

and thus, by (13), we have that the sum of probabilities in (12) converges, hence Lemma 1 completes the proof.

The Corollary is a straight consequence of the Theorem. We stated it separately because of its interesting content.

References

- [1] J. Galambos, *The distribution of the largest coefficient in continued fraction expansions*, Quart. J. Math. Oxford Ser. 23 (1972), pp. 147-151.
 [2] — *The largest coefficient in continued fractions and related problems*, Proceedings of the Conf. on Diophantine Approximation, Washington, D. C., 1972; New York 1973, pp. 101-109.
 [3] A. Khintchine, *Kettenbrüche*, Leipzig 1956.
 [4] J. Neveu, *Mathematical Foundations of the Calculus of Probability*, San Francisco 1965.
 [5] W. Philipp, *Some metrical theorems in number theory II*, Duke Math. J. 38 (1970), pp. 447-458.

DEPARTMENT OF MATHEMATICS
 TEMPLE UNIVERSITY
 Philadelphia, Pa.

Received on 14. 3. 1973

(383)

On gaps between numbers with a large prime factor, II

by

T. N. SHOREY (Bombay)*

1. In [2] the following result was proved:

THEOREM 1. Let $n > 1$ be an integer. Let a_1, \dots, a_n be rational numbers such that

(i) $a_1 > 0, \dots, a_n > 0$ are multiplicatively independent,

(ii) $|\log a_i| \leq \exp\left(-\frac{1}{A} \log S_1\right), 1 \leq i \leq n$ and $A > 1$,

(iii) The sizes of a_1, \dots, a_n do not exceed S_1 . (The size of a rational number a/b , $(a, b) = 1$, is defined as $|b| + |a/b|$.)

If $\beta_1, \dots, \beta_{n-1}$ are rational numbers of size not exceeding S_1 , then

$$|\beta_1 \log a_1 + \dots + \beta_{n-1} \log a_{n-1} - \log a_n| > \exp(-nA)^{cn^2} \log S_1$$

where $c > 0$ is an effectively computable constant which is independent of n, A and S_1 .

In this paper we shall prove the following:

THEOREM 2. Let $n > 1$ be an integer. Let $a_1, \dots, a_n, \beta_1, \dots, \beta_{n-1}$ be rational numbers satisfying the assumptions of Theorem 1. Further assume that

(iv) $a_1 = \frac{m}{m'}, a_2 = \frac{p_2}{p_2'}, \dots, a_n = \frac{p_n}{p_n'}$ where $p_2, \dots, p_n, p_2', \dots, p_n'$ are

pairwise distinct prime numbers and none of them is either a factor of m or m' .

Then

$$|\beta_1 \log a_1 + \dots + \beta_{n-1} \log a_{n-1} - \log a_n| > \exp(-nA)^{c_1 n} \log S_1$$

where $c_1 > 0$ is an effectively computable constant which is independent of n, A and S_1 .

* I am very thankful to Professor H. M. Stark for sending me a preprint of his unpublished result [5]. My thanks are also due to Professor K. Ramachandra for going through the manuscript.

If n, A are large and a_1, \dots, a_n satisfy the assumption (iv) of Theorem 2, then Theorem 2 is an improvement of Theorem 1. The improvement depends on some of the ideas of Stark [5].

If Theorem 1 is replaced by Theorem 2 in [2], the method of [2] shows that the following result can be obtained in view of Jutila's result [1] that the greatest prime factor of $(u+1) \dots (u+k)$ exceeds $k(\log k)^2$ provided that $k^{3/2} \leq u \leq \exp((\log k)^{5/4})$ and k exceeds a certain absolute constant.

THEOREM 3. *Let k be a fixed natural number and let n_1, n_2, \dots be all the natural numbers (in the increasing order) which have at least one prime factor exceeding k . Define*

$$f(k) = \max_{i=1,2,\dots} (n_{i+1} - n_i).$$

Then

$$f(k) = O\left(\frac{k}{\log k} \left(\frac{\log \log \log k}{\log \log k}\right)\right).$$

This bound for $f(k)$ is sharper than that of [2], namely,

$$f(k) = O\left(\frac{k}{\log k} \left(\frac{\log \log \log \log k}{\log \log \log k}\right)^{1/2}\right).$$

We remark that the multiplicative independence of a_1, \dots, a_n follows from the assumption (iv) imposed on a_1, \dots, a_n in Theorem 2. Further (iv) can be somewhat relaxed (see Remark after the proof of Theorem 2). One would like to have $(nA)^{c_1} \log S_1$ in place of $(nA)^{c_1 n} \log S_1$ in Theorem 2. This would improve the bound for $f(k)$ to $k(\log k)^{-1-\delta}$, where $\delta > 0$ is a small constant.

2. Proof of Theorem 2. Unless otherwise specified, we shall follow the notations of [2] in this paper. The definition of \tilde{r}_1 in [2] (see after inequality (12)) is changed as follows:

$$\tilde{r}_1 = \left\lfloor \frac{E_1 n}{b} \right\rfloor + 2,$$

where E_1 is a positive constant to be suitably chosen. Allow the large constant c_1 (occurring in the definition of h in [2]) to depend on E_1 also. Assume that

$$(1) \quad \beta \leq \frac{1}{2} n^{-k} (6h\tilde{r}_1)^{-2h\tilde{r}_1 k/n^2} S_1^{-2h\tilde{r}_1 k/An^2}.$$

See that the inequalities (13) of [2] are satisfied. Proceed exactly as in [2] to conclude that

$$q(l, m_1, \dots, m_{n-1}) = 0$$

for all integers $l, 1 \leq l \leq h\tilde{r}_1$ and for all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k\tilde{r}_1$. Define

$$k\tilde{r}_{1+m} = [B^{-1}k\tilde{r}_{1+m}], \quad 0 \leq m \leq M-1.$$

(We shall choose M in such a way that $k\tilde{r}_{1+M} > 10$.)

We shall divide the (remaining) proof of Theorem 2 in three lemmas.

LEMMA 1. *Assume that β satisfies (1). Then for any rational number $a/p, 0 \leq a/p \leq h$ with $0 < p \leq h$ and non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$, we have*

$$q(a/p, m_1, \dots, m_{n-1}) = 0.$$

Proof. Put

$$f(z) = \Phi_{m_1, \dots, m_{n-1}}(z, \dots, z),$$

with $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$ and $m_i \geq 0$ ($1 \leq i \leq n$). For every z with $|z| = 2h\tilde{r}_1$, we have the interpolation formula:

$$\begin{aligned} \frac{1}{2\pi i} \int_{r:|\zeta|=A} \frac{f(\zeta)F(z)}{(\zeta-z)F(\zeta)} d\zeta \\ = f(z) + \sum_{r=1}^{h\tilde{r}_1} \sum_{m=0}^{k\tilde{r}_1-k\tilde{r}_{1+1}} \frac{f^{(m)}(r)}{m!2\pi i} \int_{r:|\zeta-r|=1/2} \frac{(\zeta-r)^m F(z)}{(\zeta-z)F(\zeta)} d\zeta \end{aligned}$$

where

$$F(\zeta) = \prod_{u=1}^{h\tilde{r}_1} (z-u)^{k\tilde{r}_1-k\tilde{r}_{1+1}}, \quad \Delta = 5h\tilde{r}_1 \exp\left(\frac{1}{A} \log S_1\right).$$

This interpolation formula gives that for every $z, |z| = 2h\tilde{r}_1$,

$$(2) \quad |f(z)| \leq w \left(S_1^{6nLh\tilde{r}_1} (2S_1L)^{7k} \exp\left(-\frac{h\tilde{r}_1 k\tilde{r}_1}{An^2} \log S_1\right) + \beta n^k S_1^{8nLh\tilde{r}_1} (2S_1L)^{8k} (6h\tilde{r}_1)^{2h\tilde{r}_1 k\tilde{r}_1/n^2} \right).$$

(For this, one can refer to the similar details following formula (7) of [2].)

Hence by maximum-modulus principle, (2) holds for all rational numbers $z = a/p$ with $0 \leq a/p \leq h$. Assume that $q(a/p, m_1, \dots, m_{n-1}) \neq 0$ for some rational number a/p with $0 \leq a/p \leq h, 0 < p \leq h$ and for some non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$. Notice that

$$\begin{aligned} \left| (\log a_1)^{-m_1} \dots (\log a_{n-1})^{-m_{n-1}} \Phi_{m_1, \dots, m_{n-1}}\left(\frac{a}{p}, \dots, \frac{a}{p}\right) - q\left(\frac{a}{p}, m_1, \dots, m_{n-1}\right) \right| \\ \leq (L+1)^n S_1^{4nLh} (2S_1L)^{5k} S_1^{nLh} (2S_1L)^k 2Lh\beta \leq \beta S_1^{6nLh} (2S_1L)^{6k}. \end{aligned}$$

Since $p \leq h$, $q(a/p, m_1, \dots, m_{n-1})$ is a non-zero algebraic number of degree $\leq h^n$. The absolute value of each of the conjugates of $q(a/p, m_1, \dots, m_{n-1})$ does not exceed

$$S_1^{6nLh} (2S_1L)^{6k}.$$

The denominator of $q(a/p, m_1, \dots, m_{n-1})$ does not exceed

$$S_1^{nLh^2} (2S_1L)^k.$$

Hence

$$(3) \quad \left| \Phi_{m_1, \dots, m_{n-1}} \left(\frac{a}{p}, m_1, \dots, m_{n-1} \right) \right| \geq w(S_1^{-7nLh^{n+2}} (2S_1L)^{-7kh^n} - \beta S_1^{6nLh} (2S_1L)^{6k}).$$

The contradiction is obtained by showing that (2) and (3) are inconsistent. For this it is enough to have

$$\exp \left(\frac{h \tilde{r}_1 k \tilde{r}_1}{An^2} \log S_1 \right) > S_1^{15nLh^{n+2} + 15nLh \tilde{r}_1} (2S_1L)^{15kh^n} \left\{ 1 + \beta n^k (6h \tilde{r}_1)^{2h \tilde{r}_1 k \tilde{r}_1 / n^2} \exp \left(\frac{h \tilde{r}_1 k \tilde{r}_1}{An^2} \log S_1 \right) \right\}.$$

Since β satisfies (1), it is sufficient to show that

$$\exp \left(\frac{h \tilde{r}_1 k \tilde{r}_1}{An^2} \log S_1 \right) > S_1^{16nLh^{n+2} + 15nLh \tilde{r}_1} (2S_1L)^{15kh^n}.$$

This can easily be established. For proof, one can refer to similar details just after inequality (13) of [2]. This completes the proof of Lemma 1.

Remark. Let $p'(\lambda_1, \dots, \lambda_n)$, $0 \leq \lambda_i \leq L$ ($i = 1, \dots, n$) be integers satisfying

$$|p'(\lambda_1, \dots, \lambda_n)| \leq S_1^{4nLh} (2S_1L)^{5k}.$$

(See inequality next to (3) of [2]). Consider

$$q' = q'(z, m_1, \dots, m_{n-1}) = \sum_{\lambda_1=0}^L \dots \sum_{\lambda_n=0}^L p'(\lambda_1, \dots, \lambda_n) \alpha_1^{\lambda_1 z} \dots \alpha_n^{\lambda_n z} \gamma_1^{m_1} \dots \gamma_{n-1}^{m_{n-1}}.$$

Suppose that $q'(l, m_1, \dots, m_{n-1}) = 0$ for all integers l , $1 \leq l \leq h$, and for all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k$. If β satisfies (1), then our argument shows that

$$q'(a/p, m_1, \dots, m_{n-1}) = 0$$

for all rational numbers a/p , $0 \leq a/p \leq h$ with $p \leq h$ and all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k_{\tilde{r}_1+1}$.

We shall call that q' is associated to $p'(\lambda_1, \dots, \lambda_n)$.

LEMMA 2. Assume that β satisfies (1). Let $p'(\lambda_1, \dots, \lambda_n)$ be integers, $0 \leq \lambda_i \leq L$ ($i = 1, \dots, n$), satisfying

$$|p'(\lambda_1, \dots, \lambda_n)| \leq S_1^{4nLh} (2S_1L)^{5k}.$$

Let $q' = q'(z, m_1, \dots, m_{n-1})$ be associated to $p'(\lambda_1, \dots, \lambda_n)$. Assume that for x_0 , $0 < x_0 \leq 1$, we have

$$q'(x_0 + l, m_1, \dots, m_{n-1}) = 0$$

for all integers $0 \leq l \leq h-1$ and all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k$ ($= k_1$). Then

$$q'(a/p, m_1, \dots, m_{n-1}) = 0$$

for all rational numbers a/p , $0 \leq a/p \leq h$, $0 < p \leq h$ and all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k_{\tilde{r}_1+2}$.

Proof. It is sufficient to prove that

$$q'(l, m_1, \dots, m_{n-1}) = 0$$

for all integers l , $1 \leq l \leq h$, and non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k_2$. (Then the lemma would follow by the above remark.) Define

$$f(z) = \Phi'_{m_1, \dots, m_{n-1}}(z, \dots, z), \quad m_1 + \dots + m_{n-1} \leq k_2$$

where

$$\Phi'(z_1, \dots, z_{n-1}) = \sum_{\lambda_1=0}^L \dots \sum_{\lambda_n=0}^L p'(\lambda_1, \dots, \lambda_n) \alpha_1^{\lambda_1 z_1} \dots \alpha_{n-1}^{\lambda_{n-1} z_{n-1}}.$$

For z with $|z| = 2h$, we have

$$\frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(\zeta) F(z)}{(\zeta - z) F(\zeta)} d\zeta = f(z) + \sum_{r=0}^{h-1} \sum_{m=0}^{k_1-k_2} \frac{f^{(m)}(r+x_0)}{m! 2\pi i} \int_{\Gamma_r} \frac{(\zeta - r - x_0)^m F(z)}{(\zeta - z) F(\zeta)} d\zeta$$

where Γ_r denotes the circle with centre $(r+x_0)$ and radius $\frac{1}{2}$,

$$F(\zeta) = \prod_{u=0}^{h-1} (\zeta - u - x_0)^{k_1 - k_2 + 1} \quad \text{and} \quad \Delta = 5h \exp \left(\frac{1}{A} \log S_1 \right).$$

For every z , $|z| = 2h$, the above formula gives

$$(4) \quad |f(z)| \leq w \left(S_1^{6nLh_1} (2S_1L)^{7k} \exp \left(-\frac{h_1 k_1}{An^2} \log S_1 \right) + \beta n^k S_1^{8nLh_1} (2S_1L)^{8k} (6h_1)^{2h_1 k_1 n^2} \right).$$

Hence by maximum-modulus principle, (4) holds for $z = l$ with $1 \leq l \leq h$. Further for every $l, 1 \leq l \leq h_1$,

$$(5) \quad |f(l)| \geq w(S_1^{-nLh_1}(2S_1L)^{-2k} - \beta S_1^{7nLh_1}(2S_1L)^{7k})$$

provided that

$$q'(l, m_1, \dots, m_{n-1}) \neq 0.$$

(See inequality (6) of [2] with $S = S_1$.)

(4) and (5) are inconsistent, if

$$\exp\left(\frac{h_1 k_1}{An^2} \log S_1\right) > S_1^{9nLh_1}(2S_1L)^{10k} \{1 + 2\beta n^k (2h_1)^{2h_1 k_1/n^2} S_1^{2h_1 k_1/An^2}\}.$$

This inequality is secured in [2] (see the inequality that appears after (12) in [2]).

Hence Lemma 2 is proved.

LEMMA 3. Assume that β satisfies (1). Then there exists a prime $p, h^{1/2} < p \leq h$, with the following property: There exist integers j_1, \dots, j_n , with $0 \leq j_i < p$ ($1 \leq i < n$) and $j_n = 1$, such that

$$\alpha_1^{j_1} \dots \alpha_n^{j_n} = \eta^p,$$

where η is rational.

The proof of Lemma 3 depends on the following:

LEMMA 3'. Let $\alpha_1, \dots, \alpha_n$ be non-zero elements of an algebraic number field K and let $\alpha_1^{1/p}, \dots, \alpha_n^{1/p}$ denote fixed p -th roots for some prime p . Further let $K' = K(\alpha_1^{1/p}, \dots, \alpha_n^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of K' of degree p or we have

$$\alpha_1^{j_1} \dots \alpha_n^{j_n} = \eta^p$$

for some η in K and some integers j_1, \dots, j_n with $0 \leq j_i < p$ ($1 \leq i < n$) and $j_n = 1$.

This is Lemma 5 of [3].

Proof of Lemma 3. Let $p, h^{1/2} < p \leq h$, be a prime for which the lemma is not true. Then by Lemma 3', $K'(\alpha_n^{1/p})$ is an extension of $K' = Q(\alpha_1^{1/p}, \dots, \alpha_{n-1}^{1/p})$ (Q denotes the field of rational numbers) of degree p . Let $p_1, \dots, p_r, h^{1/2} < p_i \leq h$ ($1 \leq i \leq r$) be all the primes for which Lemma 3 is not valid. For convenience, write $p_0(\lambda_1, \dots, \lambda_n)$ for $p(\lambda_1, \dots, \lambda_n)$ which are determined in [2]. Set $q_0 = q_0(z, m_1, \dots, m_{n-1}) = q(z, m_1, \dots, m_{n-1})$ (associated to $p(\lambda_1, \dots, \lambda_n)$). Then by Lemma 1, we have

$$q_0\left(\frac{a}{p'}, m_1, \dots, m_{n-1}\right) = 0$$

for all rational numbers $0 \leq a/p' \leq h, 0 < p' \leq h$ and all non-negative

integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$. In particular $q_0(a/p_1, m_1, \dots, m_{n-1}) = 0$ with $0 \leq a/p_1 \leq h$ and $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$, i.e.

$$\sum_{\lambda_n=0}^L \left(\sum_{\lambda_1=0}^L \dots \sum_{\lambda_{n-1}=0}^L p_0(\lambda_1, \dots, \lambda_n) \alpha_1^{a/p_1} \dots \alpha_{n-1}^{a/p_1} \gamma_1^{m_1} \dots \gamma_{n-1}^{m_{n-1}} \right) \alpha_n^{a/p_1} = 0,$$

with $0 \leq a/p_1 \leq h$ and $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}, m_i \geq 0$. If $a \not\equiv 0 \pmod{p_1}$, the above sum is still zero when λ_n is summed over any single residue class $(\text{mod } p_1)$. Therefore for $0 \leq a/p_1 \leq h, a \not\equiv 0 \pmod{p_1}$ and $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$, we have

$$\sum_{\lambda_n=\lambda_n(p_1)}^L \left(\sum_{\lambda_1=0}^L \dots \sum_{\lambda_{n-1}=0}^L p_0(\lambda_1, \dots, \lambda_n) \alpha_1^{a/p_1} \dots \alpha_{n-1}^{a/p_1} \gamma_1^{m_1} \dots \gamma_{n-1}^{m_{n-1}} \right) \alpha_n^{a/p_1} = 0$$

where $A_n, 0 \leq A_n \leq L$, is any integer and $\lambda_n \equiv A_n \pmod{p_1}$ stands for $\lambda_n \equiv A_n \pmod{p_1}$. Define

$$p_1(\lambda_1, \dots, \lambda_n) = \begin{cases} p_0(\lambda_1, \dots, \lambda_n) & \text{if } \lambda_n \equiv A_n \pmod{p_1}, \\ 0 & \text{otherwise} \end{cases}$$

and call q_1 the function associated with $p_1(\lambda_1, \dots, \lambda_n)$. Hence

$$q_1\left(\frac{1+lp_1}{p_1}, m_1, \dots, m_{n-1}\right) = 0,$$

with $0 \leq l \leq h-1$ and $m_1 + \dots + m_{n-1} \leq k\tilde{r}_{1+1}$ with $m_i \geq 0$. By Lemma 2, we obtain

$$q_1\left(\frac{a}{p'}, m_1, \dots, m_{n-1}\right) = 0,$$

with $0 \leq a/p' \leq h, 0 < p' \leq h$ and $m_1 + \dots + m_{n-1} \leq k_2\tilde{r}_{1+2}, m_i \geq 0$. Define

$$p_2(\lambda_1, \dots, \lambda_n) = \begin{cases} p_1(\lambda_1, \dots, \lambda_n) & \text{if } \lambda_n \equiv A_n \pmod{p_2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proceed as above and conclude that

$$q_2\left(\frac{1+lp_2}{p_2}, m_1, \dots, m_{n-1}\right) = 0,$$

with $0 \leq l \leq h-1$ and $m_1 + \dots + m_{n-1} \leq k_2\tilde{r}_{1+2}, m_i \geq 0$. Proceed by induction and conclude that

$$(6) \quad q_r\left(\frac{1+lp_r}{p_r}, m_1, \dots, m_{n-1}\right) = 0$$

for all integers $l, 0 \leq l \leq h-1$ and all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k_{r(\tilde{r}_1+2)}$. Notice that

$$p_r(\lambda_1, \dots, \lambda_n) = \begin{cases} p_0(\lambda_1, \dots, \lambda_n) & \text{if } \lambda_n \equiv A_n(p_i) \ (1 \leq i \leq r), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$p_1 \dots p_r > h^{r/2} > L \quad \text{if } r = [4nE].$$

Therefore

$$p_r(\lambda_1, \dots, \lambda_n) = 0 \text{ if } \lambda_n \neq A_n \quad \text{and} \quad p_r(\lambda_1, \dots, A_n) = p_0(\lambda_1, \dots, A_n).$$

In (6), set $l = 0$ and we obtain (writing p for p_r)

$$(7) \quad \sum_{\lambda_1=0}^L \dots \sum_{\lambda_{n-1}=0}^L p_0(\lambda_1, \dots, A_n) \alpha_1^{l_1/p} \dots \alpha_{n-1}^{l_{n-1}/p} \gamma_1^{m_1} \dots \gamma_{n-1}^{m_{n-1}} = 0.$$

This is true for all non-negative integers m_1, \dots, m_{n-1} with $m_1 + \dots + m_{n-1} \leq k_{r(\tilde{r}_1+2)}$. Observe that

$$k_{r(\tilde{r}_1+2)} > nL,$$

if $E > 1$ and c_1 is large enough. Hence (7) is valid for all integers m_1, \dots, m_{n-1} with $0 \leq m_i \leq L$ ($1 \leq i \leq n$). Notice that the determinant

$$(\lambda_{n-1} + A_n \beta_{n-1})^{m_{n-1}}, \quad 0 \leq \lambda_{n-1} \leq L, \quad 0 \leq m_{n-1} \leq L$$

does not vanish. Hence if $A_{n-1}, 0 \leq A_{n-1} \leq L$, is an arbitrary integer, (7) gives

$$\sum_{\lambda_1=0}^L \dots \sum_{\lambda_{n-2}=0}^L p_0(\lambda_1, \dots, A_{n-1}, A_n) \alpha_1^{l_1/p} \dots \alpha_{n-2}^{l_{n-2}/p} \gamma_1^{m_1} \dots \gamma_{n-2}^{m_{n-2}} = 0,$$

for all integers m_1, \dots, m_{n-2} with $0 \leq m_i \leq L$. Proceeding similarly, we obtain that $p_0(A_1, \dots, A_n) = 0$. Notice that A_1, \dots, A_n are arbitrary. Hence $p_0(\lambda_1, \dots, \lambda_n) = 0$ for all $(\lambda_1, \dots, \lambda_n)$, which is a contradiction.

Hence the number of primes $p, h^{1/2} < p \leq h$, for which Lemma 3 is not valid is at most $4nE$. But the number of primes between $h^{1/2}$ and h exceeds $8nE$, if c_1 is large enough. Hence there must exist a prime $p, h^{1/2} < p \leq h$, satisfying

$$(8) \quad \alpha_1^{j_1} \dots \alpha_n^{j_n} = \eta^p$$

for some $\eta = a/b, (a, b) = 1$, in \mathcal{Q} and for some integers j_1, \dots, j_n with $0 \leq j_i < p$ ($1 \leq i < n$) and $j_n = 1$. This completes the proof of Lemma 3.

Proof of Theorem 2. Assume that β satisfies (1). By Lemma 3, (8) holds. But this is not possible, because of the restrictions (iv) (on $\alpha_1, \dots, \alpha_n$) in Theorem 2. Hence

$$\beta > \frac{1}{2} n^{-k} (6h_{r_1}^{\tilde{r}_1})^{-2h_{r_1}^{\tilde{r}_1} k/n^2} S_1^{-2h_{r_1}^{\tilde{r}_1} k/A_n^2} > \exp(- (nA)^{c_2 n} \log S_1)$$

where $c_2 > 0$ is an effectively computable constant which is independent of n, A and S_1 . This completes the proof of Theorem 2.

Remark. Instead of assuming (iv), (§ 1, Theorem 2) it is sufficient to suppose that $\alpha_1, \dots, \alpha_n$ are such that no relation of the type (8) is possible.

Added in proof. As the main purpose of this paper is to improve the upper estimate of [2] for $f(E)$, Theorem 2 is not stated in all its generality. The linear forms of the type of Theorem 2 (when α_i are close to 1) were considered for the first time in [4] to prove the results announced in [3].

References

[1] M. Jutila, *On numbers with large prime factor II*, to appear in Journ. Indian Math. Soc.
 [2] K. Ramachandra and T. N. Shorey, *On gaps between numbers with a large prime factor*, Acta Arith. 24 (1973), pp. 99-111.
 [3] T. N. Shorey, *On a theorem of Ramachandra*, Acta Arith. 20 (1972), pp. 215-221.
 [4] — *Linear forms in the logarithms of algebraic numbers with small coefficients I*, to appear in Journ. of Indian Math. Soc.
 [5] H. M. Stark, *Further advances in the theory of linear forms in logarithms*, Diophantine Approximation and Its Applications, New York and London 1973, pp. 255-293.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
 Colaba, Bombay 5, India

Received on 19. 4. 1973

(396)