On Galois representations associated to Hilbert modular forms*

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Abstract

In this paper, we prove that, to any Hilbert cuspidal eigenform, one may attach a compatible system of Galois representations. This result extends the analogous results of Deligne and Deligne–Serre for elliptic modular forms. The principal work on this conjecture was carried out by Carayol and Taylor, but their results left one case remaining, which we complete in this paper. We also investigate the compatibility of our results with the local Langlands correspondence, and prove that whenever the local component of the automorphic representation is not special, then the results coincide.

Introduction

The Langlands philosophy suggests that, to every Hilbert cuspidal eigenform over a totally real field F, one should be able to attach a compatible system of 2-dimensional representations of $\mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$. When $\mathrm{F} = \mathbb{Q}$, this result is well-known, and is due to Deligne [3] and to Deligne–Serre [6]; Deligne's paper proved the result whenever the weight of the modular form is at least 2, and the paper of Deligne and Serre proved the result when the form has weight 1.

For Hilbert modular forms, the analogue of the study of Deligne was carried out by Carayol [2] whenever the weights (for now there is a vector of weights) are all at least 2, and $[\mathrm{F} : \mathbb{Q}]$ is odd (and also for many forms when F has even degree), and by Taylor [13] and Blasius–Rogawski [1] when all of the weights are at least 2 and F is a number field of even degree. In this paper, we attach compatible systems of Galois representations to forms for which some of the weights are 1. The method is to give congruences

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between the given Hilbert modular forms and those of higher weight, in a similar spirit to [15]. (The case where all of the weights are 1 is well-known, and due to Rogawski–Tunnell [11] and to Ohta [10].)

I should like to take this opportunity to thank Richard Taylor for suggesting this problem, and for his help during the time that the work on this paper was carried out.

1 The adelic viewpoint

We now introduce our notation, which largely follows that of Hida [7].

Let F denote a totally real number field of degree d over $\mathbb{Q}$ and let I denote the set $\{\tau_1, \dots, \tau_d\}$ of embeddings $\tau : \mathrm{F} \hookrightarrow \mathbb{R}$. We will be interested in Hilbert modular forms of weight k (a d-tuple), and level $\mathfrak{n}$ (an ideal of $\mathcal{O}_\mathrm{F}$). Let ∂ denote the different of F over $\mathbb{Q}$.

Let G denote the linear algebraic group $\mathrm{Res}_{\mathrm{F}/\mathbb{Q}}(\mathrm{GL}_2)$. Write $\mathbb{A}^\infty$ for the finite adeles of $\mathbb{Q}$.

We make the convention that, for a set A (possessing a suitable notion of positivity), A_+ will denote the totally positive elements of A except that we will use GL_2^+ to denote the elements of GL_2 with positive determinant.

We introduce some notation concerning the weight of a Hilbert modular form.

Let $t = (1, 1, \dots, 1) \in \mathbb{Z}^I$. We say that $m \in \mathbb{Z}^I$ is *parallel* if $m \in \mathbb{Z}.t$. Likewise, we say $m \in \mathbb{Z}^I$ is *parallel* to $n \in \mathbb{Z}^I$ if $m - n$ is parallel.

Fix $k \in \mathbb{Z}^I_{\geq 1}$ such that

$$k_{\tau_1} \equiv k_2 \equiv \cdots \equiv k_{\tau_d} \pmod 2.$$

Define $v \in \mathbb{Z}^I_{\geq 0}$ such that some $v_\tau = 0$, and $k + 2v$ is parallel. To fix the transformation law, we choose w parallel to $v + k$. (We will usually make the choice $v + k - t$ for w, but choose to work in more generality during this section.) Also write $\widehat{w}$ for $k - w$.

If $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$, and $u = u^\infty u_\infty \in G(\mathbb{A}^\infty) \times G(\mathbb{A}_\infty)_+$, then, for k and w as above, we may define the transform $f|_{k,w}u : G(\mathbb{A}) \longrightarrow \mathbb{C}$. Then, as in [7], we have the notion of a space of *modular* (resp. *cusp*) *forms*, $M_{k,w}(U)$ (resp. $S_{k,w}(U)$) for an open compact subgroup U of $G(\mathbb{A}^\infty)$.

The compact open subgroups U in which we will be most interested are the adelic analogues of the classical group $\Gamma_1(N)$.

Define $U_0 = \prod_\mathfrak{q} \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},\mathfrak{q}})$, where $\mathfrak{q}$ runs over the finite primes of F. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_\mathrm{F}$, and define:

$$U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0 \,\middle|\, c \in \mathfrak{n}, a - 1 \in \mathfrak{n} \right\},$$

$$V_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0 \middle| c \in \mathfrak{n}, d-1 \in \mathfrak{n} \right\}.$$

Definition 1.1 Denote by $S_{k,w}(\mathfrak{n})$ the space $S_{k,w}(U_1(\mathfrak{n}))$, and by $S^*_{k,w}(\mathfrak{n})$ the space $S_{k,\widehat{w}}(V_1(\mathfrak{n}))$. If f lies in one of these spaces, we may say that f has *level* $\mathfrak{n}$.

We define an algebra of Hecke operators as in [7]. If U and U' are open compact subgroups in $G(\mathbb{A}^\infty)$, and if $x \in G(\mathbb{A}^\infty)$, then we define the *Hecke operator*

$$[UxU'] : M_{k,w}(U) \longrightarrow M_{k,w}(U')$$

by $f \mapsto \sum f|_{k,w}x_i$, where $UxU' = \coprod Ux_i$.

Definition 1.2 When $U = U_1(\mathfrak{n})$, we define the following Hecke operators:

- If $\mathfrak{q}$ is a prime of F,

$$T_\mathfrak{q} = \left[U \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{q} \end{pmatrix} U \right],$$

where $\pi_\mathfrak{q}$ is an element of $\mathbb{A}_\mathrm{F}^\infty$ which is 1 everywhere except at $\mathfrak{q}$, where it is a uniformiser.

- If $\mathfrak{a}$ is a fractional ideal of F which satisfies $(\mathfrak{a}, \mathfrak{n}) = 1$, set

$$S_\mathfrak{a} = \left[U \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} U \right],$$

where $\alpha = \prod_\mathfrak{q} \pi_\mathfrak{q}^{v_\mathfrak{q}(\mathfrak{a})}$.

Define the *Hecke algebra* $\mathbb{T}_{k,w}(\mathfrak{n})$ to be the $\mathbb{Z}$-algebra in $\mathrm{End}(S_{k,w}(\mathfrak{n}))$ generated by the Hecke operators $T_\mathfrak{q}$ for $\mathfrak{q}$ a prime of F and the operators $S_\mathfrak{a}$ for $\mathfrak{a}$ an integral ideal of F prime to $\mathfrak{n}$.

Lemma 1.3 ([7], Proposition 2.3) *There is a natural isomorphism*

$$\begin{array}{rcl} S_{k,w}(U) & \cong & S_{k,\widehat{w}}(U^\iota) \\ f(x) & \mapsto & f^*(x) = f(x^{-\iota}) \end{array}$$

where ι denotes the main involution on GL_2. Furthermore, this isomorphism satisfies the following relation:

$$(f|[UxU])^* = f^*|[U^\iota x^{-\iota} U^\iota]$$

for all $x \in U$.

In particular, we define the Hecke operators on $S^*_{k,w}(\mathfrak{n})$ to be compatible with this formula, and define the Hecke algebra $\mathbb{T}^*_{k,w}(\mathfrak{n})$ to be the $\mathbb{Z}$-algebra in $\mathrm{End}(S^*_{k,w}(\mathfrak{n}))$ generated by the operators $T_\mathfrak{q}$ and $S_\mathfrak{a}$, where $\mathfrak{q}$ and $\mathfrak{a}$ run through the same indexing set as in the definition of $\mathbb{T}_{k,w}(\mathfrak{n})$.

Finally, we define the operator $< \mathfrak{a} >$ on $M^*_{k,w}(\mathfrak{n})$ as a scalar multiple of $S_\mathfrak{a}$. To do this, we give the following identity (see [7],(3.9)) of operators on $M^*_{k,w}(\mathfrak{n})$:

$$S_\mathfrak{a} = N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})^{[2w-k]} < \mathfrak{a} >$$

where, if $r \in \mathbb{Z}.t$, we write $r = [r].t$.

2 The classical viewpoint

Although the adelic viewpoint seems the most convenient in which to introduce the notion of Hecke operator, we will also need to discuss congruences. For this, a classical approach seems easier.

We say that a subgroup Γ of $\mathrm{GL}_2(\mathrm{F}) \cap G(\mathbb{A}^\infty)G(\mathbb{A}_\infty)_+$ is a *congruence subgroup* if it contains

$$\Gamma_\mathfrak{n} = \{\gamma \in \mathrm{SL}_2(\mathcal{O}_\mathrm{F}) | \gamma - I_2 \in \mathfrak{n}.\mathrm{M}_2(\mathcal{O}_\mathrm{F})\}$$

for some integral ideal $\mathfrak{n}$, and $\Gamma/(\Gamma \cap \mathrm{F})$ is commensurable with $\mathrm{SL}_2(\mathcal{O}_\mathrm{F})/\{\pm 1\}$.

If $f : \mathfrak{h}^I \longrightarrow \mathbb{C}$, and $\gamma \in \mathrm{GL}_2^+(\mathrm{F})$, we have a transform $f\|_{k,w}\alpha : \mathfrak{h}^I \longrightarrow \mathbb{C}$ leading to definitions (see [7]) of *modular* (resp. *cusp*) *forms*, $M_{k,w}(\Gamma)$ (resp. $S_{k,w}(\Gamma)$) for congruence subgroups Γ. A modular form has a Fourier expansion

$$f(z) = \sum_\xi a(\xi, f) e_\mathrm{F}(\xi z)$$

where $a(\xi, f) \in \mathbb{C}$ and $e_\mathrm{F}(\xi z) = \exp(2\pi i \sum_{i=1}^d \xi^{\tau_i} z_i)$. ξ runs over 0 (if f is not a cusp form) and all totally positive elements of a lattice in F.

We indicate the relationship between the two definitions of Hilbert modular forms given thus far; in particular, we now begin to discuss the adelic q-expansion, integrality and congruences.

Let h be the number of ideal classes of F mod $\mathcal{P}_\infty$, the product of all of the archimedean primes.

Take h elements $t_1, \ldots, t_h$ of $\mathbb{A}^*_\mathrm{F}$ such that $t_i^\infty \in \widehat{\mathcal{O}}_\mathrm{F}$ $(= \mathcal{O}_\mathrm{F} \otimes_\mathbb{Z} \widehat{\mathbb{Z}})$, and $t_1\mathcal{O}_\mathrm{F}, \ldots, t_h\mathcal{O}_\mathrm{F}$ form a complete set of representatives for such ideal classes; note that, by the Strong Approximation Theorem, we can choose t_i such that $(t_i)_S = 1$ for any finite set of places. We will always insist that all infinite places lie in S; in addition, we will always include all of the primes dividing the level. Write

$$x_i = \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}, \qquad x_i^{-\iota} = \begin{pmatrix} 1 & 0 \\ 0 & t_i^{-1} \end{pmatrix}.$$

Let E denote the set of totally positive units in $\mathcal{O}_{\mathrm{F}}$. Define

$$\Gamma_i(\mathfrak{n}) = x_i^{-\iota} E.U_1(\mathfrak{n}) G(\mathbb{A}_\infty)_+ x_i^{\iota} \cap G(\mathbb{Q}) = x_i E.V_1(\mathfrak{n}) G(\mathbb{A}_\infty)_+ x_i^{-1} \cap G(\mathbb{Q}).$$

We have canonical isomorphisms ([7]):

$$\begin{aligned} S_{k,w}(\mathfrak{n}) &\longrightarrow \bigoplus_{i=1}^{h} S_{k,w}(\Gamma_i(\mathfrak{n})) \\ S^*_{k,w}(\mathfrak{n}) &\longrightarrow \bigoplus_{i=1}^{h} S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n})) \end{aligned}$$

Conversely, given an h-tuple $(f_1, \ldots, f_h) \in \bigoplus_{i=1}^{h} M_{k,\widehat{w}}(\Gamma_i(\mathfrak{n}))$, the associated function $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$ in $M^*_{k,w}(\mathfrak{n})$ is given by:

$$f(\alpha x_i \gamma) = (f_i \|_{k,w} \gamma_\infty)(z_0) \text{ for all } \alpha \in G(\mathbb{Q}), \gamma \in U_1(\mathfrak{n}) G(\mathbb{A}_\infty)_+.$$

Then, in particular,

$$f(x_i y) = (f_i \|_{k,w} y)(z_0) \text{ for all } y \in G(\mathbb{A}_\infty)_+.$$

A routine verification, using this last formula, shows that $(fg)_i = f_i g_i$.

It is this correspondence which allows us to form the adelic q-expasion; see [12] and [7], §4. We will identify the q-expansion of a modular form $f \in M^*_{k,w}(\mathfrak{n})$ with the h Fourier expansions of the corresponding $f_i \in M_{k,\widehat{w}}(\Gamma_i(\mathfrak{n}))$. This means that f will have h constant terms in its Fourier expansion, and one term corresponding to all other totally positive ideals. It is easy to see that the Fourier expansions of f and g, two adelic modular forms, will multiply together to give the Fourier expansion of fg (as this is true for each of the component modular forms on $\Gamma_i(\mathfrak{n})$).

As $\Gamma_i(\mathfrak{n})$ contains elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for all $b \in (t_i \mathcal{O}_{\mathrm{F}})$, f_i has a Fourier expansion of the form

$$f_i(z) = \sum_{\xi \in (t_i^{-1} \partial^{-1})_+ \cup \{0\}} a_i(\xi) e_{\mathrm{F}}(\xi z).$$

Remark 2.1 For elliptic modular forms, the Fourier coefficients of an eigenform generate a number field. For Hilbert modular forms, this is only necessarily true ([12], 2.8, 2.9) when $k_{\tau_1}, \ldots, k_{\tau_d}$ all have the same parity, as we insisted earlier.

We begin now to discuss questions of integrality and congruence.

Definition 2.2 Denote by $\Phi(v)$ the subfield of $\overline{\mathbb{Q}}$ generated by

$$x^v = \prod_{\tau} \tau(x)^{v_\tau}$$

for all $x \in \mathrm{F}$. Denote its ring of integers by $\mathcal{O}(v)$.

The character

$$(-)^v : \mathrm{F}^\times \longrightarrow \Phi(v)^\times$$

extends by continuity to a character on $\mathbb{A}_\mathrm{F}^\times$, which we denote in the same way.

Definition 2.3 Let A be an $\mathcal{O}(v)$-algebra contained inside $\mathbb{C}$. If, for all $x \in \mathbb{A}_\mathrm{F}^\infty$, the ideal $(x^v\mathcal{O}(v))A$ is generated by a single element in A, we say A satisfies the *Hida condition* (after [7],(3.1)). A number field containing $\Phi(v)$ will be said to satisfy the *Hida condition* if its ring of integers satisfies the Hida condition.

When dealing with questions of A-integrality, we will always insist that A satisfies the Hida condition. If k is parallel, so that $v = 0$, this imposes no restrictions on the rings A that we may consider; but, in any case, for every number field K, there is a finite extension K_0 of K satisfying the Hida condition.

Definition 2.4 For $\mathfrak{q}$ a prime ideal of $\mathcal{O}_\mathrm{F}$, we fix, once and for all, a choice $\{\mathfrak{q}^v\}$ of generator in A of the ideal x^vA, where $\mathfrak{q} = x\mathcal{O}_\mathrm{F}$. If $\mathfrak{a}$ is any ideal, define $\{\mathfrak{a}^v\}$ by $\prod_\mathfrak{q}\{\mathfrak{q}^v\}^{v_\mathfrak{q}(\mathfrak{a})}$.

As in [7], 4.1 (but with $w = k + v - \epsilon.t$ for *any* $\epsilon \in \mathbb{Z}$), a form $f \in S^*_{k,w}(\mathfrak{n})$ has a Fourier expansion:

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y^\infty|_\mathbb{A}^\epsilon y_\infty^{k-w} \sum_{\xi \in \mathrm{F}_+^\times} a(\xi y\partial, f)\{(\xi y\partial)^v\}\xi^{-v} e_\mathrm{F}(\sqrt{-1}\xi y_\infty) e_\mathrm{F}(\xi x),$$

where $a(\xi y\partial, f) = 0$ unless $\xi y\partial$ is an integral ideal. There are no non-cuspidal modular forms if k is not parallel, but even in this case, one has an adelic q-expansion ([12]) which takes account of the constant terms.

Definition 2.5 The space of A-*integral* (adelic) cusp forms will be defined as:

$$S^*_{k,w}(\mathfrak{n};A) = \{f \in S^*_{k,w}(\mathfrak{n}) | a(\mathfrak{a}, f) \in A \text{ for each integral ideal } \mathfrak{a}\}.$$

As in [7], 4.3, we then have:

$$f_i(z) = c_{v,i} \sum_{\xi \in (t_i^{-1}\partial^{-1})_+} a(\xi t_i \partial, f)\{\xi^v\}\xi^{-v} e_{\mathrm{F}}(\xi z),$$

where $c_{v,i} = N_{\mathrm{F}/\mathbb{Q}}(t_i)^{-\epsilon}\{(t_i\partial)^v\}$.

Hida [7], 4.12, proves that if A is an integrally closed domain containing $\mathcal{O}(v)$ satisfying the Hida condition, and such that if A is finite flat over $\mathcal{O}(v)$ or $\mathbb{Z}_\ell$ for some prime ℓ, then $S^*_{k,w}(\mathfrak{n};A)$ is stable under the action of $< \mathfrak{a} >$ for every integral ideal $\mathfrak{a}$ prime to $\mathfrak{n}$, at least in the case $w = k + v - t$, the only case in which we shall use this result.

Definition 2.6 The space of A-*integral* cusp forms $S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n});A)$ on $\Gamma_i(\mathfrak{n})$ consists of the forms in $S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n}))$ whose Fourier expansion coefficients lie in A.

Notice that now

$$S^*_{k,w}(\mathfrak{n};A) = \bigoplus_{i=1}^{h} c_{v,i} S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n});A),$$

as $\{\xi^v\}\xi^{-v}$ is a unit in A.

We define K-rationality similarly for number fields $K/\Phi(v)$ satisfying the Hida condition. We remark that $S^*_{k,w}(\mathfrak{n};K)$ is stable under the action of the Hecke operators $T_\mathfrak{q} \in \mathbb{T}^*_{k,w}(\mathfrak{n})$, at least when $w = k + v - t$ (this is [7], 4.8).

We finally conclude this section with a brief discussion of congruence.

Take a number field $K \supset \Phi(v)$ satisfying the Hida condition. Write $\mathcal{O}_K$ for its ring of integers. For each integral ideal α of K, we say that Hilbert modular forms f and g in $M^*_{k,w}(\mathfrak{n};K)$ are *congruent* modulo α if, for all integral ideals $\mathfrak{a}$ of F, $a(\mathfrak{a}, f) - a(\mathfrak{a}, g) \in \alpha$, that is to say, if the Fourier expansions are congruent modulo α. Likewise, we say that f is a *modulo* α *eigenfunction* of the Hecke operator T if there exists $a \in \mathcal{O}_K$ such that $f|_{k,w}T \equiv a.f \pmod{\alpha}$.

3 The main conjecture

The following conjecture is well-known.

Conjecture 3.1 *Let f be a Hilbert cuspidal eigenform in $S^*_{k,w}(\mathfrak{n}; K)$ where K is a field containing $\Phi(v)$, and let λ be a prime of $\mathcal{O}_K$ lying above a prime ℓ of $\mathbb{Z}$. Then there is a continuous representation:*

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathcal{O}_{K,\lambda}),$$

which is unramified outside $\mathfrak{n}\ell$, and such that if $\mathfrak{q}$ is a prime of $\mathcal{O}_\mathrm{F}$ not dividing $\mathfrak{n}\ell$, then

$$\begin{aligned} \operatorname{tr}\rho_\lambda(\mathsf{Frob}_\mathfrak{q}) &= \theta_f(T_\mathfrak{q}) \\ \det\rho_\lambda(\mathsf{Frob}_\mathfrak{q}) &= \theta_f(S_\mathfrak{q})N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}), \end{aligned}$$

where $\theta_f(T)$ denotes the eigenvalue of the Hecke operator T on f. Furthermore, for all primes $\mathfrak{q}\not|\ell$, the restriction of ρ_λ to the local Weil group at $\mathfrak{q}$ should coincide with the λ-adic representation $\sigma^\lambda(\pi_\mathfrak{q})$ attached by the local Langlands correspondence to the $\mathfrak{q}$-component of the automorphic representation corresponding to f.

We recall that this was proven by Carayol [2], when d is odd and $k \geq 2t$ (and also for many forms when d is even). The method is essentially geometric, involving a deep analysis of the cohomology groups attached to certain Shimura curves. Taylor ([13]) completed the proof for all forms with $k \geq 2t$ and d even, using a congruence method.

Earlier, several authors had obtained partial results, notably Ohta and Blasius-Rogawski (whose method complements that of Taylor, by constructing the representations geometrically). Both describe the representations at good primes. Hida [8] has attached systems of Galois representations to homomorphisms from his universal nearly ordinary Hecke algebras. These algebras are universal for $k \geq 2t$, and thus his results specialise to the results of Ohta and Blasius-Rogawski. He is also able to analyse the behaviour of the representations at bad primes if the forms are ordinary.

When $k = t$, the conjecture is also known, and is due to Ohta [10] and Rogawski-Tunnell [11], but the case where some, but not all, of the components of k are 1 seems unresolved. In the remainder of the paper, we develop a method to solve this case, and to prove the conjecture above for these forms, except at primes where $\pi_\mathfrak{q}$ is special. The method involves congruences, as no motivic description of the representations seems to be known at the current time.

Finally, we remark that Ribet, in a letter to Carayol (unpublished), has proven that if the representations exist, then they are irreducible.

4 Construction of a certain modular form

In order to prove the first part of Conjecture 3.1 for all Hilbert cuspidal eigenforms, we will try to exploit congruence methods, as in Taylor [13]. However, we will construct the congruences between forms of partial weight one, and forms of a higher weight. We will do this by finding a suitable modular form congruent to 1 modulo λ, and multiplying by suitable powers of this form.

Several authors have constructed forms with some of the properties that we need, and to prove everything that we require, we will adapt a form constructed by Hida, in [16], 1.4.2. Indeed, we will copy the proof almost exactly, but will make a slight adaptation towards the end.

Definition 4.1 Fix, as indicated in §2, elements $t_1, \ldots, t_h \in \mathbb{A}_{\mathrm{F}}^{\times}$ such that

- $t_i^{\infty} \in \widehat{\mathcal{O}}_{\mathrm{F}}$,
- $t_1\mathcal{O}_{\mathrm{F}}, \ldots, t_h\mathcal{O}_{\mathrm{F}}$ form a complete set of representatives for the ideal classes of F mod $\mathcal{P}_\infty$,
- $(t_i)_{N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})\ell\partial\infty} = 1$.

Then define t_i^* by the relation

$$t_i^{-1}\partial^{-1} = t_i^*\mathcal{O}_{\mathrm{F}}.$$

Note that $(t_i^*)_\infty = 1$.

Definition 4.2 Let $\mathfrak{a}$ be a fractional ideal and $\mathfrak{b}$ be an integral ideal of $\mathcal{O}_{\mathrm{F}}$. Then define the subgroup

$$\Gamma(\mathfrak{a}, \mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathrm{F}) \middle| a, d \in \mathcal{O}_{\mathrm{F}}, b \in \mathfrak{a}^{-1}, c \in \mathfrak{ab}, ad - bc \in \mathcal{O}_{\mathrm{F}}^{\times} \right\}.$$

Remark 4.3 It is easy to see that $\Gamma(t_i^{-1}\mathcal{O}_{\mathrm{F}}, \mathfrak{c}) = \Gamma_i(\mathfrak{c})$ for any integral ideal $\mathfrak{c}$.

Theorem 4.4 *There exists a modular form*

$$E \in M^*_{\kappa,\kappa}(\mathfrak{c}; \mathbb{Z}_\ell)$$

satisfying the following conditions:

1. $\kappa = 2^s(\ell - 1).t \in \mathbb{Z}^I$ *for some* $s \geq 1$.

2. *$\mathfrak{c}$ may be chosen to be any one of infinitely many prime ideals of $\mathcal{O}_{\mathrm{F}}$ coprime to ∂.*

3. *$E \equiv 1$ (mod ℓ).*

4. *$E| < \mathfrak{a} >= E$ for all integral ideals $\mathfrak{a}$ prime to $\mathfrak{c}$.*

Proof. For this, we follow Hida's proof of [16], 1.4.2, very closely, but with a couple of minor alterations.

If ℓ is odd, we let $\mathrm{K} = \mathrm{F}(\zeta_\ell)^+$, and $\mathrm{M} = \mathrm{F}(\zeta_\ell)$, and if $\ell = 2$, we let $\mathrm{K} = \mathrm{F}$, and $\mathrm{M} = \mathrm{F}(\sqrt{-1})$. Then, if $\mathfrak{b}$ denotes an integral ideal of $\mathcal{O}_{\mathrm{M}}$, and $\xi \in \mathrm{K}_+$, we form the following theta series:

$$g'(z) = \sum_{\alpha \in \mathfrak{b}^{-1}} e^{\pi i \mathrm{tr}(\xi \alpha \overline{\alpha} z)}.$$

Let U denote the set $\{u \in (\mathcal{O}_{\mathrm{F}}^\times)_+\}/\{u\overline{u}|u \in \mathcal{O}_{\mathrm{M}}^\times\}$, of order 2^{s-1}, say.

Form

$$g(z) = \prod_{u \in U} g'(uz).$$

Then Shimura has proven ([16]) that $g^2 \in M^K_{2^s,0}(\Gamma(\mathfrak{c}, D))$ is a modular form (over K) associated to the group $\Gamma(\mathfrak{c}, D)$, where $D = \mathrm{Disc}(\mathrm{M}/\mathrm{K})$, and where

$$\mathfrak{c} = (\xi) N_{\mathrm{M}/\mathrm{K}}(\mathfrak{b}^{-1}) \partial_{\mathrm{K}}.$$

To obtain a form over F of the desired weight, restrict g^2, and raise to a suitable power t. Note that such a form must be congruent to 1 modulo ℓ exactly as in [16].

For $i = 1, \ldots, h$, we will construct a form E_i over F of the desired weight using this recipe.

For infinitely many prime ideals $\mathfrak{q}$ of $\mathcal{O}_{\mathrm{K}}$, we can find suitable $\mathfrak{b}_i$ and ξ_i so that

$$\mathfrak{c}_i = t_i^* \partial \mathfrak{q}.$$

Let $E_i = g_i^{2t(i)}$ be the form on F of the desired weight. E_i is, by Shimura's result, a form in

$$M_{\kappa,0}(\Gamma(t_i^* \partial, \mathfrak{q}_0)) = M_{\kappa,0}(\Gamma(t_i^{-1} \mathcal{O}_{\mathrm{F}}, \mathfrak{q}_0)) = M_{\kappa,0}(\Gamma_i(\mathfrak{q}_0)),$$

where $\mathfrak{q}_0 = \mathfrak{q} \cap \mathcal{O}_{\mathrm{F}}$. Fix $\mathfrak{c}$ to be such a $\mathfrak{q}_0$.

We choose $\mathfrak{b}_i$ and ξ_i in an arbitrary manner (so as to get a form of level $\mathfrak{c}$), as i runs through a set of representatives for C/C^2, where C is the group of strict ideal classes. If

$$t_i \mathcal{O}_{\mathrm{F}} = (t_j \mathcal{O}_{\mathrm{F}})(t_k \mathcal{O}_{\mathrm{F}})^{-2},$$

then we choose $\mathfrak{b}_j = \mathfrak{b}_i t_k^{-1}$ and $\xi_j = \xi_i \delta^{-1}$, where, as in [16], 1.4.2, δ is a totally positive generator of the ideal $(t_i^{-1} t_j t_k^2)$. Now take the corresponding E_j, and let E denote the adelic modular form corresponding to the h-tuple $\{E_1, \ldots, E_h\}$.

To prove that $E|_{\kappa,0} < \mathfrak{a} >= E$ for all integral ideals $\mathfrak{a}$ prime to $\mathfrak{c}$, one calculates exactly as in [7], §4.

One computes $E|S_{\mathfrak{a}}(x)$ as $\left(E \Big|_{\kappa,0} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^{-1} \right)(x)$, where a denotes the finite idele corresponding to $\mathfrak{a}$. Writing

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

as in [7], one first computes the effect of the matrix $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ on each E_i (using the results of [7], 4.9, and [5], §7, to compute the effect of this matrix on theta series), and then the effect of the matrix $\begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}$ exactly as in the proof of [7], 4.11. We omit the details of this calculation.

5 Towards the conjecture

Our method, which will resemble that of [13] and [15] in certain aspects, is based on a congruence argument. We will use the form constructed above to provide congruences between the given form of partial weight one, and forms of higher weight.

To prove Conjecture 3.1, let $f \in S^*_{k,w}(\mathfrak{n}; K)$, as above, and let λ be a prime of K.

At this stage, we will fix $w = k + v - t$; for more general w, one may twist (see [2], §3) to reduce the problem to this situation.

We first exploit the remark of Hida that every number field $K/\Phi(v)$ possesses an extension of finite degree which satisfies the Hida condition. We may thus, enlarging K if necessary, assume that K satisfies the Hida condition. This allows us to speak of integral forms.

In fact, one can show that the integral forms span the rational forms, that is,

$$S^*_{k,w}(\mathfrak{n}; \mathcal{O}_K) \otimes_{\mathcal{O}_K} K = S^*_{k,w}(\mathfrak{n}; K).$$

Equivalently, the denominators in the Fourier expansion of a rational form are bounded.

To see this, note that one can also define a slightly larger space of Hilbert modular forms as functions on the moduli space of Hilbert-Blumenthal abelian varieties with given polarisation and level structure. We omit the details, but refer the reader to [7] or [9]. One obtains the q-expansion of such Hilbert modular forms by evaluation at a "generalised Tate curve", a scheme defined over an integral ring Z (defined in [9]). It follows that K-rational Hilbert modular forms in this definition have their q-expansion lying in $Z \otimes K$, and thus have bounded denominators. As there is a q-expansion preserving embedding from $S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n}); K)$ to some space of these geometrically defined Hilbert modular forms ([7], (4.14)), it follows that the q-expansions of forms in $S_{k,\widehat{w}}(\Gamma_i(\mathfrak{n}); K)$ have bounded denominators, and thus the same result holds for $S^*_{k,w}(\mathfrak{n}; K)$.

Thus we may work with $f \in S^*_{k,w}(\mathfrak{n}; \mathcal{O}_K)$.

Then, with the form E as above, we have:

$$f \in S^*_{k,w}(\mathfrak{n}; \mathcal{O}_K) \hookrightarrow S^*_{k,w}(\mathfrak{n}; \mathcal{O}_{K,\lambda}),$$

$$E \in M^*_{\kappa,\kappa}(\mathfrak{c}; \mathbb{Z}_\ell) \hookrightarrow M^*_{\kappa,\kappa}(\mathfrak{c}; \mathcal{O}_{K,\lambda}).$$

Recall that we may choose $\mathfrak{c}$ to be one of infinitely prime ideals of $\mathcal{O}_{\mathrm{F}}$–we shall always insist that $(\mathfrak{c}, \mathfrak{n}\partial) = 1$.

Let $A = \mathcal{O}_{K,\lambda}$, and let m be the maximal ideal of A.

Definition 5.1 Write $f_n = f.E^{\ell^n} \in S^*_{k_n,w_n}(\mathfrak{m})$, where $k_n = k + \kappa\ell^n$, $w_n = w + \kappa\ell^n$ and $\mathfrak{m} = \mathfrak{n}\mathfrak{c}$.

Lemma 5.2 $f_n \equiv f \pmod{\mathsf{m}^{n+1}}$.

Proof. It is clear that

$$E^{\ell^n} \equiv 1 \pmod{\mathsf{m}^{n+1}}.$$

Then

$$a_i(\xi, f_n) = \sum_{\zeta+\theta=\xi} a_i(\zeta, f) a_i(\theta, E^{\ell^n}),$$

and we conclude that

$$\begin{aligned} c_{v,i} a(\xi t_i \partial, f_n)\{\xi^v\}\xi^{-v} &= \sum_{\zeta+\theta=\xi, \theta\neq 0} c_{v,i} a(\zeta t_i \partial, f)\{\zeta^v\}\zeta^{-v} a_i(\theta, E^{\ell^n}) \\ &\qquad + c_{v,i} a(\xi t_i \partial, f)\{\xi^v\}\xi^{-v} a_i(0, E^{\ell^n}). \end{aligned}$$

But

$$\begin{aligned} a_i(0, E^{\ell^n}) &\equiv 1 \pmod{\mathsf{m}^{n+1}} \\ a_i(\theta, E^{\ell^n}) &\equiv 0 \pmod{\mathsf{m}^{n+1}} \qquad \text{for } \theta \neq 0. \end{aligned}$$

Also, $\{\zeta^v\}\zeta^{-v}$ is a unit in A for all ζ. Combining all of these facts (first dividing by $c_{v,i}$), we obtain

$$a(\xi t_i \partial, f_n) \equiv a(\xi t_i \partial, f) \pmod{\mathsf{m}^{n+1}},$$

and so

$$f_n \equiv f \pmod{\mathsf{m}^{n+1}},$$

as required.

The crucial proposition is the following:

Proposition 5.3 *For every operator T of the form $S_{\mathfrak{q}}$ with $\mathfrak{q} \not| \mathfrak{m}\ell$, or $T_{\mathfrak{q}}$ with either $\mathfrak{q} \not| \mathfrak{m}\ell$ or $\mathfrak{q}|\mathfrak{n}$ and $\mathfrak{q} \not| \ell$, we have*

$$f|T \equiv f_n|T \ (mod\ \mathsf{m}^{n+1}).$$

Proof. Obviously

$$f| < \mathfrak{a} > \equiv f| < \mathfrak{a} > \pmod{\mathsf{m}^{n+1}},$$

where $\mathfrak{a}$ is an integral ideal prime to $\mathfrak{m}\ell$. We noted earlier that $f| < \mathfrak{a} >$ remained integral. By the same method as the proof of Lemma 5.2, we obtain

$$f| < \mathfrak{a} > .E^{\ell^n} \equiv f| < \mathfrak{a} > \pmod{\mathsf{m}^{n+1}}.$$

Then, using Theorem 4.4,

$$f| < \mathfrak{a} > .(E| < \mathfrak{a} >)^{\ell^n} \equiv f| < \mathfrak{a} > \pmod{\mathsf{m}^{n+1}}.$$

As the map $f \mapsto f| < \mathfrak{a} >$ is a homomorphism, we conclude that

$$f_n| < \mathfrak{a} > \equiv f| < \mathfrak{a} > \pmod{\mathsf{m}^{n+1}}.$$

Next, we see that $\ell - 1|[\kappa]$, and so

$$N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})^{[\ell^n \kappa]} \equiv 1 \pmod{\mathsf{m}^{n+1}}.$$

Then

$$N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})^{[k+2v-t+\ell^n\kappa]} f_n| < \mathfrak{a} > \equiv N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})^{[k+2v-t]} f| < \mathfrak{a} > \pmod{\mathsf{m}^{n+1}}.$$

But $N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})S_{\mathfrak{a}} = N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})^{[2w-k+t]} < \mathfrak{a} >$ as operators on $S^*_{k,w}$. Hence

$$N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a}) f_n|S_{\mathfrak{a}} \equiv N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a}) f|S_{\mathfrak{a}} \pmod{\mathsf{m}^{n+1}},$$

which gives the first assertion of the proposition (as $N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})$ is a unit when $\mathfrak{q}\not|\ell$).

It remains to demonstrate the result for the Hecke operators $T_\mathfrak{q}$, with $\mathfrak{q}\not|\ell$ and $\mathfrak{q}|\mathfrak{n}$ or $\mathfrak{q}\not|\mathfrak{m}$. But we have (using a result of Hida [7], 4.2):

$$\begin{aligned} a(\mathfrak{b}, f|T_\mathfrak{q})\{\mathfrak{q}^v\} &= \sum_{\substack{\mathfrak{b}+\mathfrak{q}\subset\mathfrak{a}\\ \mathfrak{a}+\mathfrak{n}=\mathcal{O}_\mathrm{F}}} N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})a(\mathfrak{b}\mathfrak{q}\mathfrak{a}^{-2}, f|S_\mathfrak{a})\{\mathfrak{a}^{2v}\} \\ &= \sum_{\substack{\mathfrak{b}+\mathfrak{q}\subset\mathfrak{a}\\ \mathfrak{a}+\mathfrak{m}=\mathcal{O}_\mathrm{F}}} N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})a(\mathfrak{b}\mathfrak{q}\mathfrak{a}^{-2}, f|S_\mathfrak{a})\{\mathfrak{a}^{2v}\} \\ &\equiv \sum_{\substack{\mathfrak{b}+\mathfrak{q}\subset\mathfrak{a}\\ \mathfrak{a}+\mathfrak{m}=\mathcal{O}_\mathrm{F}}} N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{a})a(\mathfrak{b}\mathfrak{q}\mathfrak{a}^{-2}, f_n|S_\mathfrak{a})\{\mathfrak{a}^{2v}\} \pmod{\mathsf{m}^{n+1}} \\ &\equiv a(\mathfrak{b}, f_n|T_\mathfrak{q})\{\mathfrak{q}^v\} \pmod{\mathsf{m}^{n+1}}, \end{aligned}$$

the second equality holding precisely because $\mathfrak{q}|\mathfrak{n}$ or $\mathfrak{q}\not|\mathfrak{m}$. Thus we conclude that

$$f|T_\mathfrak{q} \equiv f_n|T_\mathfrak{q} \pmod{\mathsf{m}^{n+1}},$$

(as $\{\mathfrak{q}^v\}\in A^\times$) as required.

We now begin to prove Conjecture 3.1 in the case where f is a Hilbert modular form of partial weight 1.

6 The main theorem

The form of the following theorem is taken from Taylor [14].

Theorem 6.1 *Let $f\in S^*_{k,w}(\mathfrak{n};\mathcal{O}_K)$ be a Hilbert cuspidal eigenform of partial weight one. Then, if λ is a prime of $\mathcal{O}_K$, there exists a continuous representation*

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathcal{O}_{K,\lambda}),$$

which is unramified outside $\mathfrak{n}\ell$, and satisfying

1. $\det\rho_\lambda = \chi$, *where χ is the continuous character unramified outside $\mathfrak{n}\ell$, which is defined by*

$$\chi(\mathsf{Frob}_\mathfrak{q}) = \theta_f(S_\mathfrak{q})N_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})$$

for each prime $\mathfrak{q}$ of F not dividing $\mathfrak{n}\ell$, and where $\theta_f(T)$ denotes the eigenvalue of T on the eigenform f.

2. *if $\mathfrak{q}\not|\mathfrak{n}\ell$ is a prime of F, then*

$$\mathrm{tr}\,\rho_\lambda(\mathsf{Frob}_\mathfrak{q}) = \theta_f(T_\mathfrak{q}).$$

3. *if* $\mathfrak{q}$ *is a prime of* F *such that* $\mathfrak{q}\not|\ell$ *but* $\mathfrak{q}|\mathfrak{n}$, *then either*
   (a) $\theta_f(T_\mathfrak{q}) = 0$, *or*
   (b) *for every* $\sigma_\mathfrak{q}$ *in the decomposition group at* $\mathfrak{q}$ *lying above* $\mathsf{Frob}_\mathfrak{q}$, *we have:*
$$\operatorname{tr}\rho_\lambda(\sigma_\mathfrak{q}) = \theta_f(T_\mathfrak{q}) + \chi(\sigma_\mathfrak{q})\theta_f(T_\mathfrak{q})^{-1}.$$

Proof. Recall that a pseudorepresentation in the sense of Wiles [16] is essentially the trace of a 2-dimensional representation. A nice summary of the main definitions and results of the theory of pseudorepresentations is given in [13], §2.

The results of Carayol and Taylor referred to above imply (see [13], Proposition 1) that there is a continuous pseudorepresentation

$$r_n : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathbb{T}_{k_n,w_n}(\mathfrak{m}) \otimes \mathbb{Z}_\ell,$$

where Σ denotes the set of primes of F dividing $\mathfrak{m}\ell$, such that

$$\begin{array}{rcll} \operatorname{tr} r_n(\mathsf{Frob}_\mathfrak{q}) & = & T_\mathfrak{q} & \text{for } \mathfrak{q}\not|\mathfrak{m}\ell \\ T_\mathfrak{q}^{s_\mathfrak{q}}(T_\mathfrak{q}^2 - T_\mathfrak{q}\operatorname{tr} r_n(\sigma_\mathfrak{q}) - \chi(\sigma_\mathfrak{q}))^2 & = & 0 & \text{for } \mathfrak{q}|\mathfrak{n} \text{ but } \mathfrak{q}\not|\ell \end{array}$$

where $s_\mathfrak{q}$ is equal to the highest power of $\mathfrak{q}$ dividing $\mathfrak{m}\lambda$. In particular, $s_\mathfrak{q}$ is independent of n.

But, as above, there is a natural isomorphism between $\mathbb{T}_{k_n,w_n}(\mathfrak{m})$ and $\mathbb{T}^*_{k_n,w_n}(\mathfrak{m})$. We also define the map:

$$\alpha_n : \mathbb{T}^*_{k_n,w_n}(\mathfrak{m}) \longrightarrow \mathcal{O}_{K,\lambda}/\lambda^{n+1},$$

defined by $T \mapsto \theta_f(T) \pmod{\mathsf{m}^{n+1}}$; that this makes sense is essentially the content of Proposition 5.3.

Consider the composition of these maps:

$$R_n = \alpha_n \circ r_n.$$

Then this is a continuous pseudorepresentation

$$R_n : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathcal{O}_{K,\lambda}/\lambda^{n+1},$$

which satisfies

$$\operatorname{tr} R_n(\mathsf{Frob}_\mathfrak{q}) \equiv \theta_f(T_\mathfrak{q}) \pmod{\lambda^{n+1}} \quad \text{for } \mathfrak{q}\not|\mathfrak{m}\ell,$$

and

$$\theta_f(T_\mathfrak{q})^{s_\mathfrak{q}}(\theta_f(T_\mathfrak{q}^2) - \theta_f(T_\mathfrak{q})\operatorname{tr} R_n(\sigma_\mathfrak{q}) - \chi(\sigma_\mathfrak{q}))^2 \equiv 0 \pmod{\lambda^{n+1}} \quad \text{for } \mathfrak{q}|\mathfrak{n}, \mathfrak{q}\not|\ell.$$

Clearly this set of pseudorepresentations, as n varies, are compatible in that:

$$R_n \equiv R_{n+1} \pmod{\lambda^{n+1}},$$

(as they agree on $\mathsf{Frob}_{\mathfrak{q}}$ for $\mathfrak{q} \not| \mathfrak{m}$, a dense set), and so, using one of the fundamental properties of pseudorepresentations ([16]), the limit

$$R : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathcal{O}_{K,\lambda}$$

is also a pseudorepresentation, and clearly satisfies

$$\begin{aligned} \mathrm{tr}\, R(\mathsf{Frob}_{\mathfrak{q}}) &= \theta_f(T_{\mathfrak{q}}) \quad \text{for } \mathfrak{q} \not| \mathfrak{m}\ell, \\ \theta_f(T_{\mathfrak{q}})(\theta_f(T_{\mathfrak{q}}^2) - \theta_f(T_{\mathfrak{q}})\mathrm{tr}\, R(\sigma_{\mathfrak{q}}) - \chi(\sigma_{\mathfrak{q}})) &= 0 \quad \text{for } \mathfrak{q} | \mathfrak{n} \text{ but } \mathfrak{q} \not| \ell. \end{aligned}$$

As R is a pseudorepresentation, it lifts to a genuine representation

$$\rho_\lambda : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{GL}_2(K_\lambda),$$

with the same trace as R.

Thus the theorem is proven for all primes $\mathfrak{q}|\mathfrak{n}$ or $\mathfrak{q} \not| \mathfrak{m}$ (i.e., $\mathfrak{q} \not| \mathfrak{c}$). It suffices now to pick two different prime ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ for the level of E; the traces of the two representations thus obtained agree at all primes $\mathfrak{q} \not| \ell\mathfrak{c}_1\mathfrak{c}_2$, and so the representations are thus isomorphic. But together, the two representations define the trace at all primes $\mathfrak{q} \not| \ell$, and coincide with the statement of the theorem.

That ρ_λ may be valued in $\mathrm{GL}_2(\mathcal{O}_{K,\lambda})$ follows as ρ_λ stabilises a lattice in the usual way (as $\mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F})$ is compact).

Finally we remark that although we earlier replaced K by an extension satisfying the Hida condition, the representation that we have just constructed has a model defined over the field generated by its traces. This is exactly the original field of definition of f, and thus the representations are then genuinely valued in the completions of the field of definition for f as required.

This concludes the proof.

Having constructed the desired representations, we now investigate the compatibility of these representations with those obtained from the local Langlands correspondence, as in the second half of Conjecture 3.1.

7 The local Langlands correspondence

There is a natural bijective correspondence (the *local Langlands correspondence*) between isomorphism classes of admissible irreducible representations (over $\mathbb{C}$) of $\mathrm{GL}_2(\mathrm{F}_\mathfrak{p})$ and isomorphism classes of 2-dimensional F-semisimple representations of the local Weil-Deligne group at $\mathfrak{p}$.

Write $\omega_\mathfrak{p}$ for the (quasi-)character $|\ |_\mathfrak{p}$ of $\mathrm{F}_\mathfrak{p}^\times$, and, consequently of the local Weil group $W_{\mathrm{F}_\mathfrak{p}}$, where we will normalise the isomorphism of class field theory so that the arithmetic Frobenius elements correspond to uniformisers. As before, $\mathsf{Frob}_\mathfrak{p}$ will continue to denote the arithmetic Frobenius element.

As $\pi_\mathfrak{p}$ will be used to denote the local component of an automorphic representation, we will use $\mathfrak{p}$ to denote the uniformiser of the ideal $\mathfrak{p}$. This abuse of notation should cause no confusion.

As in [4], there are several natural ways in which to write the local Langlands correspondence, and we shall follow Deligne in using the *Hecke correspondence* (see [4], 3.2.6). For each representation $\pi_\mathfrak{p}$ of $\mathrm{GL}_2(\mathrm{F}_\mathfrak{p})$ as above, we will write $\sigma(\pi_\mathfrak{p})$ for the associated F-semisimple representation of the Weil-Deligne group.

Throughout the remainder of the paper, π will denote the automorphic representation associated to a Hilbert cuspidal eigenform $f \in S^*_{k,w}(\mathfrak{n}; K)$ as in the previous section.

The representations $\pi_\mathfrak{p}$ are then realised over K, and for each finite place λ of K of residue characteristic prime to that of $\mathfrak{p}$, there corresponds a continuous λ-adic representation $\sigma^\lambda(\pi_\mathfrak{p})$ of $W_{\mathrm{F}_\mathfrak{p}}$. Then one should have a global correspondence:

Conjecture 7.1 *The compatible system $\{\rho_\lambda\}$ of continuous 2-dimensional λ-adic representations constructed in Theorem 6.1 is such that for every finite place $\mathfrak{p}$ of F, and for every finite place λ of K with different residue characteristic to that of $\mathfrak{p}$, one has*

$$(\rho_\lambda|_{W_{\mathrm{F}_\mathfrak{p}}})^{F-ss} \textit{ is equivalent to } \sigma^\lambda(\pi_\mathfrak{p}).$$

Presumably one expects that the representations $\rho_\lambda|_{W_{\mathrm{F}_\mathfrak{p}}}$ will be F-semisimple. However, throughout the remainder of this paper, we will ignore questions of F-semisimplicity—thus when we say that two representations are equivalent ("$\rho_1 \sim \rho_2$"), we shall mean that their F-semisimplifications are equivalent.

The main result is the following, which we shall prove case-by-case:

Theorem 7.2 *Conjecture 7.1 holds whenever $\pi_\mathfrak{p}$ is not special.*

Remark 7.3 When $\pi_\mathfrak{p}$ is special, one has

$$\sigma^\lambda(\pi_\mathfrak{p}) \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

where χ_1 and χ_2 are characters determined by $\pi_\mathfrak{p}$, and where $*$ is predicted to be non-trivial. Our methods can prove that

$$\rho_\lambda|_{W_{F_\mathfrak{p}}} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

but we cannot show that $*$ should be non-trivial.

We consider seven possibilities for $\pi_\mathfrak{p}$ in Theorem 7.2:

- principal series
  - **(P1)** both defining characters are unramified
  - **(P2)** exactly one of the defining characters is unramified
  - **(P3)** both defining characters are ramified
- special representation
  - **(S1)** the defining character is unramified
  - **(S2)** the defining character is ramified
- supercuspidal representation
  - **(C1)** monomial
  - **(C2)** extraordinary

Taylor ([13] and [14]) shows that Theorem 6.1 suffices to determine exactly the restriction of the representation $\rho_\lambda|_{W_{F_\mathfrak{p}}}$ in the cases (P1)–(P3), and that this restriction is then equivalent to $\sigma^\lambda(\pi_\mathfrak{p})$.

The case (S1) is similar to case (P2), in that we know that for every $\sigma_\mathfrak{p}$ lying above $\mathsf{Frob}_\mathfrak{p}$ in the decomposition group, one has

$$\operatorname{tr}\rho_\lambda(\sigma_\mathfrak{p}) = \theta_f(T_\mathfrak{p}) + \chi(\sigma_\mathfrak{p})\theta_f(T_\mathfrak{p})^{-1},$$

as in Theorem 6.1 (3b).

One also knows that in the case of a special representation with defining character μ, $\theta_f(T_\mathfrak{p}) = \mu_\lambda(\sigma_\mathfrak{p})\omega_\mathfrak{p}(\sigma_\mathfrak{p})^{-1/2}$. Knowing that $\chi(\sigma_\mathfrak{p}) = \mu_\lambda(\sigma_\mathfrak{p})^2\omega_\mathfrak{p}^{-1}$ (from the definition of χ and the fact that the eigenvalue of $S_\mathfrak{p}$ is equal to the central character of $\pi_\mathfrak{p}$), one deduces that

$$\operatorname{tr}\rho_\lambda(\sigma_\mathfrak{p}) = \mu_\lambda(\sigma_\mathfrak{p})\omega_\mathfrak{p}(\sigma_\mathfrak{p})^{-1/2} + \mu_\lambda(\sigma_\mathfrak{p})\omega_\mathfrak{p}(\sigma_\mathfrak{p})^{1/2}.$$

One can then use the classification of 2-dimensional representations to deduce that

$$\rho_\lambda \sim \begin{pmatrix} \mu_\lambda \omega_{\mathfrak{p}}^{-1/2} & * \\ 0 & \mu_\lambda \omega_{\mathfrak{p}}^{1/2} \end{pmatrix}$$

as claimed, although it is not clear that $*$ should be non-trivial.

The case (S2) reduces to the case (S1) after a twist, exactly as Taylor ([14]) reduces case (P3) to either case (P1) or (P2). We omit the details.

Next, we deal with the monomial supercuspidal case (C1). Then there is a quadratic extension $\mathrm{K}/\mathrm{F}_{\mathfrak{p}}$, and a character $\mu : \mathrm{K}^\times \longrightarrow \mathbb{C}^\times$ such that $\pi_{\mathfrak{p}}$ is associated to μ. Let $\mathrm{Gal}(\mathrm{K}/\mathrm{F}_{\mathfrak{p}}) = \{1, \sigma\}$.

It is a standard argument (using Krasner's lemma) that there exists L/F totally real and quadratic with $\mathfrak{p}$ inert such that $\mathrm{L}_{\mathfrak{p}} = \mathrm{K}$.

Let π_{L} denote the base change of π to L; it is known that $(\pi_{\mathrm{L}})_{\mathfrak{p}} = I(\mu, \mu \circ \sigma)$, where $\mu \neq \mu \circ \sigma$. Let $\rho_{\mathrm{L},\lambda}$ be the 2-dimensional representation of $\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})$ associated to π_{L}.

One knows that for principal series representations, base change corresponds to restriction of the corresponding representations of the local Weil groups. But, by cases (P1)–(P3), if $\mathfrak{q}$ is a prime such that $\pi_{\mathfrak{q}}$ is principal series, then the corresponding representation $\sigma^\lambda(\pi_{\mathfrak{q}})$ of the local Weil group actually coincides with $\rho_\lambda|_{W_{\mathrm{F}_{\mathfrak{q}}}}$. Similarly, if $\mathfrak{q}'$ is a prime of L above $\mathfrak{q}$, the representation $\sigma^\lambda(\pi_{\mathrm{L},\mathfrak{q}'})$ coincides with $\rho_{\mathrm{L},\lambda}|_{W_{\mathrm{L}_{\mathfrak{q}'}}}$. Then

$$\rho_\lambda|_{W_{\mathrm{L}_{\mathfrak{q}'}}} = \rho_{\mathrm{L},\lambda}|_{W_{\mathrm{L}_{\mathfrak{q}'}}},$$

and so globally one has $\rho_{\mathrm{L},\lambda} = \rho_\lambda|_{\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})}$, as these representations agree almost everywhere. Then (by (P1)–(P3))

$$\rho_{\mathrm{L},\lambda}|_{W_{\mathrm{L}_{\mathfrak{p}}}} \sim \begin{pmatrix} \mu_\lambda & 0 \\ 0 & (\mu \circ \sigma)_\lambda \end{pmatrix}.$$

Thus

$$\rho_\lambda|_{W_{\mathrm{L}_{\mathfrak{p}}}} \sim \begin{pmatrix} \mu_\lambda & 0 \\ 0 & (\mu \circ \sigma)_\lambda \end{pmatrix},$$

and so

$$\begin{aligned} \rho_\lambda|_{W_{\mathrm{F}_{\mathfrak{p}}}} &\sim \mathrm{Ind}_{W_{\mathrm{L}_{\mathfrak{p}}}}^{W_{\mathrm{F}_{\mathfrak{p}}}} \mu_\lambda \\ &= \sigma^\lambda(\pi_{\mathfrak{p}}) \end{aligned}$$

as required.

Finally there remains the case (C2) where $\pi_{\mathfrak{p}}$ is extraordinary cuspidal, but the argument of Carayol ([2], 12.3) goes through unchanged, and reduces the problem, after a cubic base-change, to the monomial supercuspidal case, which was solved in the above paragraph.

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