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Research Article

On k-Gamma and k-Beta Distributions and Moment Generating Functions

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The main objective of the present paper is to define k-gamma and k-beta distributions and moments generating function for the said distributions in terms of a new parameter $k > 0$. Also, the authors prove some properties of these newly defined distributions.

1. Basic Definitions

In this section we give some definitions which provide a base for our main results. The definitions (1.1–1.3) are given in [1] while (1.4–1.6) are introduced in [2]. Also, we have taken some statistics related definitions (1.7–1.11) from [3–5].

1.1. Pochhmmer Symbol. The factorial function is denoted and defined by

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1); & \text{for } n \geq 1,\ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases} \quad (1)$$

The function $(a)_n$ defined in relation (1) is also known as Pochhmmer symbol.

1.2. Gamma Function. Let $z \in \mathbb{C}$; the Euler gamma function is defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{(z)_n} \quad (2)$$

and the integral form of gamma function is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathbb{R}(z) > 0. \quad (3)$$

From the relation (3), using integration by parts, we can easily show that

$$\Gamma(z+1) = z\Gamma(z). \quad (4)$$

The relation between Pochhammer symbol and gamma function is given by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \quad (5)$$

1.3. Beta Function. The beta function of two variables is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \operatorname{Re}(x), \operatorname{Re}(y) > 0 \quad (6)$$

and, in terms of gamma function, it is written as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (7)$$

1.4. Pochhammer k-Symbol. For $k > 0$, the Pochhammer k-symbol is denoted and defined by

$$(a)_{n,k} = \begin{cases} a(a+k)(a+2k)\cdots(a+(n-1)k); & \text{for } n \geq 1,\ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases} \quad (8)$$

1.5. k-Gamma Function. For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function is defined as

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n! k^n (nk)^{z/k-1}}{(z)_{n,k}} \tag{9}$$

and the integral representation of k-gamma function is

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt. \tag{10}$$

1.6. k-Beta Function. For $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, the k-beta function of two variables is defined by

$$B_k(x, y) = \frac{1}{k}\int_0^\infty t^{x/k-1}(1-t)^{y/k-1} dt \tag{11}$$

and, in terms of k-gamma function, k-beta function is defined as

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{12}$$

Also, the researchers [6–10] have worked on the generalized k-gamma and k-beta functions and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{13}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \tag{14}$$

$$\Gamma_k(k) = 1, \quad k > 0. \tag{15}$$

Using the above relations, we see that, for $x, y > 0$ and $k > 0$, the following properties of k-beta function are satisfied by authors (see [6, 7, 11]):

$$\beta_k(x+k, y) = \frac{x}{x+y}\beta_k(x, y), \tag{16}$$

$$\beta_k(x, y+k) = \frac{y}{x+y}\beta_k(x, y), \tag{17}$$

$$\beta_k(xk, yk) = \frac{1}{k}\beta(x, y), \tag{18}$$

$$\beta_k(x, k) = \frac{1}{x}, \qquad \beta_k(k, y) = \frac{1}{y}. \tag{19}$$

Note that when $k \to 1$, $\beta_k(x, y) \to \beta(x, y)$.

For more details about the theory of k-special functions like k-gamma function, k-beta function, k-hypergeometric functions, solutions of k-hypergeometric differential equations, contegious functions relations, inequalities with applications and integral representations with applications involving k-gamma and k-beta functions and so forth. (See [12–17].)

1.7. Probability Distribution and Expected Values. In a random experiment with n outcomes, suppose a variable X assumes the values $x_1, x_2, x_3, \ldots, x_n$ with corresponding probabilities $P_1, P_2, P_3, \ldots, P_n$; then this collection is called probability distribution and $\Sigma p_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a, b]$, then $\int_a^b f(x)dx = 1$.

In statistics, there are three types of moments which are (i) moments about any point $x = a$, (ii) moments about $x = 0$, and (iii) moments about mean position of the given data. Also, expected value of the variate is defined as the first moment of the probability distribution about $x = 0$ and the rth moment about mean of the probability distribution is defined as $E(x_i - \overline{x})^r$ where $\overline{x}$ is the mean of the distribution.

Also, $E(x)$ shows the expected value of the variate x and is defined as the first moment of the probability distribution about $x = 0$; that is,

$$\mu_1' = E(x) = \int_a^b x f(x)\, dx. \tag{20}$$

1.8. Gamma Distribution. A continuous random variable Z is said to have a gamma distribution with parameter $m > 0$, if its probability distribution function is defined by

$$f(z) = \begin{cases} \dfrac{1}{\Gamma(m)} z^{m-1} e^{-z}, & 0 \leqq z < \infty, \\ 0, & \text{elsewhere} \end{cases} \tag{21}$$

and its distribution function $F(z)$ is defined by

$$F(z) = \begin{cases} \displaystyle\int_0^z \frac{1}{\Gamma(m)} z^{m-1} e^{-z} dz, & z \geq 0, \\ 0, & z < 0, \end{cases} \tag{22}$$

which is also called the incomplete gamma function.

1.9. Moment Generating Function of Gamma Distribution. The moment generating function of Z is defined by

$$\begin{aligned} M_0(t) = E\left(e^{tZ}\right) &= \int_0^\infty e^{tZ} f(z)\, dz \\ &= \int_0^\infty \frac{1}{\Gamma(m)} z^{m-1} e^{-z(1-t)} dz. \end{aligned} \tag{23}$$

1.10. Beta Distribution of the First Kind. A continuous random variable Z is said to have a beta distribution with two parameters m and n, if its probability distribution function is defined by

$$f(z) = \begin{cases} \dfrac{1}{B(m, n)} z^{m-1}(1-z)^{n-1}, & 0 \leqq z \leqq 1;\ m, n > 0 \\ 0, & \text{elsewhere.} \end{cases} \tag{24}$$

This distribution is known as a beta distribution of the first kind and a beta variable of the first kind is referred to as $\beta_1(m,n)$. Its distribution function $F(z)$ is given by

$$F(z) = \begin{cases} 0, & z < 0, \\ \int_0^z \frac{1}{B(m,n)} z^{m-1}(1-z)^{n-1}dz, & 0 \leqq z \leqq 1;\ m,n > 0, \\ 0, & z > 1. \end{cases} \quad (25)$$

1.11. Beta Distribution of the Second Kind. A continuous random variable Z is said to have a beta distribution of the second kind with parameters m and n, if its probability distribution function is defined by

$$f(z) = \begin{cases} \frac{1}{\beta(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}}, & 0 \leqq z < \infty;\ m,n > 0, \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

and its probability distribution function is given by

$$F(z) = \int_0^\infty \frac{1}{\beta(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}} dz, \quad 0 \leqq z < \infty;\ m,n > 0. \quad (27)$$

2. Main Results: k-Gamma and k-Beta Distributions

In this section, we define gamma and beta distributions in terms of a new parameter $k > 0$ and discuss some properties of these distributions in terms of k.

Definition 1. Let Z be a continuous random variable; then it is said to have a k-gamma distribution with parameters $m > 0$ and $k > 0$, if its probability density function is defined by

$$f_k(z) = \begin{cases} \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^k/k}, & 0 \leqq z < \infty,\ k > 0, \\ 0, & \text{elsewhere} \end{cases} \quad (28)$$

and its distribution function $F_k(z)$ is defined by

$$F_k(z) = \begin{cases} \int_0^z \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^k/k} dz, & z > 0, \\ 0, & z < 0. \end{cases} \quad (29)$$

Proposition 2. *The newly defined $\Gamma_k(m)$ distribution satisfies the following properties.*

(i) *The k-gamma distribution is the probability distribution that is area under the curve is unity.*

(ii) *The mean of k-gamma distribution is equal to a parameter m.*

(iii) *The variance of k-gamma distribution is equal to the product of two parameters mk.*

Proof of (i). Using the definition of k-gamma distribution along with the relation (10), we have

$$\int_0^\infty f_k(z)\,dz = \frac{1}{\Gamma_k(m)} \int_0^\infty z^{m-1} e^{-z^k/k} dz = \frac{\Gamma_k(m)}{\Gamma_k(m)} = 1. \quad (30)$$

□

Proof of (ii). As mean of a distribution is the expected value of the variate, so the mean of the k-gamma distribution is given by

$$\bar{z} = E_k(Z) = \frac{1}{\Gamma_k(m)} \int_0^\infty z \cdot z^{m-1} e^{-z^k/k} dz. \quad (31)$$

Using the definition of k-gamma function and the relation (13), we have

$$\bar{z} = \frac{1}{\Gamma_k(m)} \int_0^\infty z^m e^{-z^k/k} dz = \frac{\Gamma_k(m+k)}{\Gamma_k(m)} = m\frac{\Gamma_k(m)}{\Gamma_k(m)} = m. \quad (32)$$

□

Proof of (iii). As variance of a distribution is equal to $E(x^2) - (E(x))^2$, so the variance of k-gamma distribution is calculated as

$$\text{Var}_k(Z) = E_k\left(Z^2\right) - \left(E_k(Z)\right)^2. \quad (33)$$

Now, we have to find $E_k(Z^2)$, which is given by

$$\begin{aligned} E_k\left(Z^2\right) &= \frac{1}{\Gamma_k(m)} \int_0^\infty z^2 \cdot z^{m-1} e^{-z^k/k} dz \\ &= \frac{1}{\Gamma_k(m)} \int_0^\infty z^{m+1} e^{-z^k/k} dz \\ &= \frac{\Gamma_k(m+2k)}{\Gamma_k(m)} = \frac{(m+k)\, m\Gamma_k(m)}{\Gamma_k(m)} \\ &= m(m+k). \end{aligned} \quad (34)$$

Thus we obtain the variance of k-gamma distribution as

$$\sigma_k^2 = m(m+k) - m^2 = mk, \quad (35)$$

where σ_k^2 is the notation of variance present in the literature. □

2.1. k-Beta Distribution of First Kind. Let Z be a continuous random variable; then it is said to have a k-beta distribution of the first kind with two parameters m and n, if its probability distribution function is defined by

$$f_k(z) = \begin{cases} \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}, & 0 \leqq z \leqq 1;\ m,n,k > 0, \\ 0, & \text{elsewhere.} \end{cases} \quad (36)$$

In the above distribution, the beta variable of the first kind is referred to as $\beta_{1,k}(m,n)$ and its distribution function $F_k(z)$ is given by

$$F_k(z)=\begin{cases}0, & z<0,\\ \displaystyle\int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz, & 0\leqq z\leqq 1;\\ & m,n>0,\\ 0, & z>1.\end{cases} \tag{37}$$

Proposition 3. *The k-beta distribution $\beta_{1,k}(m,n)$ satisfies the following basic properties.*

(i) *k-beta distribution is the probability distribution that is the area of $\beta_{1,k}(m,n)$ under a curve $f_k(z)$ is unity.*

(ii) *The mean of this distribution is $m/(m+n)$.*

(iii) *The variance of $\beta_{1,k}(m,n)$ is $mnk/((m+n)^2(m+n+k))$.*

Proof of (i). By using the above definition of k-beta distribution, we have

$$\int_0^z F_k(z)\,dz=\int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz, \qquad 0\leqq z\leqq 1;\quad m,n>0. \tag{38}$$

By the relation (11), we get

$$\int_0^z F_k(z)\,dz=\int_0^1 \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz = \frac{B_k(m,n)}{B_k(m,n)}=1. \tag{39}$$

□

Proof of (ii). The mean of the distribution, $\mu'_{1,k}$, is given by

$$\mu'_{1,k}=E_k(Z)=\int_0^z zF_k(z)\,dz = \int_0^z \frac{1}{kB_k(m,n)} z\cdot z^{m/k-1}(1-z)^{n/k-1}dz, \qquad 0\leqq z\leqq 1;\quad m,n>0. \tag{40}$$

Using the relations (12), (13), and (16), we have

$$\mu'_{1,k}=\int_0^1 \frac{1}{kB_k(m,n)} z^{m/k}(1-z)^{n/k-1}dz=\frac{B_k(m+k,n)}{B_k(m,n)} = \frac{\Gamma_k(m+k)\,\Gamma_k(n)\,\Gamma_k(m+n)}{\Gamma_k(m)\,\Gamma_k(n)\,\Gamma_k(m+n+k)}=\frac{m}{m+n}. \tag{41}$$

□

Proof of (iii). The variance of $\beta_{1,k}(m,n)$ is given by

$$\sigma_k^2=(\mathrm{Var})_k=E_k\left(Z^2\right)-\left(E_k(Z)\right)^2, \tag{42}$$

$$\begin{aligned}E_k\left(Z^2\right)&=\int_0^1 \frac{1}{kB_k(m,n)} z^{m/k+1}(1-z)^{n/k-1}dz\\ &=\frac{B_k(m+2k,n)}{B_k(m,n)}\\ &=\frac{\Gamma_k(m+2k)\,\Gamma_k(n)\,\Gamma_k(m+n)}{\Gamma_k(m)\,\Gamma_k(n)\,\Gamma_k(m+n+2k)}\\ &=\frac{m(m+k)}{(m+n)(m+n+k)}.\end{aligned} \tag{43}$$

Thus substituting the values of $E_k(Z^2)$ and $E_k(Z)$ in (42) along with some algebraic calculations we have the desired result. □

2.2. k-Beta Distribution of the Second Kind. A continuous random variable Z is said to have a k-beta distribution of the second kind with parameters m and n, if its probability distribution function is defined by

$$f_k(z)=\begin{cases}\dfrac{1}{k\beta_k(m,n)}\dfrac{z^{m/k-1}}{(1+z)^{(m+n)/k}}, & 0\leqq z<\infty;\ m,n,k>0,\\ 0, & \text{otherwise}.\end{cases} \tag{44}$$

Note. The k-beta distribution of the second kind is denoted by $\beta_{2,k}(m,n)$.

Theorem 4. *The k-beta function of the second kind represents a probability distribution function that is*

$$\int_0^\infty f_k(z)\,dz=1. \tag{45}$$

Proof. We observe that

$$\int_0^\infty f_k(z)\,dz=\int_0^\infty \frac{1}{k\beta_k(m,n)}\frac{z^{m/k-1}}{(1+z)^{(m+n)/k}}dz. \tag{46}$$

Let $1+z=1/y$, so that $dz=-dy/y^2$; thus by using the relation (11), the above equation gives

$$=\frac{1}{\beta_k(m,n)}\frac{1}{k}\int_0^1 y^{n/k-1}(1-y)^{m/k-1}dy=\frac{\beta_k(m,n)}{\beta_k(m,n)}=1. \tag{47}$$

□

3. Moment Generating Function of k-Gamma Distribution

In this section, we derive the moment generating function of continuous random variable Z of newly defined k-gamma distribution in terms of a new parameter $k > 0$, which is illustrated as

$$M_{0,k}(t) = E_k\left(e^{tZ^k}\right) = \int_0^\infty \frac{1}{\Gamma_k(m)} e^{tz^k} z^{m-1} e^{-z^k/k} dz$$
$$= \frac{1}{\Gamma_k(m)} \int_0^\infty z^{m-1} e^{(-z^k/k)(1-kt)} dz. \tag{48}$$

Let $u = z(1-kt)^{1/k}$, so that $z = u/(1-kt)^{1/k}$ and $dz = du/(1-kt)^{1/k}$. Then substituting these values in (48), we obtain

$$M_{0,k}(t) = \frac{1}{(1-kt)^{(m-1)/k}\Gamma_k(m)} \int_0^\infty u^{m-1} e^{-u^k/k} \frac{du}{(1-kt)^{1/k}}$$
$$= \frac{1}{(1-kt)^{m/k}\Gamma_k(m)} \int_0^\infty u^{m-1} e^{-u^k/k} du$$
$$= \frac{\Gamma_k(m)}{(1-kt)^{m/k}\Gamma_k(m)} = (1-kt)^{-m/k}, \quad |kt| < 1. \tag{49}$$

Now differentiating r times $M_{0,k}(t)$ with respect to t and putting $t = 0$, we get

$$\mu'_{r,k} = m(m+k)(m+2k)\cdots(m+(r-1)k). \tag{50}$$

Thus when $r = 1$, we obtain $\mu'_{1,k} = m$, when $r = 2$, $\mu'_{2,k} = m(m+k)$, and hence $\mu_{2,k} = \mu'^2_{1,k} - \mu'_{2,k} = mk$ = variance of the k-gamma distribution proved in Proposition 2.

3.1. Higher Moment in terms of k. The rth moment in terms of k is given by

$$\mu'_{r,k}$$
$$= E(Z^r) = \frac{1}{kB_k(m,n)} \int_0^1 z^r \cdot z^{m/k-1}(1-z)^{n/k-1} dz$$
$$= \frac{1}{kB_k(m,n)} \int_0^1 z^{m/k+r-1}(1-z)^{n/k-1} dz$$
$$= \frac{B_k(m+rk,n)}{B_k(m,n)} = \frac{\Gamma_k(m+rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(m+rk+n)}$$
$$= \frac{m(m+k)(m+2k)\cdots(m+(r-1)k)}{(m+n)(m+n+k)(m+n+2k)\cdots(m+n+(r-1)k)}. \tag{51}$$

Theorem 5. *The moments of the higher order of k-beta distribution of the second kind are given as*

$$\mu'_{r,k} = \frac{m(m+k)(m+2k)\cdots(m+(r-1)k)}{(n-k)(n-2k)\cdots(n-rk)}. \tag{52}$$

Proof. Consider

$$\mu'_{r,k} = E(Z^r) = \int_0^\infty \frac{1}{k\beta_k(m,n)} \frac{z^{m/k-1+r}}{(1+z)^{(m+n)/k}} dz. \tag{53}$$

Changing the variables as $z = (1-y)/y \Rightarrow dz = (-1/y^2)dy$, above equation becomes

$$= \frac{1}{k\beta_k(m,n)} \int_0^1 y^{n/k-r-1}(1-y)^{m/k+r-1} dy. \tag{54}$$

Replacing $(1-y)$ by t, we have

$$\mu'_{r,k} = \frac{1}{\beta_k(m,n)} \frac{1}{k} \int_0^1 t^{m/k+r-1}(1-t)^{n/k-r-1} dt$$
$$= \frac{\beta_k(m+rk, n-rk)}{\beta_k(m,n)}$$
$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n)}$$
$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)}{\Gamma_k(m)\Gamma_k(n)}. \tag{55}$$

Now using $\Gamma_k(n-rk) = \Gamma_k(n)/(n-k)(n-2k)\cdots(n-rk)$ in the above equation we get the desired result. □

4. Conclusion

In this paper the authors conclude that we have the following.

(i) If k tends to 1, then k-gamma distribution and k-beta distribution tend to classical gamma and beta distribution.

(ii) The authors also conclude that the area of k-gamma distribution and k-beta distribution for each positive value of k is one and their mean is equal to a parameter m and $m/(m+n)$, respectively. The variance of k-gamma distribution for each positive value of k is equal to k times of the parameter m. In this case if $k = 1$, then it will be equal to variance of gamma distribution. The variance of k-beta distribution for each positive value of k is also defined.

(iii) In this paper the authors introduced moments generating function and higher moments in terms of a new parameter $k > 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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