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On Gale and Shapley "College Admissions and the Stability of Marriage"*

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ABSTRACT

In this note, we start to claim that established marriages can be heavily destabilized when the population of existing couples is enriched by the arrival of new candidates to marriage. Afterwards, we discuss briefly how stability concepts can be extended to account for entry and exit phenomena affecting the composition of the marriage market.

Keywords: Matching; Stability; Marriage Model; Divorce Cascades

1. Introduction

Marriage stability within a heterosexual and monogamist community formed by one set of men and one set of women is investigated by [1]. Stability (henceforth *G-S stability*) in the marriage market is satisfied when women and men are matched so that there are no woman and nor man who are not married to each other though they would prefer each other to their actual mates. [1] show that, in such a market, there always exists a stable marriage matching.[1]

Yet, divorces are a factual evidence and get more and more frequent! In this note, we start to claim that established marriages can be heavily destabilized when the population of existing couples is enriched by the arrival of new candidates to marriage. Using a striking example with n men and n women, we show that the entrance of a new couple in the marriage market completely disrupts, however large n may be, the unique stable matching prevailing before. Afterwards, we discuss briefly how stability concepts can be extended to account for entry and exit phenomena affecting the composition of the marriage market. [2] look at how stable matchings can be destabilized by new entrants. Our paper extends this discussion by introducing a measure of external stability.

2. Divorce Cascades: An Example

Consider a marriage market with the same number n of women and men. In this market, both women (as well as men) have homogeneous preferences over the set of potential mates. More precisely, both men and women can be ranked in such a way that if a man is higher in the ranking of men, then each woman strictly prefers him to all others ranked below him, and similarly for the preferences of men over the set of women. Clearly, the unique G-S stable matching for this market consists in matching i-th ranked man with the i-th ranked woman, $i = 1,\dots n$.

Consider now that a new single woman and a new single man arrive in this market. Moreover, assume that this new single women—the Beauty—is now ranked by all existing men at the top of the set of women, while the new single man—the Beast—is ranked by every woman at the bottom of the set of men. In other words, the entry of this couple is as if "*The Beauty and the Beast*" had entered the market! Of course now the "top man" starts **tobe** strongly interested in the Beauty and wishes to divorce from the past "top woman" to marry the Beauty. Thus, the past "top woman" now becomes available to the 2nd-ranked man, who prefers her to his current mate. Consequently, a new divorce occurs as he wishes to marry the past "top woman" who is now willing to accept him. And so on and so forth, until the past "bottom man" decides to divorce from the "bottom woman," who now, poor woman, is left with the Beast! Accordingly, and whatever large the number n, the entry of a single pair of persons of opposite sex entails a cascade of divorces and fully disrupts all the n previous G-S stable matches: this entry generates a domino effect, where the

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[1]This result illustrates a well-known French proverb, which says "Chaque casserole a son couvercle".

disruption of one couple automatically provokes the disruption of the next one in the ranking, until all previous matches are destroyed.

3. External Stability

The above example evokes the possibility of defining a concept of external stability related to a matching which is G-S stable. In this example, we start with a situation that is G-S stable and we slightly modify the market by allowing one further man and one further woman to enter the market. For the particular set-up of preferences introduced above, this marginal change entails a completely different new G-S stable matching in the resulting market. For this reason, we argue that the initial matching is externally unstable. For different specifications of preferences, the degree of external instability could be of course different, to the extent that the number of disrupted couples due to the entry can vary between 0 and n. This justifies the following definition.

Definition 1.

Given a marriage market and a G-S stable matching in this market, this matching is said to be *k-externally stable* whenever, at any G-S stable matching in the marriage market consisting of one further man and one further woman, at least k existing past matches are not disrupted in the new matching.

In the example above, the initial matching is clearly 0-*externally stable* since all existing initial matches are disrupted at the stable matching in the new marriage market. Now assume that the new man puts the new woman at the top of his ranking and vice versa, while each existing mate ranks the entrant of the opposite sex in the very bottom of his/her previous ordering. It is clear that this entry should not disrupt any existing couple in the initial G-S stable matching. Accordingly, in the last case, any G-S stable matching in the initial marriage market is *n-externally stable.*

The degree of external stability of G-S stable matchings clearly depends on the individual rankings of existing marriage mates. Given the richness of possible preference rankings, one can hardly make very general statements about the degree of external stability. Nevertheless, when restricting to more particular types of preference rankings, one might get some insights into this question. To illustrate this point, we consider in the following section, two specific categories of preference set-ups, namely, vertical and horizontal set-ups.

4. Vertical and Horizontal Set-Ups

In real life, individuals' preferences regarding potential marriage mates are highly subjective and heterogeneous. In this section, we illustrate that some preference set-ups engender marriage markets with higher degree of *k-external stability* than others, inspiring ourselves from the literature on product differentiation to contrast the cases of vertical and horizontal preference set-ups.

In the economic writings devoted to product differenttiation, it is usual to refer to economic contexts involving vertical and horizontal differentiation. In the case of vertical differentiation, consumers' preferences are such that consumers unanimously agree about goods' merits, ranking goods in the same manner when they are sold at the same price. In the case of horizontal differentiation, consumers are no longer unanimous with respect to goods' relative merits and, accordingly, when goods are sold at the same price, consumers do not rank them similarly: Some consumers prefer one variant to the other while the reverse is true for the others. A particular preference set-up meeting this condition corresponds to the case of single peaked preferences in which each consumer has a specific ideal variety, like in the Hotelling's Main Street example.

In this section, we transpose this approach to the marriage market, contrasting vertical and single peaked preference set-ups.

Definition 2.

A vertical set-up corresponds to a profile of preference relations over the potential marriage partners, such that, for every agent k, $k = 1, \cdots n$, we have $i \succ_k j \Leftrightarrow i > j$ for every $i, j = 1, \cdots n,\ \ i \neq j$.

Hence, the vertical set-up case corresponds to the set of preference relations adopted in our example, with both men and women being ranked in such a way that if a woman (man) is higher in the ranking of women (men), then each man (woman) strictly prefers her (him) to all others ranked below her (him).

Definition 3.

A single peaked preference set-up corresponds to a profile of preferences over the potential marriage partners such that, for every agent k, $k = 1, \cdots n$, we have $i \succ_k j \Leftrightarrow |k-i| < |k-j|$ and, whenever $|k-i| = |k-j|$, $i \succ_k j \Leftrightarrow i > j$ for every $i, j = 1, \cdots n,\ \ i \neq j$)

This definition simply transposes the usual notion of distance used in continuous models *à la Hotelling* to define preference relations characterized by the fact that the further distant an option is from the ideal option of an individual, the least preferred it is. In this case, the ranking of potential marriage mates is no longer unanimous and the top men and top women will differ from individual to individual. For example, this is the case when women (as well as men) are differentiated with respect to some characteristic and, consequently, they rank potential marriage partners according to the similarity with their own characteristic. This implies that each woman (resp. man) has her (his) specific individual ranking over marriage partners, with marriage mates whose characterristic is closer to her (his) own being better positioned in her (his) individual ranking. The following table illustrates the characteristics of single peaked preferences set-up, considering a marriage market with three men and three women:

$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$		$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$
m_1	m_2	m_3	and	w_1	w_2	w_3
m_3	m_3	m_2		w_3	w_3	w_2
m_2	m_1	m_1		w_2	w_1	w_1

Transposing this preference set-up to a marriage market with *n* women and *n* men and ranking men and women according to the characteristic with respect to which they differ, it can be easily seen that the unique G-S stable matching for this market consists in matching the *i*th-ranked man with the *i*th-ranked woman, $i = 1, \cdots n$.

Vertical and Horizontal Rankings: External Stability

In the previous section, we argued that the degree of external stability in marriage markets is determined by the nature of individuals' preference set-ups over potential marriage partners. In this section, we illustrate this point by contrasting the cases of vertical and single peaked preference set-ups, which exhibit substantially different properties in terms of external stability. In fact, single peaked preference set-ups produce stable matchings that are more externally-stable than the ones corresponding to vertical preference set-ups.

Let us consider a marriage market composed of a same number of men and women. Assume that preference relations over potential marriage mates are described by the vertical preference set-up. Finally, assume that an additional man and an additional woman enter this market. If all existing women rank this additional man between the existing men j and $j + 1$, while all existing men rank this additional woman between the existing women $j + h$ and $j + h + 1$, then it is easy to see that all the h marriages corresponding to the existing matches of individuals ranked in-between the positions of the new members of the market are now disrupted.

Accordingly, the degree of external stability of marriage markets with vertical preferences is determined by the relative positioning of the new woman and the new man on individuals' preference rankings after entry: the larger the gap in their positioning, the more externally-unstable such a marriage market. In the limit, when we consider the largest possible gap (n), as in the example of "The Beauty and the Beast", all the existing matches at the initial G-S stable matching are disrupted at the new G-S stable matching.

In contrast, single peaked preferences are substantially more stable. Independently of the relative positioning of the new members of the marriage market, the entrance of a new woman and a new man in the market disrupts, at-most, two matches of the initial G-S stable matching (all the remaining couples stay together at the new G-S stable matching). Hence, whereas in the case of vertical preference set-ups (where the degree of external in stability varies from 0 to n), in the case of single peaked preferences, the maximum degree of instability is equal to 2.

5. Conclusion

In this note, we have suggested that the notion of G-S stability in a marriage market could be advantageously complemented with a concept of stability related to the possibility of entry in this market. Starting from an illustrative example, we show how entry can heavily destabilize stablished marriages. We propose a tentative concept which could capture the extent of external stability of a marriage market.

REFERENCES

[1] D. Gale and L. S. Shapley, "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, Vol. 69, No. 1, 1962, pp. 9-15. doi:10.2307/2312726

[2] Y. Blum, A. E. Roth and U. G. Rothblum, "Vacancy Chains and Equilibration in Senior-Level Labor Markets," *Journal of Economic Theory*, Vol. 76, No. 2, 1997, pp. 362-411. doi:10.1006/jeth.1997.2307

