Delta^0(\alpha,\beta) = \left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta & 0 \\ 0 & \alpha \end{smallmatrix}\right)\right).$$

Note that this is a morphism in $\mathcal{M}^0_{S[[x,y]]}(f + xy)$. Thus we obtain an additive functor

$$\Delta^0 : \mathcal{M}^0_S(f) \to \mathcal{M}^0_{S[[x,y]]}(f + xy).$$

Since there is a commutative diagram

$$\begin{array}{ccccc} S[[x,y]]^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & x \\ y & -f \end{smallmatrix}\right)} & S[[x,y]]^2 & \xrightarrow{\left(\begin{smallmatrix} f & x \\ y & -1 \end{smallmatrix}\right)} & S[[x,y]]^2 \\ \cong\downarrow\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) & & \cong\downarrow\left(\begin{smallmatrix} 1 & 0 \\ y & -1 \end{smallmatrix}\right) & & \cong\downarrow\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \\ S[[x,y]]^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & f+xy \end{smallmatrix}\right)} & S[[x,y]]^2 & \xrightarrow{\left(\begin{smallmatrix} f+xy & 0 \\ 0 & 1 \end{smallmatrix}\right)} & S[[x,y]]^2 \end{array}$$

with isomorphic vertical maps, both $\Delta^0(f, 1)$ and $\Delta^0(1, f)$ are isomorphic to $(f + xy, 1) \oplus (1, f + xy)$. Hence Δ^0 induces an additive functor

$$\Delta : \underline{\mathcal{M}}_S(f) \to \underline{\mathcal{M}}_{S[[x,y]]}(f + xy).$$

By virtue of Theorem 2.10, we get an additive functor

$$\underline{\mathcal{G}}(R) \to \underline{\mathcal{G}}(B).$$

We also denote it by Δ.

The same proof as that of [21, Lemma (12.9)] shows the following result.

Lemma 3.11. *Let*

$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right) : \left(\left(\begin{smallmatrix} \phi & x \\ y & -\psi \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi & x \\ y & -\phi \end{smallmatrix}\right)\right) \to \left(\left(\begin{smallmatrix} \phi' & x \\ y & -\psi' \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi' & x \\ y & -\phi' \end{smallmatrix}\right)\right)$$

be a morphism in $\mathcal{M}^0_{S[[x,y]]}(f+xy)$. *Assume that all the entries of the matrix* a' *are in the maximal ideal* $(x, y)S[[x, y]]$ *of the local ring* $S[[x, y]]$. *Then one has an equivalence*

$$\left(\begin{pmatrix} \psi' & x & a' & b' \\ y & -\phi' & c' & d' \\ 0 & 0 & \phi & x \\ 0 & 0 & y & -\psi \end{pmatrix}, \begin{pmatrix} \phi' & x & -a & -b \\ y & -\psi' & -c & -d \\ 0 & 0 & \psi & x \\ 0 & 0 & y & -\phi \end{pmatrix}\right) \sim \left(\left(\begin{smallmatrix} \psi' & x \\ y & -\phi' \end{smallmatrix}\right), \left(\begin{smallmatrix} \phi' & x \\ y & -\psi' \end{smallmatrix}\right)\right) \oplus \left(\left(\begin{smallmatrix} \phi & x \\ y & -\psi \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi & x \\ y & -\phi \end{smallmatrix}\right)\right)$$

of matrix factorizations of $f + xy$ *over* $S[[x, y]]$.

The theorem below is the main result of this section, which is a generalized version of Knörrer's periodicity theorem [11, Theorem 3.1].

Theorem 3.12. (1) *The functor* $\Delta : \underline{\mathcal{G}}(R) \to \underline{\mathcal{G}}(B)$ *is fully faithful.*

(2) *Suppose that* $\frac{1}{2}, \sqrt{-1} \in S$ *and that* R *is henselian. Then the functor* $\Delta : \underline{\mathcal{G}}(R) \to \underline{\mathcal{G}}(B)$ *is an equivalence.*

Proof. Both of the assertions can be proved similarly to the proof of [21, Theorem (12.10)]. For the first assertion, we use Remark 2.14 and Lemma 3.11. As to the second assertion, note from the assumption that $B = S[[x, y]]/(f + xy) = S[[u, v]]/(f + u^2 + v^2)$ where $u = \frac{x+y}{2}, v = \frac{x-y}{2\sqrt{-1}}$. Apply Proposition 3.9, Corollary 3.10 and Lemma 3.7. □

By analogous arguments to the proof of [21, Corollary (12.11)] and Proposition 2.13, we obtain a corollary of Theorem 3.12.

Corollary 3.13. *Suppose that* R *is henselian.*

(1) *Let* $g : M \to N$ *be a homomorphism of totally reflexive* R*-modules such that* M *is nonfree and indecomposable. Then* g *is a split monomorphism (respectively, split epimorphism) if and only if so is* $\Delta^0 g$.

(2) *Assume that* $\frac{1}{2}, \sqrt{-1} \in S$. *Let* M *be a nonfree indecomposable totally reflexive* R*-module. Then* $\Delta^0 M$ *is a nonfree indecomposable totally reflexive* B*-module.*

4. The ascent and descent of G-regular property

We investigate ascent and descent of the G-regular property, modeling our study on the situations where ascent and descent of the regular property is known to hold. First, the G-regular property descends through flat local homomorphisms.

Proposition 4.1. *Let $R \to S$ be a flat local ring homomorphism. If S is G-regular, then so is R.*

Proof. Let M be a totally reflexive R-module. Then $M \otimes_R S$ is a totally reflexive S-module by [4, Theorem 8.7(6)]. Since S is G-regular, $M \otimes_R S$ is a free S-module. Applying [4, Theorem 8.7(6)] again, we see that M is a free R-module. Thus R is also G-regular. □

Proposition 4.2. *Let R be a local ring and $\boldsymbol{x} = x_1, \dots, x_n$ an R-sequence. If $R/(\boldsymbol{x})$ is G-regular, then so is R.*

Proof. We may assume that $n = 1$. Let M be a totally reflexive R-module. According to Lemma 1.5(3), M/x_1M is a totally reflexive $R/(x_1)$-module, and so it is a free $R/(x_1)$-module by assumption. The R-module M is torsionfree since it is reflexive, so x_1 is an M-regular element. By [7, Lemma 1.3.5], M is a free R-module. It follows that R is a G-regular local ring. □

Remark 4.3. The converse of Proposition 4.2 does not necessarily hold. In fact, let k be a field and let $R = k[[t]]$ be a formal power series ring. Then R is regular, so R is G-regular by Proposition 1.8(1). The element t^2 of R is R-regular. However, since $R/(t^2) = k[[t]]/(t^2)$ is a singular Gorenstein local ring, it is not G-regular by Proposition 1.8(1) again.

Corollary 4.4. *Let n be a positive integer. A local ring R is G-regular if and only if so is the formal power series ring $R[[X_1, \dots, X_n]]$.*

Proof. The "if" part follows from Proposition 4.1, and the "only if" part follows from Proposition 4.2. □

Corollary 4.5. *Let $R \to S$ be a flat local ring homomorphism, and let $\mathfrak{m}$ denote the unique maximal ideal of R. If R is regular and $S/\mathfrak{m}S$ is G-regular, then S is also G-regular.*

Proof. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a regular system of parameters of R. The residue ring $S/\boldsymbol{x}S = S/\mathfrak{m}S$ is a G-regular local ring. Since S is flat over R, the sequence $\boldsymbol{x}$ is S-regular. It follows from Proposition 4.2 that S is G-regular. □

Proposition 4.6. *Let $(R, \mathfrak{m})$ be a G-regular local ring and $x \in \mathfrak{m}$ an R-regular element. Then $R/(x)$ is G-regular if and only if $x \notin \mathfrak{m}^2$.*

Proof. The "if" part: Suppose that $R/(x)$ is not G-regular. Then there exists a nonfree totally reflexive $R/(x)$-module N. We can assume without loss of generality that N is indecomposable. Hence N has no free $R/(x)$-summand. Proposition 2.13 implies that there is a reduced matrix factorization (ϕ, ψ) of x over R such that $\operatorname{Coker} \phi = N$. Thus all the entries of the matrices ϕ, ψ are in the maximal ideal $\mathfrak{m}$ of R. The equality $\phi\psi = xI$, where I is the identity matrix, shows that x is an element in $\mathfrak{m}^2$.

The "only if" part: Suppose that $x \in \mathfrak{m}^2$. Then one can write $x = \sum_{i=1}^{r} y_i z_i$ for some $r \geq 1$ and $y_i, z_i \in \mathfrak{m}$. Let $e_1, \dots, e_r$ be the canonical basis of the free R-module $F := R^r$. We define two R-linear maps μ, ν from the exterior algebra $\bigwedge F$ of F to itself by

$$\mu(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{j=1}^{s} (-1)^{j-1} y_{i_j} (e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_s}),$$

$$\nu(w) = \left(\sum_{j=1}^{r} z_j e_j \right) \wedge w.$$

Note that $\bigwedge F$ is a free R-module of rank 2^r. Setting $\phi = \mu + \nu$, we see that (ϕ, ϕ) is a matrix factorization of x over R; see [21, Lemma (8.14)]. Since the images of μ, ν are contained in the maximal ideal $\mathfrak{m}$, the image of the R-linear map ϕ are contained in $\mathfrak{m}(\bigwedge F)$, namely, the matrix factorization (ϕ, ϕ) is reduced. It follows by Proposition 2.13 that $\operatorname{Coker} \phi$ is a nonfree totally reflexive $R/(x)$-module. Hence $R/(x)$ is not a G-regular local ring. □

Corollary 4.7. *A local ring $(R,\mathfrak{m})$ is G-regular if and only if so is its $\mathfrak{m}$-adic completion $\widehat{R}$.*

Proof. The "if" part follows from Proposition 4.1. Let us show the "only if" part; suppose that R is a G-regular local ring. Let $x_1,\dots,x_n$ be a system of generators of the maximal ideal $\mathfrak{m}$ of R. Then there is an isomorphism

$$\widehat{R} \cong R[[X_1,\dots,X_n]]/(X_1-x_1,\dots,X_n-x_n),$$

where $X_1,\dots,X_n$ are indeterminates over R. Corollary 4.4 and Proposition 4.6 imply that the local ring $\widehat{R}$ is G-regular. □

5. Sufficient conditions for G-regular property

In this section, we give some sufficient conditions for a given local ring to be G-regular. We also construct several examples of G-regular local rings.

A sufficient condition is given by the following result, which was proved by Avramov and Martsinkovsky [5, Examples 3.5]. See also [22, Corollary 2.5].

Lemma 5.1. *Every Golod local ring which is not a hypersurface is G-regular. In particular, every non-Gorenstein Cohen-Macaulay local ring with minimal multiplicity is G-regular.*

Example 5.2. According to Lemma 5.1, for examples, the local algebras

$$k[[x,y]]/(x^2,xy,y^2),\quad k[[x,y,z]]/(x^2-yz,y^2-xz,z^2-xy),\quad k[[t^3,t^4,t^5]](\subseteq k[[t]])$$

over a field k, where x,y,z,t are indeterminates over k, are G-regular, since all of them are non-Gorenstein Cohen-Macaulay local rings with minimal multiplicity.

In the above example, the first ring shows that a G-regular local ring is not necessarily a domain, while every regular local ring is a domain.

The following result is due to Yoshino [22, Theorem 3.1]. Using its contrapositive we obtain some sufficient conditions for a local ring to be G-regular.

Lemma 5.3. *Let $(R,\mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^3=0\neq\mathfrak{m}^2$ with a coefficient field k. Suppose that R is not G-regular. Then R is a standard graded Koszul k-algebra, and the Hilbert series $H_R(t)$ of the ring R, the Poincaré series $P_{R/\mathfrak{m}}(t)$ of the R-module $R/\mathfrak{m}$, and the Bass series $I^R(t)$ of the R-module R are as follows:*

$$H_R(t)=1+(r+1)t+rt^2,\quad P_{R/\mathfrak{m}}(t)=\frac{1}{1-(r+1)t+rt^2},\quad I^R(t)=\frac{r-t}{1-rt}.$$

Here, $r=\dim_k \operatorname{Hom}_R(k,R)$ is the type of R.

Taking advantage of this lemma, we can construct G-regular local rings that do not have minimal multiplicity.

Example 5.4. Let k be a field.

(1) The ring

$$R=k[[x,y]]/(x^3,xy,y^3)$$

is an artinian non-Gorenstein G-regular local ring which does not have minimal multiplicity by the Hilbert series computation of Lemma 5.3.

(2) The ring

$$S=k[[x,y,z]]/(x^3-y^2z,y^3-x^2z,z^2-xy)$$

is a 1-dimensional Cohen-Macaulay non-Gorenstein G-regular local ring not having minimal multiplicity. Indeed, $S/zS=R$ is a G-regular local ring by (1), and z is a nonzerodivisor of S. Hence Proposition 4.2 shows that the local ring S is also G-regular.

The G-regular local rings constructed above are all Cohen-Macaulay. Now, let us construct an example of a non-Cohen-Macaulay G-regular local ring.

Example 5.5. Let us consider the local algebra

$$R = k[[x, y]]/(x^2, xy)$$

over a field k. This is a non-Cohen-Macaulay G-regular local ring.

In fact, suppose that R is not G-regular. Then there exists a nonfree totally reflexive R-module M. We can assume without loss of generality that M is indecomposable. The first syzygy $N = \Omega_R M$ of M is also a nonfree indecomposable totally reflexive R-module by [16, Proposition 7.1]. Note that there is a free R-module F such that N is contained in $\mathfrak{m}F$. Hence we have $xN \subseteq x\mathfrak{m}F = (x^2, xy)F = 0$. Thus N is an $R/(x)$-module. Since $R/(x) = k[[y]]$ is a principal ideal domain, the structure theorem (for finitely generated modules over a principal ideal domain) and the indecomposability of N show that N is isomorphic as an $R/(x)$-module to either $R/(x)$ or $R/(x, y^n)$ for some $n \geq 1$. But there is an exact sequence

$$0 \to k \to R \to R/(x) \to 0,$$

which implies that the R-module $R/(x)$ is of infinite G-dimension by Lemma 1.5(4) and (1). Also, we have $\mathrm{Hom}_R(R/(x, y^n), R) \cong (0 :_R (x, y^n)) = (x) \cong k$, which implies that $R/(x, y^n)$ is not a reflexive R-module for any $n \geq 2$. When $n = 1$, we have $R/(x, y^n) = k$, which has infinite G-dimension as an R-module by Lemma 1.5(1). Since the R-module N is totally reflexive, we get a contradiction, and we conclude that R is a G-regular local ring.

6. Some problems

Question 6.1. Let $R \to S$ be a flat local homomorphism of local rings. Suppose that both R and $S/\mathfrak{m}S$ are G-regular, where $\mathfrak{m}$ is the maximal ideal of R. Then is S also G-regular?

Partial answers to the above question have been obtained in Corollaries 4.4, 4.5 and 4.7.

It is natural to ask if a localization of a G-regular local ring at a prime ideal is G-regular or not. This does not have an affirmative answer in general, as we see in the following example.

Example 6.2. Let k be a field, and let

$$R = k[[x, y, z]]/(x^2, xz, yz).$$

The element $y - z$ is a nonzerodivisor of R, and we have $\mathfrak{m}^2 = (y - z)\mathfrak{m}$. Hence R is a 1-dimensional Cohen-Macaulay non-Gorenstein local ring with minimal multiplicity, so R is G-regular by Lemma 5.1. Localizing R at the prime ideal $\mathfrak{p} = (x, z)$, we have

$$R_\mathfrak{p} \cong k[[x, y]]_{(x)}/(x^2),$$

which is a singular Gorenstein local ring. Proposition 1.8(1) says that $R_\mathfrak{p}$ is not G-regular.

Let R be a local ring. An R-module M is called *bounded* if the set of the Betti numbers of M admits an upper bound. An R-module M is said to be *periodic of period* n, where n is a positive integer, if the nth syzygy $\Omega_R^n M$ is isomorphic to M. We just say that M is *periodic* if M is either free or periodic of period n for some integer $n \geq 1$. We say that M is *eventually periodic* if there exists an integer $r \geq 0$ such that $\Omega_R^r M$ is periodic.

A well-known theorem of Eisenbud [10] asserts that every bounded module over a complete intersection local ring is eventually periodic of period 2. To be precise, let S be a regular local ring, $\boldsymbol{x} = x_1, \dots, x_n$ an S-sequence, and $R = S/(\boldsymbol{x})$ the residue ring. Then Eisenbud's theorem says that any bounded R-module in $\mathcal{C}(R)$ is eventually periodic of period 2. The following question asks whether the G-regular version of this result holds.

Question 6.3. Let S be a G-regular local ring, $\boldsymbol{x} = x_1, \dots, x_n$ an S-sequence, and $R = S/(\boldsymbol{x})$ the residue ring. (Namely, let R be a "G-complete intersection.") Then are all bounded totally reflexive R-modules eventually periodic (of period 2)?

Let $(S, \mathfrak{n})$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ the residue ring. Then a celebrated theorem of Tate [20] asserts that the ideal I is principal if the residue field of R is bounded as an R-module. Combining this with Eisenbud's matrix factorization theorem, we see that I is a principal ideal if and only if every R-module in $\mathcal{C}(R)$ is bounded, if and only if every R-module in $\mathcal{C}(R)$ is periodic. The question below asks if the G-regular version of this holds.

Question 6.4. Let R be a local ring over which every totally reflexive module is periodic. Then (under some adequate assumptions) does there exist a G-regular local ring S and an S-regular element $f \in S$ such that $R \cong S/(f)$? (Namely, is R a "G-hypersurface"?)

A partial answer to this question can be found in [22, Theorem 4.2]. The converse statement holds by Theorem 2.10 and Proposition 2.4. To be precise, let S be a G-regular local ring, $f \in S$ an S-regular element, and $R = S/(f)$ the residue ring. Then every totally reflexive R-module is periodic.

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