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On Gauge Invariance of Yang-Mills Theories with Matter Fields

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Abstract. -We continue the investigation of quantized Yang-Mills theories coupled to matter fields in the framework of causal perturbation theory. In this approach, which goes back to Epstein and Glaser, one works with free fields throughout, so that all expressions are mathematically well-defined. The general proof of the Cg-identities (C-number identities expressing gauge invariance) is completed. We attach importance to the correct treatment of the degenerate terms and to the Cg-identities with external matter legs. Moreover, the compatibility of all Cg-identities with P-, T-, C-invariance and pseudo-unitarity is shown.

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1. Introduction

In this paper we complete the study in refs. [1,2,3,4] of gauge invariance for Yang-Mills theories coupled to matter fields in the framework of causal perturbation theory. This approach, which goes back to Epstein and Glaser [5], has the merit that only well-defined quantities, namely free fields, appear, in contrast to the ill-defined interacting fields in the usual Lagrangian formalism. The central objects in this approach are the n-point distributions T_n which appear in the formal power series of the S-matrix

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n \, T_n(x_1, ..., x_n) g(x_1) ... g(x_n), \tag{1.1}$$

where $g(x)$ is a tempered switching function. The T_n's may be viewed as mathematically well-defined time-ordered products. They are constructed inductively from the given first order

$$T_1(x) = T_1^A(x) + T_1^u(x) + T_1^\psi(x), \tag{1.2}$$

with

$$T_1^A(x) \stackrel{\text{def}}{=} \frac{ig}{2} f_{abc} : A_{\mu a}(x) A_{\nu b}(x) F_c^{\nu\mu}(x) :, \tag{1.3}$$

$$T_1^u(x) \stackrel{\text{def}}{=} -igf_{abc} : A_{\mu a}(x) u_b(x) \partial^\mu \tilde{u}_c(x) :, \tag{1.4}$$

$$T_1^\psi(x) \stackrel{\text{def}}{=} i\, j_{\mu a}(x) A_a^\mu(x), \tag{1.5}$$

where the matter current $j_{\mu a}$ is defined by

$$j_{\mu a}(x) \stackrel{\text{def}}{=} \frac{g}{2} : \overline{\psi}_\alpha(x) \gamma_\mu (\lambda_a)_{\alpha\beta} \psi_\beta(x) : . \tag{1.6}$$

Herein, g is the coupling constant, f_{abc} are the structure constants of the gauge group SU(N) and $\frac{-i}{2}\lambda_a$, $a = 1, ..., N^2 - 1$ denote the generators of the fundamental representation of SU(N). The gauge potentials A_a^μ, $F_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu A_a^\nu - \partial^\nu A_a^\mu$ and the ghost fields u_a, $\tilde{u}_a$ are massless and fulfil the wave equation. The matter fields ψ and $\overline{\psi} \stackrel{\text{def}}{=} \psi^+ \gamma^0$ may be massless or massive and ψ satisfies the free Dirac equation

$$i\gamma_\nu \partial^\nu \psi_\alpha(x) = M_{\alpha\beta} \psi_\beta(x), \tag{1.7}$$

with a diagonal mass matrix $M_{\alpha\beta} = m_\alpha \delta_{\alpha\beta}$, $m_\alpha \geq 0$. We want the matter current $j_{\mu a}$ to be conserved

$$\partial^\mu j_{\mu a}(x) = 0, \tag{1.8}$$

which requires $\lambda_a M = M^+ \lambda_a (= M\lambda_a)$, $\forall a = 1, ..., N^2 - 1$. By means of Schurs lemma, we conclude that $j_{\mu a}$ is conserved, if and only if the mass is colour independent

$$M_{\alpha\beta} = m\delta_{\alpha\beta}, \quad m \geq 0. \tag{1.9}$$

Therefore, we only consider matter fields fulfilling (1.9).

The most important property of the S-matrix (1.1) to be proven is gauge invariance, which means roughly speaking that the commutator of the T_n-distributions with the gauge charge

$$Q \stackrel{\rm def}{=} \int_{t=const.} d^3x \, (\partial_\nu A_a^\nu \overleftrightarrow{\partial}_0 u_a) \tag{1.10}$$

is a (sum of) divergence(s). In first order this holds true

$$[Q, T_1^A(x) + T_1^u(x)] = i\partial_\nu (T_{1/1}^{A\nu}(x) + T_{1/1}^{u\nu}(x)), \tag{1.11}$$

where

$$T_{1/1}^{A\nu}(x) \stackrel{\rm def}{=} igf_{abc} : A_{\mu a}(x) u_b(x) F_c^{\nu\mu}(x) :, \tag{1.12}$$

$$T_{1/1}^{u\nu}(x) \stackrel{\rm def}{=} -\frac{ig}{2} f_{abc} : u_a(x) u_b(x) \partial^\nu \tilde{u}_c(x) :, \tag{1.13}$$

and, by means of the current conservation (1.8),

$$[Q, T_1^\psi(x)] = i\partial_\nu T_{1/1}^{\psi\nu}(x) \tag{1.14}$$

with

$$T_{1/1}^{\psi\nu}(x) \stackrel{\rm def}{=} i\, j_a^\nu(x) u_a(x). \tag{1.15}$$

Note that $[Q, T_1^A]$ is not a divergence. In order to have gauge invariance in first order, we are forced to introduce the ghost coupling T_1^u (1.4). We define gauge invariance in arbitrary order by

$$[Q, T_n(x_1, ..., x_n)] = i \sum_{l=1}^{n} \partial_\mu^{x_l} T_{n/l}^\mu (x_1, ..., x_n). \tag{1.16}$$

The divergences on the r.h.s. of (1.16) are given by n-th order T-distributions from a different theory which contains, in addition to the usual Yang-Mills couplings (1.2) a so-called Q-vertex, defined by $T_{1/1}^\nu = T_{1/1}^{A\nu} + T_{1/1}^{u\nu} + T_{1/1}^{\psi\nu}$ (1.12), (1.13), (1.15). (See ref. [2] for more details.) The operator gauge invariance (1.16) can be expressed by the *Cg-identities*, the C-number identities for gauge invariance [2,3]. It is a remarkable fact that the gauge invariance of quantized non-abelian theories can be formulated by the simple condition (1.16) involving only the well-defined asymptotic fields (which are free fields) and that this condition does not connect different orders of the perturbation series. To clarify the latter point, note that the 4-gluon interaction is not a first order term, it is of second order, namely the free normalization term

$$Cig^2 : A_{\mu a}(x_1) A_{\nu b}(x_1) A_d^\mu(x_2) A_e^\nu(x_2) : f_{abc} f_{dec} \delta(x_1 - x_2) \tag{1.17}$$

of $T_2(x_1, x_2)$. Gauge invariance (1.16) in second order fixes the value of the normalization constant C uniquely and (1.17) agrees with the usual 4-gluon interaction (see [1] or sect. 3(b)).

Our gauge invariance (1.16) implies the invariance of the S-matrix (1.1) (in the formal adiabatic limit $g \to 1$) with respect to the following gauge transformations of the free fields

$$\phi(x) \longrightarrow \phi_\lambda(x) = e^{-i\lambda Q'} \phi(x) e^{i\lambda Q'}, \quad \phi = A_a^\mu, F_a^{\mu\nu}, u_a, \tilde{u}_a, \psi, \overline{\psi}, \tag{1.18}$$

where

$$Q' \stackrel{\rm def}{=} (-1)^{Q_g} Q, \quad Q_g \stackrel{\rm def}{=} i \int_{t=const.} d^3x : \tilde{u}(x) \overleftrightarrow{\partial}_0 u(x) : . \tag{1.19}$$

Q_g is the ghost charge operator [6]: $[Q_g, u_a] = -u_a$, $[Q_g, \tilde{u}_a] = \tilde{u}_a$. Obviously, these transformations (1.18) are linear and the transformed fields are still free fields, as it must be in causal perturbation theory. Note, that the latter fact excludes a local transformation $\psi_\alpha(x) \longrightarrow e^{i\Lambda(x)}\psi_\alpha(x)$. The infinitesimal versions $\delta\phi = \partial_\lambda|_{\lambda=0}\phi_\lambda$ of (1.18) are proportional to the (anti)commutators

$$\delta A_a^\mu = -i(-1)^{Q_g}[Q, A_a^\mu] = (-1)^{Q_g}\partial^\mu u_a, \quad \delta F_a^{\mu\nu} = -i(-1)^{Q_g}[Q, F_a^{\mu\nu}] = 0,$$

$$\delta u_a = -i(-1)^{Q_g}\{Q, u_a\} = 0, \quad \delta\partial^\mu \tilde{u}_a = -i(-1)^{Q_g}\{Q, \partial^\mu \tilde{u}_a\} = -(-1)^{Q_g}\partial_\nu F_a^{\mu\nu},$$

$$\delta\psi_\alpha = -i(-1)^{Q_g}[Q, \psi_\alpha] = 0, \quad \delta\overline{\psi}_\alpha = -i(-1)^{Q_g}[Q, \overline{\psi}_\alpha] = 0. \tag{1.20}$$

These transformations are the *free field version of the famous BRS-transformations* [7]. Remark: The transformations (1.18) do not completely agree with the transformations in ref.[1]: $\phi(x) \longrightarrow e^{-i\lambda Q}\phi(x)e^{i\lambda Q}$. However, due to $[Q', T_n(X)] = (-1)^{Q_g}[Q, T_n(X)]$, our gauge invariance (1.16) implies the invariance of the S-matrix with respect to *both* kinds of transformations.

Because we consider transformations of the free fields only and, especially, since we can not transform ψ, $\overline{\psi}$, the reader may think that (1.16) is only a restricted gauge invariance. But the main purpose of gauge invariance is to eliminate the unphysical degrees of freedom in the proof of unitarity on the physical subspace. However, there are no unphysical particles in the matter sector. Therefore, it causes no troubles that we can not transform ψ, $\overline{\psi}$. In fact, we succeded in proving the physical unitarity by means of (1.16) (sect. 5 of ref. [4]). Moreover, the gauge invariance (1.16) yields the usual Ward identities in QED (refs. [2,12,15]). For Yang-Mills theories we have proven that *our Cg-identities (expressing (1.16)) imply the usual Slavnov-Taylor identity [8,9,10]* in the case of two external legs (sect. 3.3 of ref. [2]). The extension of this result to three and four external legs is in progress [11]. We emphazise that the *Cg-identities contain more information than the Slavnov-Taylor identities*, because in the latter the *inner* vertices are integrated out with $g(x) \equiv 1$. In order to derive the Slavnov-Taylor identities from our Cg-identities, the latter integration must be carried out and we have to eliminate the distributions with one Q-vertex [2,11]. However, this elimination can not be done completely in the case of an external pair $(\overline{\psi}, \psi)$. But in this case also *Taylor [8] is forced to introduce the Q-vertex (1.15) to formulate the Slavnov-Taylor identities.* He needs the C-number distributions $\bar{t}^1_{\overline{\psi}\psi u(A)}$, $\bar{t}^2_{\overline{\psi}\psi u(A)}$ (see subsect. 4(b)), which are precisely the distributions with Q-vertex which we cannot eliminate [11].

We do not consider it as a weakness that we study the simple gauge transformations (1.18) only. By contrast, it is a virtue that the invariance with respect to these simple transformations implies the full content of the usual gauge invariance. However, we have not yet studied the independence on the *gauge fixing*.

The proof of (1.16) follows the inductive construction of the $T_n, T^\nu_{n/l}$. The crucial step is the causal distribution splitting $d_n = r_n - a_n$. The problem is that we do not have a general formula for a covariant splitting solution r_n at our disposal [2]. In QED the non-vanishing mass of the fermions ψ, $\overline{\psi}$ seems to guarantee the existence of the central (or symmetrical) splitting solution, which is obtained by a dispersion integral from d_n (refs. [5,12]). However, in the case of non-abelian gauge theories, a mass $m > 0$ (1.9) of the matter fields does not help us in this respect because of the self-interaction of the gauge bosons.

The general proof of gauge invariance (1.16) in refs. [1,2,3,4] is not complete concerning the degenerate terms and the coupling to matter fields. We shall close these gaps and prove the compatibility of gauge invariance with discrete symmetries and pseudo-unitarity.

2. Outline of the Proof of Gauge Invariance

The proof follows the inductive construction of the $T_n, T_{n/l}$. Therefore, it is by induction on n, too, like most proofs in causal perturbation theory. The operator gauge invariance (corresponding to (1.16)) of A'_n, R'_n and $D_n = A'_n - R'_n$,

$$[Q, D_n(x_1, ..., x_n)] = i \sum_{l=1}^{n} \partial_\mu^{x_l} D^\mu_{n/l}(x_1, ..., x_n), \tag{2.1}$$

has been proven in section 3.1 of ref. [2] in a straightforward way. This proof is very instructive because it shows that our definition (1.16) of gauge invariance is adapted to the inductive construction of the T_n's. However, the distribution splitting $D_n = R_n - A_n$ is done in terms of the numerical distributions $d_n = r_n - a_n$, because they depend on the relative coordinates only and, therefore, are responsible for the support properties. We see that we have to express the operator gauge invariance (2.1) by the *Cg-identities* for D_n, the C-number identities for gauge invariance, which imply the operator gauge invariance (2.1) of D_n.

However, there is a serious problem: Consider the $l = 1$ term on the r.h.s. of (2.1)

$$\partial_\mu^{x_1} D^\mu_{n/1}(x_1, ...) = (\partial^1_\mu d^{\mu 1\nu}_{uA})(x_1, ...)u(x_1)A_\nu(x_2) + d^{\mu 1\nu}_{uA}(x_1, ...)\partial_\mu u(x_1)A_\nu(x_2) + ... \tag{2.2}$$

If $d^{\mu 1\nu}_{uA}$ contains a contribution with a factor $\delta(x_1 - x_3)$, then the terms with different field operators may compensate, due to the identity

$$[u(x_1) - u(x_3)]\partial_\mu^{x_1}\delta(x_1 - x_3) + \delta(x_1 - x_3)\partial_\mu u(x_1) = 0. \tag{2.3}$$

(Note that the contribution with $u(x_3)$ comes from the term with $x_1 \leftrightarrow x_3$ exchanged, which belongs to the $l = 3$ term in (2.1).) Even the definition of the C-number distributions in R'_n, A'_n (and therefore in $D_n = R'_n - A'_n$) has a certain ambiguity because terms $\sim (\partial\delta) : A... :$ can mix up with terms $\sim \delta : \partial A... :\sim \delta : F... :$. To get rid of these ambiguities, we choose the convention of *only* applying Wick's theorem (doing nothing else) to

$$A'_n(x_1, ...; x_n) = \sum_{Y,Z} \tilde{T}_k(Y)T_{n-k}(Z, x_n), \tag{2.4}$$

where the (already constructed) operator decompositions of $\tilde{T}_k$, T_{n-k} are inserted, and similarly for $A'_{n/l}, R'_n$ and $R'_{n/l}$. We do not change this operator decomposition in constructing $D_n, D_{n/l}, R_n, R_{n/l}, A_n, A_{n/l}, T'_n, T'_{n/l}$ and $T_n, T_{n/l}$. We call it the *natural* operator decomposition. Note that this prescription fixes the numerical distributions uniquely, up to the normalization in the causal splitting $d^{(l)}_n = r^{(l)}_n - a^{(l)}_n$. Then, according to (2.1) ((1.16) resp.), we commute with Q or take the divergence $\partial_\nu^{x_l}$ and obtain the *natural operator decomposition of (2.1) ((1.16) resp.)*.

However, due to (2.2),(2.3), the Cg-identities for D_n cannot be proven directly by decomposing (2.1). We must go another way: *Instead of proving the operator gauge invariance (1.16), we prove the corresponding Cg-identities (by induction on n), which are a stronger statement.* In this framework the Cg-identities for D_n can be proven by means of the Cg-identities for T_k, $\tilde{T}_k$ in lower orders $1 \leq k \leq n - 1$.

The Cg-identities for D_n (or T_n resp.) are obtained by collecting all terms in the natural operator decomposition of (2.1) (or (1.16) resp.) which belong to a particular combination $:\mathcal{O}:$ of external field operators. By doing this, one has to take care of the following two specifications (A) and (B):

(A) There is a speciality concerning matter fields. A divergence $\partial_\nu^{x_l}$ on the r.h.s. of (2.1) ((1.16)resp.) acting on ψ or $\overline{\psi}$, appears always in the form $\gamma_\nu \partial^\nu \psi$ or $\partial^\nu \overline{\psi} \gamma_\nu$ (see subsect. 4(b) below). Due to the Dirac equation

$$\gamma_\nu \partial^\nu \psi_\alpha = -im\psi_\alpha, \quad \partial^\nu \overline{\psi}_\alpha \gamma_\nu = im\overline{\psi}_\alpha, \tag{2.5}$$

we do not obtain a new field operator, as this is the case in pure Yang-Mills theories (e.g. $\partial_\nu u(x)$ is not proportional to $u(x)$). The terms with the divergence acting on ψ or $\overline{\psi}$ and those with the divergence acting on the numerical distribution, belong to the same Cg-identity. Therefore, *we always apply the Dirac equation (2.5), if the divergence acts on ψ or $\overline{\psi}$.*

(B) The arguments of some field operators must be changed by using δ-distributions, i.e. by applying the simple identity

$$:B(x_i)\mathcal{O}(X):\delta(x_i - x_k)... =: B(x_k)\mathcal{O}(X):\delta(x_i - x_k)... \tag{2.6}$$

where $X \stackrel{\text{def}}{=} (x_1, x_2, ...x_n)$ and $\mathcal{O}(X)$ means the external field operators besides B. Soon, this will be explained further.

By means of (A) and (B) we are now able to give a precise definition of our assertion that the Cg-identities hold: *We start with the natural operator decomposition of (1.16), applying always the Dirac equation according to (A). Using several times the identity (2.6), we can obtain an operator decomposition*

$$[Q, T_n(X)] - i\sum_{l=1}^{n} \partial_l T_{n/l}(X) = \sum_j \tau_j(X) : \mathcal{O}_j(X):, \tag{2.7}$$

(where $\tau_j(X)$ is a numerical distribution and $:\mathcal{O}_j(X):$ a normally ordered combination of external field operators) which fulfils

$$\tau_j(X) = 0, \quad \forall j. \tag{2.8}$$

The decomposition (2.7) must be invariant with respect to permutations. The latter means that the numerical distribution belonging to $:\mathcal{O}_j(\pi X):$ $(\pi \in S_n,\ \pi X \stackrel{\text{def}}{=} (x_{\pi 1}, ...x_{\pi n}))$ is obtained from the numerical distribution belonging to $:\mathcal{O}_j(X):$ (which is $\tau_j(X)$) by permuting the arguments with π (i.e. it is given by $\tau_j(\pi X)$). Otherwise, we would get contradictions in the Cg-identities (2.8). We call (2.7) the *Cg-operator decomposition* of (1.16).

A Cg-identity is uniquely characterized by its operator combination $:\mathcal{O}:$. The terms in a Cg-identity are singular of order [15]

$$|\mathcal{O}| + 1 \tag{2.9}$$

at $x = 0$, where

$$|\mathcal{O}| = 4 - b - g_u - g_{\tilde{u}} - \frac{3}{2}(g_\psi + g_{\overline{\psi}}) - d. \tag{2.10}$$

Here, $b, g_u, g_{\tilde{u}}, g_\psi, g_{\overline{\psi}}$ are the number of gluon, $u, \tilde{u}, \psi, \overline{\psi}$-operators, respectively, in $\mathcal{O}$, and d is the number of derivatives on these field operators. This was shown in ref. [2].

In the various diagrams contributing to T_n and $T^\alpha_{n/l}$, we have the *basic* external field operators $A, F, u, \partial\tilde{u}, \psi$ and $\overline{\psi}$. Going over to $[Q, T_n]$, we get one external field operator ∂u or ∂F in each non-vanishing term. In $\sum \partial^l_\alpha T^\alpha_{n/l}$, the derivative may act on the numerical distribution or on an external field operator. In the first case all external field operators are basic ones (i.e. $A, F, u, \partial\tilde{u}, \psi$ and $\overline{\psi}$). In the second case we always apply the Dirac equation (2.5) and decompose

$$\partial_\alpha A_\mu = \tfrac{1}{2} F_{\alpha\mu} + (\partial_\alpha A_\mu)_s, \tag{2.10a}$$

where $(\partial_\alpha A_\mu)_s \overset{\text{def}}{=} \frac{1}{2}(\partial_\alpha A_\mu + \partial_\mu A_\alpha)$. We see that in this second case terms with one non-basic field operator ∂u, $\partial\partial\tilde{u}, \partial F, (\partial A)_s$ and terms with only basic ones appear. (The latter are the $\frac{1}{2}F$-terms in (2.10a) and the $\partial\psi, \partial\overline{\psi}$-terms.) With that we are able to define the various types of Cg-identities (see sect. 2 of ref. [3]):

Type Ia: $:\mathcal{O}:$ contains one non-basic field operator ∂u or ∂F. All terms of $[Q, T_n]$ belong to type Ia.

Type Ib: $:\mathcal{O}:$ contains one non-basic field operator $\partial\partial\tilde{u}$ or $(\partial A)_s$.

Type II: $:\mathcal{O}:$ consists of basic field operators only.

The following statement is a part of our assertion that the Cg-identities hold: *The natural and the Cg-operator decomposition agree for the terms of type Ia or Ib* (i.e. the terms with one non-basic field operator). The δ-identity (2.6) and the Dirac equation (2.5) are applied for terms of type II only.

Now we classify the various terms in (1.16) rsp. (2.7) in another way. There are no pure vacuum diagrams, i.e. terms with no external legs. This is obvious for $[Q, T_n]$. Concerning the divergences, note that the diagrams of $T_{n/l}$ fulfil $g_u = g_{\tilde{u}} + 1$.

The distributions $D_n, D_{n/l}, A_n, A_{n/l}, R_n, R_{n/l}$ contain *connected* diagrams only, due to their causal supports. However, disconnected diagrams appear in $A'_n, A'_{n/l}, R'_n, R'_{n/l}$ and therefore also in $T_n, T_{n/l}$. They fulfil the Cg-identities (on T_n-level) separately. This can be proven easily by means of the Cg-identities for their connected subdiagrams, which hold by the induction hypothesis.

. Let us consider a connected diagram in the natural operator decomposition of (1.16). We call it *degenerate*, if it has at least one vertex with two external legs; otherwise the connected diagram is called *non-degenerate*. Let x_i be the degenerate vertex with two external fields, say B_1, B_2. Such a degenerate term (i.e. a term belonging to such a degenerate diagram) has the following form

$$: B_1(x_i)B_2(x_i)B_3(x_{j_1})...B_r(x_{j_{r-2}}) : \Delta(x_i - x_k)t_{n-1}(x_1 - x_n, ...\overline{x_i - x_n}, ...x_{n-1} - x_n), \tag{2.11}$$

where $k \neq i$, $j_l \neq i\,(\forall l = 1, ..., r-2)$ and the coordinate with bar in t_{n-1} must be omitted. In general, there is a sum of such terms (2.11) belonging to the fixed (degenerate) operator combination $:\mathcal{O}: =: B_1(x_i)B_2(x_i)B_3(x_{j_1})...B_r(x_{j_{r-2}}):$. For $\Delta(x_i - x_k)$ the following possibilities appear:

(a) $\Delta = D_F, \partial D_F, \partial_\mu\partial_\nu D_F\,(\mu \neq \nu), \partial_\rho\partial_\mu\partial_\nu D_F\,(\mu \neq \nu \neq \rho \neq \mu), S_F$,

(b) $\Delta = \delta^{(4)}, \partial\delta^{(4)}$.

If $\Delta = \partial S_F$, we apply the Dirac equation $\gamma_\nu \partial^\nu S_F = -i\delta^{(4)} - imS_F$. The $\partial\delta^{(4)}$-terms in (b) cancel (see remark (3) below). If a degenerate term (2.11) with $\Delta = \delta^{(4)}$ (type (b)) can be transformed in a non-degenerate one by applying (possibly several times) the identity (2.6) only, we call it *δ-degenerate*; if this is not possible we call it *truly degenerate*. All other degenerate terms (i.e. the terms of type (a)) are called truly degenerate, too.

Remarks: (1) By considering the possible diagrams for a δ-degenerate term, we shall see (at the beginning of sect. 4) that all δ-degenerate terms can be transformed in non-degenerate form by applying

$$: B_1(x_i)B_2(x_i)... : \delta(x_i - x_k)... + (x_i \longrightarrow x_k) =$$

$$=: B_1(x_i)B_2(x_k)... : \delta(x_i - x_k)...+ : B_1(x_k)B_2(x_i)... : \delta(x_i - x_k)... \qquad (2.12)$$

only, where the term with x_i, x_k exchanged is taken into account. This identity (2.12) is a special case of (2.6). However, (2.12) does not suffice to transform all truly degenerate terms from the natural into the Cg-operator decomposition; (2.6) is needed for this purpose. We recall that we apply the transformation (2.12) to the δ-degenerate terms of type II only (i.e. the δ-degenerate terms without non-basic field operator). For the δ-degenerate terms of type Ia,b the natural and the Cg-operator decomposition agree.

(2) In the process of distribution splitting δ-distributions are produced (e.g. $\square D(x) = 0, \square D^{ret,av}(x) = \delta(x)$). Therefore, one might think that the distinction in δ-degenerate and truly degenerate terms is not the same for A'_n, R'_n, D_n and A_n, R_n, T_n. For second order degenerate diagrams there are really more δ's in A_2, R_2, T_2 than in A'_2, R'_2, D_2. However, δ-degenerate terms do not appear at all in this order (see sect. 3). In higher orders $n \geq 3$ we take the natural splitting (see sect. 4(b) of ref. [13]) for the degenerate terms (2.11). This splitting does not generate new δ-distributions.

(3) There are terms in the natural operator decomposition of (1.16) which have a propagator $\Delta(x_i - x_k) = \partial\delta(x_i - x_k)$ in (2.11). They are generated by the divergence $\partial_\nu^{x_i}$ (term $l = i$ in $\sum_l \partial_l T_{n/l}$) acting on

$$g^{\mu\lambda}(\partial^\nu\partial^\kappa D^F(x_i - x_k) - \frac{1}{2}g^{\nu\kappa}\delta(x_i - x_k)) + \text{antisymmetrizations.} \qquad (2.13)$$

The second term is the 4-gluon interaction (1.17), which propagates in the inductive construction from second order (see subsect. 3(b)) to higher orders. Doing the necessary antisymmetrizations in Lorentz indices, the $\partial\delta(x_i - x_k)$- terms coming from the first term in (2.13) *cancel* with the ones coming from the 4-gluon interaction term (see subsection 2(b) of ref. [3]). Therefore, we need not to care about such $\partial\delta$-terms.

The truly degenerate terms fulfil the Cg-indentities (on T_n-level) separately, by means of the Cg-identities for their subdiagramms (subsect. 3(a)). The latter hold by the induction hypothesis. The exception are some tree diagrams in second and third order, which need an explicit calculation (subsect. 3(b)).

The disconnected and the truly degenerate terms cancel separately in (1.16). There remain the non-degenerate and δ-degenerate ones, which are linearly dependent. Therefore, *the δ-degenerate terms of type II must be transformed in non-degenerate form by using (2.12).* In this way we obtain completely *new* Cg-identities, in contrast to the disconnected and the truly degenerate Cg-identities, which rely on Cg-identities in lower orders.

Therefore, it is not astonishing that the difficult part of the proof of the Cg-identities concerns the non-degenerate $: \mathcal{O} :$ of type II (including δ-degenerate terms). In sect. 4 we prove the Cg-identities for the non-degenerate and δ-degenerate terms. In part (a) we prove them for A'_n, R'_n (and therefore also for D_n) by means of the Cg-identities in lower orders. In the process of distribution splitting the Cg-identities can be violated by local terms only which are singular of order $|\mathcal{O}| + 1$ (see ref.[2,15]), i.e. the possible anomaly has the form

$$a(x_1, ...x_n) = \sum_{|b|=0}^{|\mathcal{O}|+1} C_b D^b \delta^{4(n-1)}(x_1 - x_n, ...). \qquad (2.14)$$

We see that we only have to consider Cg-identities with

$$|\mathcal{O}| \geq -1. \tag{2.15}$$

This is only possible for Cg-identities with 2-,3-,4-legs and one Cg-identity with 5-legs- $(: \mathcal{O} :=: uAAAA :)$. For the latter, the colour structure excludes an anomaly (2.14) (sect. 4 of ref. [4]). The Cg-identities with 2-,3- and 4-legs without external operators ψ or $\overline{\psi}$ are deduced and proven in refs. [2,3] and [4]. In sect. 4(b) we list the Cg-identities with $: \mathcal{O} :$ containing ψ or $\overline{\psi}$. Their proof is given in sect. 4(c): First we restrict the constants C_b in the ansatz (2.14) for the possible anomaly by means of covariance, the colour structure (see appendix A) and invariance with respect to permutations of the inner vertices and C-invariance. Then, we remove the possible anomaly by finite renormalizations of the t-distributions in the Cg-identity. If a certain distribution t appears in several Cg-identities, the different normalizations of t must be compatible. For certain Cg-identities $(: \mathcal{O} :=: uAA :, : uAAA :, : uu\partial\tilde{u}A :)$ the removal of the anomaly is only possible, if one has some additional information about the infrared behavior of the divergences with respect to inner vertices (sects. 2 and 3 of ref.[4]). Luckily, this additional input is not needed for the Cg-identities with $: \mathcal{O} :$ containing $\psi, \overline{\psi}$. C-invariance does mainly the job of restricting the constants C_b (2.14) in the latter case.

To complete the inductive step, one has to prove the Cg-identities for $\tilde{T}_n$ which is the n-point distribution of the inverse S-matrix. This was proven in a simple and short way at the end of sect. 3.4 in ref. [2].

The latter proof is implicitly repeated in section 5, where we prove the compatibility of the various normalization conditions, which are covariance, P-,T-,C-invariance, pseudo-unitarity on the whole Fock-space (ref. [4]) and gauge invariance.

The reader may wonder why we shall spend so much words about changing the arguments of some field operators by means of the simple δ-identity (2.6). However, the linear independence of different field operator combinations $: \mathcal{O}(X) :$ is lost, if δ-distributions are present (see (2.6) and (2.3)). Therefore, by means of (2.6), one must combine terms, having different $: \mathcal{O}(X) :$ in the natural operator decomposition of $[Q, T_n] - i\sum_l \partial_l T_{n/l}$. If these combinations are not done in the right way, the Cg-identities (2.8) do not hold.

3. Truly Degenerate Terms

The prototype of a truly degenerate term in the natural operator decomposition of (1.16) reads

$$: B_1(x_1)B_2(x_1)B_3...B_r : \Delta(x_1 - x_2)..., \quad \Delta \neq \delta, \tag{3.1}$$

see fig.1. The subdiagram with vertices $\{x_2, ...x_n\}$ is an arbitrary connected diagram. For example, it is possible that x_2 is an external vertex, i.e. that we have an external field operator $B_s(x_2)$, $3 \leq s \leq r$.

Replacing in (3.1) $\Delta(x_1 - x_2)$ by $\delta(x_1 - x_2)$, the resulting term needs not to be δ-degenerate. The counter examples are the two local tree terms in figs.2 and 3 and their combinations with (3.1) which are given in figs.4,5 and 6. All terms belonging to figs.2,3,4,5 and 6 are truly degenerate, although they have a factor $\delta(x_1 - x_2)$ (or even $\delta(x_1 - x_2)\delta(x_2 - x_3)$). A tree diagram in order $n \geq 4$ has $r \geq 6$ external legs. By means of (2.9), (2.10) the corresponding numerical distribution is singular of order [2,15]

$$|\mathcal{O}| + 1 \leq 5 - r \leq -1. \tag{3.2}$$

(Remember that we are considering terms in the (natural) operator decomposition of (1.16), therefore, the singular order is $|\mathcal{O}|+1$ and not $|\mathcal{O}|$.) We conclude from (3.2) that a tree diagram in order $n \geq 4$ has non-local support, i.e. one propagator is $\Delta(x_j - x_l) \neq \delta(x_j - x_l)$.

These results lead to the presumption that *all truly degenerate terms are exactly the terms belonging to the diagrams in figs.1-6*, where the vertices may be permuted. We do not try to prove this statement because we only need the following weaker version (which holds obviously): *All truly degenerate terms are either the local tree terms symbolized in figs.2 and 3, or they have the form given by the diagram in fig.7*, with $X_1 \stackrel{\text{def}}{=} (x_1, ...x_r)$, $X_2 \stackrel{\text{def}}{=} (x_{r+1}, ...x_n)$, $\Delta(x_1 - x_{r+1}) \neq \delta(x_1 - x_{r+1})$. Again, the vertices may be permuted. The subdiagrams with vertices X_1 (X_2 respectively) are arbitrary connected diagrams. Especially, for $X_1 = (x_1)$ $(r=1)$, fig.7 agrees with fig.1. Note that there are also non-degenerate and δ-degenerate terms belonging to fig.7.

In subsect. 3(b) we prove the Cg-identities for the local tree terms of figs.2 and 3, in subsect. 3(a) we prove them for all terms of the type given by fig.7.

(a) Tree-like diagrams with non-local propagator

There is a heuristic argument for the gauge invariance of the terms symbolized by fig.7: In the process of distribution splitting the gauge invariance (2.1) of D_n can be violated by local terms only. However, the terms of fig.7 have a non-local propagator $\Delta(x_1 - x_{r+1})$. Therefore, they are non-local and cannot spoil gauge invariance.

The problem with this argument is that it relies on a unique splitting of a (numerical) distribution into a local and a non-local part, which does not exist. There is an explicit counter example: First note that instead of summing up all diagrams with permuted vertices, we may smear out one diagram with a symmetrical test function . Then, by means of the identity (we assume y and y' to be inner vertices)

$$\langle \partial_\rho D_{\rm ret}(x-y)(\partial_y^\rho - \partial_{y'}^\rho)[\delta(y-y')\bar{t}(z-x_n,\ldots,y'-x_n,\ldots)]$$

$$: B_1(x)B_2(x)B_3(z)\ldots :, \varphi(x,y,z,\ldots y',\ldots x_n)\rangle \tag{3.3a}$$

$$= -\langle \delta(x-y)\delta(y-y')\bar{t}(z-x_n,\ldots y'-x_n,\ldots) : B_1(x)B_2(x)B_3(z)\ldots :, \varphi(x,y,z,\ldots y',\ldots x_n)\rangle$$

$$-\langle \partial_\rho D_{\rm ret}(x-y)\delta(y-y')\bar{t}(z-x_n,\ldots y'-x_n,\ldots) : B_1(x)B_2(x)\ldots :, \partial_y^\rho\varphi(x,y,z,\ldots y',\ldots x_n)\rangle$$

$$+\langle \partial_\rho D_{\rm ret}(x-y)\delta(y-y')\bar{t}(z-x_n,\ldots y'-x_n,\ldots) : B_1(x)B_2(x)\ldots :, \partial_{y'}^\rho\varphi(x,y,z,\ldots y',\ldots x_n)\rangle$$

$$= -\langle \delta(x-y)\delta(y-y')\bar{t}(z-x_n,\ldots y'-x_n,\ldots) : B_1(x)B_2(x)B_3(z)\ldots :, \varphi(x,y,z,\ldots y',\ldots x_n)\rangle, \tag{3.3b}$$

which holds on symmetric test functions $\varphi(x, y, z, \ldots, y', \ldots x_n)$, the special truly degenerate term (3.3a) (see (3.1), fig.1) can be transformed into a δ-degenerate term (3.3b) (see (2.12)). Moreover, if $\bar{t}$ *is local*, the resulting term (3.3b) is local (in all variables) and, therefore, could spoil gauge invariance. Note that such transformations (3.3) are forbidden in obtaining the Cg-operator decomposition from the natural one (see (2.6),(2.7),(2.8)). (Especially, they are excluded in the definition of δ-degenerate and truly degenerate terms, which assures the uniqueness of this definition.)

In sect. 5 of ref. [3] we gave a proof for the Cg-identities of the terms (3.1) (fig.1), for pure Yang-Mills theories. The proof there is an explicit calculation of all cases and relies on the Cg-identities for the subdiagram with vertices $\{x_2, ...x_n\}$. In three steps (A),(B),(C) we now reformulate that proof and generalize it to the terms in fig.7. Moreover, matter fields are included.

(A) First we introduce some notations

$$B_0 \stackrel{\rm def}{=} 0,\ B_1 \stackrel{\rm def}{=} A,\ B_2 \stackrel{\rm def}{=} F,\ B_3 \stackrel{\rm def}{=} u,\ B_4 \stackrel{\rm def}{=} \partial\tilde{u},\ B_5 \stackrel{\rm def}{=} \psi,\ B_6 \stackrel{\rm def}{=} \overline{\psi}. \tag{3.4}$$

Furthermore, let $\Delta_{st}(x-y)$ be the 'contraction' of $B_s(x)$ with $B_t(y)$, but with the Feynman propagators $D^F(x-y)$ instead of $D^+(x-y)$, $-D^-(x-y)$; and with $S^F(x-y)$ instead of $-S^+(x-y)$, $S^-(x-y)$, e.g.

$$\Delta^{\mu\nu}_{11\,ab}(x) = ig^{\mu\nu}\delta_{ab}D^F(x),\ \Delta_{65\,\alpha\beta}(x) = -i\delta_{\alpha\beta}S^F(-x),$$

$$\Delta_{34\,ab}(x) = i\delta_{ab}\partial D^F(x),\ \Delta_{13} = 0,\ \Delta_{0t} = 0\,\forall t. \tag{3.5}$$

Note that $\Delta_{st} \neq \delta$, $\forall s,t$. (We omit the 4-gluon interaction terms (1.17), (2.13).) For simplicity, we omit colour- and Lorentz-indices. We define the index $Q(s)$ by the equation

$$[Q, B_s]_{\mp} = i\partial B_{Q(s)} \tag{3.6}$$

(commutator $[...]_-$ for $s = 0,1,2,5,6$, anticommutator $[...]_+$ for $s = 3,4$), e.g. $Q(1) = 3$, $Q(2) = 0$. For the subdiagram with vertices $X_1 = (x_1, ...x_r)$ ($X_2 = (x_{r+1}, ...x_n)$ respectively), we write

$$T_{r(/l)}(X_1) = \sum_s :T^s_{r(/l)}(X_1)B_s(x_1): + ... \quad (l = 1, ...r)$$

$$T_{n-r(/l')}(X_2) = \sum_t :B_t(x_{r+1})T^t_{n-r(/l')}(X_2): + ... \quad (l' = l - r,\ l = r+1, ...n), \tag{3.7}$$

where the dots mean terms without external leg at x_1 (x_{r+1} resp.). There may be a second external leg at x_1 (x_{r+1} resp.), but $B_s(x_1)$ ($B_t(x_{r+1})$ resp.) is the one which is contracted in fig.7. Then we have

$$T_n(X_1, X_2) = \sum_{st} :T^s_r(X_1)\Delta_{st}(x_1 - x_{r+1})T^t_{n-r}(X_2): + ...$$

$$T_{n/l}(X_1, X_2) = \sum_{st} :T^s_{r/l}(X_1)\Delta_{st}(x_1 - x_{r+1})T^t_{n-r}(X_2): + ... \text{ for } l \leq r$$

$$T_{n/l}(X_1, X_2) = \sum_{st} :T^s_r(X_1)\Delta_{st}(x_1 - x_{r+1})T^t_{n-r/l'}(X_2): + ... \text{ for } l \in \{r+1, ...n\}. \tag{3.8}$$

The terms symbolized by fig.7 are given by the natural operator decomposition of

$$[Q, T_n(X_1, X_2)] - i\sum_l \partial_l T_{n/l}(X_1, X_2) = \sum_{st}\Big[\Big\{: ([Q, T^s_r(X_1)]_{\mp} - i\sum_{l=1}^{r}\partial_l T^s_{r/l}(X_1))\cdot$$

$$\cdot\Delta_{st}(x_1 - x_{r+1}) - iT^s_{r/1}(X_1)\partial\Delta_{st}(x_1 - x_{r+1})\Big\}T^t_{n-r}(X_2): +$$

$$+ :T^s_r(X_1)\Big\{\Delta_{st}(x_1 - x_{r+1})(\pm[Q, T^t_{n-r}(X_2)]_{\mp} -$$

$$-i\sum_{l'=1}^{n-r}\partial_{l'}T^t_{n-r/l'}(X_2)) + i\partial\Delta_{st}(x_1 - x_{r+1})T^t_{n-r/1}(X_2)\Big\}: \Big] + ... \tag{3.8a}$$

(B) In order to obtain the Cg-operator decomposition of (3.8a), we consider the Cg-operator decomposition of the two subdiagrams with vertices X_1, respectively X_2. By the induction assumption, the Cg-identities are true for $(T_r(X_1), T_{r/l}(X_1))$. Considering only operator combinations : $\bar{O}_j(X_1) :=: O_j(X_1)B_{(j)}(x_1)$: (no sum over j) which contain an external field operator $B_{(j)}(x_1)$, $B_{(j)} = B_s, \partial B_s, i\partial B_{Q(s)}$, this means

$$\sum_s \Big\{ : [Q, T_r^s(X_1)]_\mp B_s(x_1) : \pm : T_r^s(X_1) i\partial B_{Q(s)}(x_1) : -$$

$$-i\sum_{l=1}^{r} : \partial_l T^s_{r/l}(X_1) B_s(x_1) : -i : T^s_{r/1}(X_1)\partial B_s(x_1) : \Big\} = \tag{3.9a}$$

$$= \sum_j \tau_{rj}(X_1) : O_j(X_1) B_{(j)}(x_1) : \tag{3.9b}$$

with

$$\tau_{rj}(X_1) = 0, \ \forall j. \tag{3.10}$$

More precisely: Starting with the natural operator decomposition of (3.9a), one obtains (3.9b) by applying the Dirac equation (2.5) and the δ-identity (2.6) only. Note that $\tau_{rj}(X_1)$ is a sum of contributions coming from the various terms in (3.9a). (Remark: A term $\tau_{rj}(X_1) : O_j(X_1)B_{(j)}(x_1)$: (no sum over j) in (3.9b) could have a contribution from a term $t_1(x_1, ..., x_k...) \sim \delta(x_k - x_1)B_{(j)}(x_k)\ (k \neq 1)$ in $[Q, T_r(X_1)] - i\sum_l \partial_l T_{r/l}(X_1)$, which has not the form of the terms in (3.9a). However, there is a second term t_2 with x_1, x_k exchanged: $t_2(x_1, ..., x_k...) = t_1(x_k, ..., x_1...) \sim \delta(x_k - x_1)B_{(j)}(x_1)$. The sum $t_1 + t_2$ must be partitioned in a symmetrical way (see the comment to (2.7),(2.8)) on the two operator combinations : $O_j(X_1)B_{(j)}(x_1)$: and : $O_j(X_1')B_{(j)}(x_k)$:, where X_1' is obtained from X_1 by exchanging x_1, x_k. Taking t_2 with : $O_j(X_1)B_{(j)}(x_1)$: and t_1 with : $O_j(X_1')B_{(j)}(x_k)$:, we really obtain (3.9).)

Now we replace in the natural operator decomposition of (3.9a) (with the Dirac equation (2.5) applied) and in (3.9b) the external leg $B_{(j)}(x_1)$ by the 'big external leg' $\sum_t \Delta_{(j)t}(x_1 - x_{r+1})T^t_{n-r}(X_2)$, where $T^t_{n-r}(X_2)$ is in the natural operator decomposition $T^t_{n-r}(X_2) = \sum_i t^t_{(n-r)i}(X_2) : O_i(X_2)$:. In detail this replacement reads

$$B_s \to \sum_t \Delta_{st} T^t_{n-r}, \quad \partial B_s \to \sum_t \partial\Delta_{st} T^t_{n-r}, \quad i\partial B_{Q(s)} \to \sum_t i\partial\Delta_{Q(s)t} T^t_{n-r}. \tag{3.11}$$

Note that ∂B_5 always appears in the form $\gamma^\nu \partial_\nu B_5 = -imB_5$, which we replace by $-im\Delta_{56}T^6_{n-r}$. The latter is meant by writing $\partial\Delta_{56}T^6_{n-r}$ in the following. Analogously we proceed with ∂B_6. For example, for $s = 2$ ($B_2 = F$) it happens that $\partial\Delta_{st}(x_1 - x_{r+1})$ contains terms $\sim \delta(x_1 - x_{r+1})$. These δ-terms are new terms, produced by the replacement (3.11). We omit them on both sides of the resulting equation, since we are interested in terms $\sim \Delta(x_1 - x_{r+1})$, $\Delta \neq \delta$ only. In this way equation (3.9) remains true

$$\sum_{st} \Big\{ : ([Q, T_r^s(X_1)]_\mp - i\sum_{l=1}^{r} \partial_l T^s_{r/l}(X_1))\Delta_{st}(x_1 - x_{r+1}) - iT^s_{r/1}(X_1)\partial\Delta_{st}(x_1 - x_{r+1}) \pm$$

$$\pm T_r^s(X_1) i\partial\Delta_{Q(s)t}(x_1 - x_{r+1}) \Big\} T^t_{n-r}(X_2) : \Big|_{\text{nod}, \not\sim \delta} =$$

$$= \sum_{ijt} \tau_{rj}(X_1)\Delta_{(j)t}(x_1 - x_{r+1})t^t_{(n-r)i}(X_2) : \mathcal{O}_j(X_1)\mathcal{O}_i(X_2) : |_{\not\sim\delta}, \qquad (3.12)$$

where 'nod' means the natural operator decomposition and '$\not\sim \delta$' that the terms $\sim \delta(x_1 - x_{r+1})$ are omitted. The numerical distributions on the r.h.s. vanish by means of (3.10). The only difference between the l.h.s. and the r.h.s. of (3.12) is that the arguments of some field operators $B(x_k)$, $k \in \{x_1, ...x_r\}$ are changed by using the δ-identity (2.6). Analogously, by means of the Cg-identities for $(T_{n-r}(X_2), T_{n-r/l}(X_2))$ and with $T^s_r(X_1) = \sum_j t^s_{rj}(X_1) : \mathcal{O}_j(X_1) :$ (natural op. dec.), one proves

$$\sum_{st} : T^s_r(X_1)\Big\{\Delta_{st}(x_1 - x_{r+1})(\pm[Q, T^t_{n-r}(X_2)]_\mp - i\sum_{l'=1}^{n-r} \partial_{l'} T^t_{n-r/l'}(X_2))+$$

$$+i\partial\Delta_{st}(x_1 - x_{r+1})T^t_{n-r/1}(X_2) - i\partial\Delta_{sQ(t)}(x_1 - x_{r+1})T^t_{n-r}(X_2)\Big\} : \Big|_{\text{nod}, \not\sim\delta} =$$

$$= \sum_{sij} t^s_{rj}(X_1)\Delta_{s(i)}(x_1 - x_{r+1})\tau_{(n-r)i}(X_2) : \mathcal{O}_j(X_1)\mathcal{O}_i(X_2) : |_{\not\sim\delta}, \qquad (3.13)$$

with

$$\tau_{(n-r)i}(X_2) = 0, \ \forall i. \qquad (3.14)$$

(C) The non-trivial step in the proof is the cancellation of the terms $\pm : T^s_r(X_1) i\partial\Delta_{Q(s)t}(x_1 - x_{r+1})T^t_{n-r}(X_2) :$ and $- : T^s_r(X_1)i\partial\Delta_{sQ(t)}(x_1 - x_{r+1})T^t_{n-r}(X_2) :$ in the sum of the l.h.sides of (3.12) and (3.13). Since the (anti)commutators of Q with F, u, ψ and $\overline{\psi}$ vanish (1.18), the following cases appear only:

(1) $s = 4$, $t = 1$: $\partial B_{Q(s)}(x_1) = \partial B_2(x_1) = \partial_\nu F^{\nu\mu}(x_1)$, $B_t(x_{r+1}) = A^\rho(x_{r+1})$ for (3.12); $B_s(x_1) = \partial^\mu \tilde{u}(x_1)$, $\partial B_{Q(t)}(x_{r+1}) = \partial B_3(x_{r+1}) = \partial^\rho u(x_{r+1})$ for (3.13). Omitting the terms $\sim \delta(x_1 - x_{r+1})$, one easily verifies $\partial\Delta_{21}(x_1 - x_{r+1}) + \partial\Delta_{43}(x_1 - x_{r+1}) = 0$.

(2) $s = 4$, $t = 2$: $\partial B_{Q(s)}(x_1) = \partial_\nu F^{\nu\mu}(x_1)$, $B_t(x_{r+1}) = F^{\rho\tau}(x_{r+1})$ for (3.12); $B_{Q(t)}(x_{r+1}) = 0$ and therefore $\Delta_{sQ(t)}(x_1 - x_{r+1}) = 0$ for (3.13). Omitting the terms $\sim \partial\delta(x_1 - x_{r+1})$, one obtains $\partial\Delta_{22}(x_1 - x_{r+1}) = 0$.

The further cases $s = 1$, $t = 4$ and $s = 2$, $t = 4$ are completely analogous to (1),(2). Taking this cancellation into account, the sum of the l.h.sides of (3.12) and (3.13) is exactly the natural operator decomposition (with the Dirac equation applied) of the terms in (3.8a) (symbolized by fig.7).

Adding up the r.h.sides of (3.12) and (3.13), the above cancellation (1),(2) (of all terms with the commutated leg contracted) happens, too. This can be seen in the following way: $: T^s_r(X_1)i\partial B_{Q(s)}(x_1) :$ in (3.9a) is the term with Q commutated with $B_s(x_1)$. Therefore, it belongs to a Cg-identity of type Ia and, by means of our induction assumption, is *unchanged* by going over from the natural operator decomposition to the Cg-operator decomposition (see sect. 2). This remains true for the terms $\pm : T^s_r(X_1)i\partial\Delta_{Q(s)t}(x_1 - x_{r+1})T^t_{n-r}(X_2) :$ in (3.12). The analogous statement holds true for the terms $- : T^s_r(X_1)i\partial\Delta_{sQ(t)}(x_1 - x_{r+1})T^t_{n-r}(X_2) :$ in (3.13). In other words, the terms cancelling above agree *identically* in the l.h.s. and r.h.s. of (3.12), respectively (3.13). Therefore, they also cancel in the sum of the r.h.sides of (3.12) and (3.13), without use of any δ-distribution. We conclude that the Cg-operator decomposition of the terms in (3.8a) is obtained by the sum of the r.h.sides of (3.12) and (3.13). Since the numerical distributions of all terms in the latter sum vanish (see (3.10),(3.14)), the Cg-identities hold. The reasoning given six, seven sentences above for $\pm : T^s_r(X_1)i\partial\Delta_{Q(s)t}(x_1 - x_{r+1})T^t_{n-r}(X_2) :$ holds true for all terms in (3.8a) (fig.7) of type Ia,b: the natural and the Cg-operator decomposition agree for the latter terms. □

(b) Tree diagrams in second and third order

In ref. [1] we considered gauge invariance in the form

$$[Q, T_n] = \text{sum of divergences}.$$

We proved it there for pure Yang-Mills theories in second order and for the tree diagrams in third order. Here, we return to these proofs by using the Cg-identity technique. Note that the Cg-identities imply gauge invariance in the stronger form (1.16), which contains the Q-vertices. Moreover, matter fields are included here.

Let us consider gauge invariance (1.16) in second order

$$[Q, T_2(x_1, x_2)] - i\sum_{l=1}^{2} \partial_\mu^{x_l} T^\mu_{2/l}(x_1, x_2). \tag{3.15}$$

In this section 3 we only consider the tree diagrams in (3.15). (The loops are non-degenerate.) We collect all terms in the natural operator decomposition of (3.15) belonging to a fixed tree operator combination $:\mathcal{O}::$ The non-local terms (i.e. the terms $\sim D^F(x_1 - x_2)$, $\partial D^F(x_1 - x_2)$, $\partial^\mu\partial^\nu D^F(x_1 - x_2)$ (no contraction of μ, ν), $\partial^\mu\partial^\nu\partial^\tau D^F(x_1 - x_2)$ (no contraction of $\mu\nu\tau$), $S^F(x_1 - x_2)$ or $S^F(x_2 - x_1)$) cancel. This was proven in subsect. (a). There remain the local terms ($\sim \delta(x_1 - x_2)$, $\sim \partial\delta(x_1 - x_2)$). In order to get them in the Cg-operator decomposition (2.7), we need to apply the identity (2.6). Since local terms can only appear for $|\mathcal{O}| \geq -1$ (see (2.9)), we merely have to consider the cases:

$$\begin{aligned}
(1)\quad &:\mathcal{O}_1(x_1, x_2) :=: A_{\mu a}(x_1)\partial_\nu u_b(x_1) A_{\rho d}(x_2) A_{\lambda e}(x_2):,\\
(2)\quad &:\mathcal{O}_2(x_1, x_2) :=: A_{\mu a}(x_1) u_b(x_1) A_{\rho d}(x_2) A_{\lambda e}(x_2):,\\
(3)\quad &:\mathcal{O}_3(x_1, x_2) :=: A_{\mu a}(x_1) u_b(x_1) A^\rho_d(x_2) F_{\rho\tau e}(x_2):,\\
(4)\quad &:\mathcal{O}_4(x_1, x_2) :=: A_{\mu a}(x_1) u_b(x_1) u_d(x_2)\partial_\rho \tilde{u}_e(x_2):,\\
(5)\quad &:\mathcal{O}_5(x_1, x_2) :=: u_a(x_1) A_{\mu b}(x_1)\overline{\psi}(x_2)...\psi(x_2):.
\end{aligned} \tag{3.16}$$

Additionally, we have the cases with x_1, x_2 exchanged in (3.16). The partition of the local terms on the two operator combinations $:\mathcal{O}_j(x_1, x_2):$ and $:\mathcal{O}_j(x_2, x_1):$ must be done in a symmetrical way (see the comment to (2.7), (2.8)). Note that the local terms, which we shall compute, do not cancel in the sum $\tau_j(x_1, x_2):\mathcal{O}_j(x_1, x_2): + \tau_j(x_2, x_1):\mathcal{O}_j(x_2, x_1):$, in all cases (3.16).

Let us consider the three tree terms

$$T_2(x_1, x_2) = \frac{-ig^2}{4} :A_{\mu a}(x_1) A_{\nu b}(x_1) A_{\rho d}(x_2) A_{\lambda e}(x_2): f_{abc} f_{dec}\{[g^{\mu\lambda}(\partial^\nu\partial^\rho D^F(x_1 - x_2) +$$

$$+ C_a g^{\nu\rho}\delta(x_1 - x_2)) - g^{\nu\lambda}(\partial^\mu\partial^\rho D^F(x_1 - x_2) + C_a g^{\mu\rho}\delta(x_1 - x_2))] - [\lambda \longleftrightarrow \rho]\} + ..., \tag{3.17}$$

$$T^\nu_{2/1}(x_1, x_2) = \frac{-ig^2}{2} :A_{\mu a}(x_1) u_b(x_1) A_{\rho d}(x_2) A_{\lambda e}(x_2): f_{abc} f_{dec}\{[g^{\mu\lambda}(\partial^\nu\partial^\rho D^F(x_1 - x_2) +$$

$$+ C_b g^{\nu\rho}\delta(x_1 - x_2)) - g^{\nu\lambda}(\partial^\mu\partial^\rho D^F(x_1 - x_2) + C_b g^{\mu\rho}\delta(x_1 - x_2))] - [\lambda \longleftrightarrow \rho]\} + ... \tag{3.18}$$

and the term of $T_{2/2}$ obtained from (3.18) by exchanging x_1 and x_2. (Note $T_{2/2}(x_1, x_2) = T_{2/1}(x_2, x_1)$.) These tree diagrams have singular order [15] $\omega = 0$ and, therefore, a free normalization term $\sim C_{a,b}\delta(x_1 - x_2)$ has been added.

Case (1): This case is of type Ia, whereas (2)-(5) are of type II. Due to $[Q, A_{\mu a}(x)] = i\partial_\mu u_a(x)$, we have two equal terms from Q commuted with (3.17) which contribute. There is also a contribution from $i\partial_\nu^{x_1} T^\nu_{2/1}(x_1, x_2)$, generated by the divergence $\partial_\nu^{x_1}$ acting on $u_b(x_1)$ in (3.18). Considering the local terms only, we see that this Cg-identity is fulfilled iff

$$C_a = C_b. \tag{3.19}$$

Case (2): There is only a contribution from the divergence $\partial_\nu^{x_1}$ acting on the numerical distribution in the term (3.18) of $T^\nu_{2/1}(x_1, x_2)$. Because of $\square D^F = \delta$, local terms appear in this Cg-identity only. One easily obtains that the latter is equivalent to

$$C_b = -\frac{1}{2}. \tag{3.20}$$

Gauge invariance fixes the values of C_a, C_b uniquely. The C_a-normalization term (in (3.17)) is the *4-gluon interaction*. It propagates into higher orders in the inductive construction of the T_n's (see sect. 4(b) of ref. [13]).

Case (3): The divergence $\partial_\nu^{x_1}$ acting on the numerical distribution in

$$T^\nu_{2/1}(x_1, x_2) = -ig^2 : A_{\mu a}(x_1) u_b(x_1) A^\rho_d(x_2) F_{\rho\tau e}(x_2) : \cdot$$
$$\cdot f_{abc} f_{dec} [g^{\mu\tau} \partial^\nu D^F(x_1 - x_2) - g^{\nu\tau} \partial^\mu D^F(x_1 - x_2)] + \ldots \tag{3.21}$$

produces a non-local and a local term. We need to consider the latter only

$$\partial_\nu^{x_1} T^\nu_{2/1}(x_1, x_2) = -\frac{ig^2}{2} : A_{\mu a}(x_1) u_b(x_1) A^\rho_d(x_2) F_{\rho\tau e}(x_2) : \cdot$$
$$\cdot g^{\mu\tau} (f_{abc} f_{dec} - f_{dbc} f_{aec}) \delta(x_1 - x_2) + \ldots, \tag{3.22}$$

where we have antisymmetrized $f_{abc} f_{dec}$ in a, d, because the other terms have this antisymmetry. There is another local contribution, coming from the C_b-term in (3.18), with the divergence acting on $A_{\mu a}(x_1)$. The resulting $\partial_\nu A_{\mu a}(x_1)$ must be antisymmetrized in $\nu\mu$, since the numerical distribution has this antisymmetry. We end up with

$$\partial_\nu^{x_1} T^\nu_{2/1}(x_1, x_2) = \frac{ig^2}{2} : A_{\mu a}(x_2) u_b(x_1) A^\rho_d(x_2) F_{\rho\tau e}(x_1) : g^{\mu\tau} f_{adc} f_{ebc} 2 C_b \delta(x_1 - x_2) + \ldots \tag{3.23}$$

for this second local contribution. Due to (3.20) and the Jacobi-identity, the sum of (3.22) and (3.23) vanishes.

Case (4): $\partial_\nu^{x_1}$ acting on the numerical distribution in

$$T^\nu_{2/1}(x_1, x_2) = ig^2 : A_{\mu a}(x_1) u_b(x_1) u_d(x_2) \partial_\rho \tilde{u}_e(x_2) : \cdot$$
$$\cdot f_{abc} f_{dec} [g^{\mu\rho} \partial^\nu D^F(x_1 - x_2) - g^{\nu\rho} \partial^\mu D^F(x_1 - x_2)] + \ldots \tag{3.24}$$

produces a non-local and a local term. The latter is

$$\partial_\nu^{x_1} T^\nu_{2/1}(x_1, x_2) = \frac{ig^2}{2} : A_{\mu a}(x_1) u_b(x_1) u_d(x_2) \partial_\rho \tilde{u}_e(x_2) : \cdot$$
$$\cdot g^{\mu\rho} (f_{abc} f_{dec} - f_{adc} f_{bec}) \delta(x_1 - x_2) + \ldots. \tag{3.25}$$

Another local contribution is generated by the divergence $\partial_\nu^{x_1}$ acting on the numerical distribution in

$$T^\nu_{2/1}(x_1,x_2) = \frac{ig^2}{2} : u_b(x_1)u_d(x_1)A_{\mu a}(x_2)\partial^\mu \tilde{u}_e(x_2) : f_{bdc}f_{aec}\partial^\nu D^F(x_1-x_2) + \ldots . \quad (3.26)$$

This second local contribution cancels with (3.25) by means of the Jacobi-identity.

Case (5): Replacing in (3.24) the open ghost line $-f_{dec} : u_d(x_2)\partial_\rho \tilde{u}_e(x_2) :$ by the open matter line $\frac{1}{2} : \overline{\psi}(x_2)\gamma_\rho \lambda_c \psi(x_2) :$, we obtain a first local contribution analogously to (3.25)

$$\partial_\nu^{x_1} T^\nu_{2/1}(x_1,x_2) = \frac{ig^2}{2} : u_a(x_1)A_{\mu b}(x_1)\overline{\psi}(x_2)\gamma^\mu \lambda_c \psi(x_2) : f_{abc}\delta(x_1-x_2) + \ldots . \quad (3.27)$$

Let us consider the two C-conjugated Compton diagrams

$$T^\nu_{2/1}(x_1,x_2) = -\frac{ig^2}{4}[: u_a(x_1)A_{\mu b}(x_2)\overline{\psi}(x_2)\gamma^\mu \lambda_b S^F(x_2-x_1)\lambda_a \gamma^\nu \psi(x_1) : +$$

$$+ : u_a(x_1)A_{\mu b}(x_2)\overline{\psi}(x_1)\gamma^\nu \lambda_a S^F(x_1-x_2)\lambda_b \gamma^\mu \psi(x_2) :] + \ldots . \quad (3.28)$$

Using

$$\partial_\nu^{x_1}(\overline{\psi}(x_1)\gamma^\nu S^F(x_1-x_2)\ldots) = -i\overline{\psi}(x_2)\delta(x_1-x_2)\ldots,$$

$$\partial_\nu^{x_1}(\ldots S^F(x_2-x_1)\gamma^\nu \psi(x_1)) = \ldots i\delta(x_2-x_1)\psi(x_2), \quad (3.29)$$

we obtain two further local contributions in $\partial_\nu^{x_1} T^\nu_{2/1}(x_1,x_2)$. These three local terms cancel by means of

$$2if_{abc}\lambda_c + \lambda_b\lambda_a - \lambda_a\lambda_b = 0. \quad (3.30)$$

In *QED* the ghost fields couple to the matter fields only. Therefore, the term (3.27) is absent. The sum (3.28) of the two C-conjugated Compton diagrams is already gauge invariant there.

We turn to the *tree diagrams in third order*, which have five external legs. In the natural operator decomposition of

$$[Q, T_3(x_1,x_2,x_3)] - i\sum_{l=1}^{3} \partial_\mu^{x_l} T^\mu_{3/l}(x_1,x_2,x_3)\Big|_{5-\text{legs}} \quad (3.31)$$

the non-local terms $\sim \Delta^{(1)}(x_{\pi 1}-x_{\pi 2})\Delta^{(2)}(x_{\pi 2}-x_{\pi 3})$ or $\sim \Delta^{(3)}(x_{\pi 1}-x_{\pi 2})\delta(x_{\pi 2}-x_{\pi 3})$ (for an arbitrary $\pi \in S_3$; $\Delta^{(1)},\Delta^{(2)},\Delta^{(3)} \neq \delta$) cancel (see subsect. (a)). For the $\Delta^{(3)}\delta$-terms the cancellation works only, if $x_{\pi 2}$ is replaced by $x_{\pi 3}$ (or vice versa) in the arguments of some field operators, using $\delta(x_{\pi 2}-x_{\pi 3})$. These replacements are determined by the Cg-identities for the subdiagrams with vertices $\{x_{\pi 2}, x_{\pi 3}\}$ (see (2.6), the tree diagrams in second order and subsect. (a)). There remain the local terms $\sim \delta^8(x_1-x_3, x_2-x_3)$. Due to (2.9),(2.10), they must have the field operators $: uAAAA :$. (In all other cases one has a derivative on the external field operators and, therefore, $|\mathcal{O}|+1$ is smaller than zero.) We write the local terms in the following symmetrical form

$$C^{\mu\nu\lambda\tau}_{abcde}\delta^8(x_1-x_3,x_2-x_3)[: \mathcal{O}_{6\,abcde\,\mu\nu\lambda\tau}(x_1) : + : \mathcal{O}_{6\,abcde\,\mu\nu\lambda\tau}(x_2) : + : \mathcal{O}_{6\,abcde\,\mu\nu\lambda\tau}(x_3) :], \quad (3.32)$$

with

$$: \mathcal{O}_{6\,abcde\,\mu\nu\lambda\tau}(x) :=: u_a(x)A_{\mu b}(x)A_{\nu c}(x)A_{\lambda d}(x)A_{\tau e}(x) : . \quad (3.33)$$

The constant $C^{\mu\nu\lambda\tau}_{abcde}$ must be invariant with respect to permutations of $(\mu, b), (\nu, c), (\lambda, d)$, (τ, e). There is no such tensor $C^{\mu\nu\lambda\tau}_{abcde}$ which is Lorentz- and SU(N)-invariant (see section 4 of ref. [4]). This finishes the proof of the Cg-identities for third order tree diagrams.

Remark: The Cg-operator decomposition of (3.31) which we constructed, has the following property: All local terms belong to a : $\mathcal{O}_6(x_i)$:, $i = 1, 2, 3$. For example, the operator combination : $\mathcal{O}_7$: $\stackrel{\text{def}}{=}$: $u(x_1)A(x_1)A(x_2)A(x_3)A(x_3)$: contains non-local terms only. This is not necessary: One can construct another Cg-operator decomposition of (3.31) in which all local terms are partitioned in a symmetrical way on the operator combinations of the non-local terms (e.g. : $\mathcal{O}_7$:). Then, : $\mathcal{O}_6$: would not appear. We followed this latter procedure in our Cg-operator decomposition (3.16) of the second order tree terms, with the exception of : $\mathcal{O}_2$:. (Only local terms exist with operators :uAAA:.)

4. Non-Degenerate and δ-Degenerate Terms

In this section we prove the Cg-identities belonging to non-degenerate operator combinations : $\mathcal{O}$: in (2.7) (including δ-degenerate terms if : $\mathcal{O}$: is of type II) and the Cg-identities with δ-degenerate : $\mathcal{O}$: in (2.7) which are of type Ia or Ib.

First we characterize the diagrams of the δ-degenerate terms in (1.16). In (3.2) we realized that the tree diagrams can have local support in second and third order only. Obviously, this result holds true even for arbitrary subdiagrams: *A term in (1.16) has at most two $\delta^{(4)}$-distributions (as propagators) in direct neighbourhood.* Neither a chain of three (or more) $\delta^{(4)}$, nor three $\delta^{(4)}$ joining the same vertex, do appear. Consequently, all diagrams of the δ-degenerate terms have the following form:

(1) Figure 1 with $\Delta(x_1 - x_2)$ replaced by $\delta(x_1 - x_2)$, or

(2) Figure 4 with $\Delta(x_2 - x_3)$ replaced by $\delta(x_2 - x_3)$.

Every δ-degenerate term can be transformed in non-degenerate form by applying (2.12) once in (1) and twice in (2). Remark: In (3.32) we proved that the tree terms with local support cancel in third order. We argued by means of the permutation symmetry of the external field operators. However, this symmetry is lost if this tree diagram is a subdiagram. Therefore, case (2) must be considered, too.

(a) Proof of the Cg-identities for A'_n, R'_n and D_n

By means of the Cg-identities for T_k and $\tilde{T}_k$ in lower orders $1 \leq k \leq n-1$, we are going to prove the Cg-identities for A'_n. The proof for R'_n is completely analogous. Together, we shall obtain the Cg-identities for $D_n = R'_n - A'_n$. This proof was given already in sect. 3.4 of ref. [2]. However, we did not care about the degenerate terms and the coupling to matter fields was not included there.

(A) In the natural operator decomposition of $[Q, A'_n] - i\sum_l \partial_l A'_{n/l}$, we always apply the Dirac equation (2.5) and obtain

$$[Q, A'_n(X)] - i\sum_l \partial_l A'_{n/l}(X) = \sum_j \alpha'^{(1)}_{nj}(X) : \mathcal{O}^{(1)}_j(X) :, \tag{4.1}$$

where $\alpha'^{(1)}_{nj}(X)$ does not vanish in general. By means of (2.12), we transform the δ-degenerate terms of type II in non-degenerate ones. Furthermore, we omit all truly degenerate terms.

In this way we obtain the Cg-operator decomposition (see (2.7),(2.8)) of the non-degenerate and δ-degenerate terms

$$[Q, A'_n(X)] - i\sum_l \partial_l A'_{n/l}(X)\Big|_{\text{non-deg.}} = \sum_j \alpha'^{(2)}_{nj}(X) : \mathcal{O}^{(2)}_j(X) : . \tag{4.2}$$

All $:\mathcal{O}^{(2)}_j(X):$ of type II are non-degenerate. Our aim is to prove

$$\alpha'^{(2)}_{nj}(X) = 0, \quad \forall j. \tag{4.3}$$

For this purpose we construct in (B) another operator decomposition of $[Q, A'_n] - i\sum_l \partial_l A'_{n/l}$, which fulfils (4.3) by construction. In part (C), we shall prove the agreement of the two operator decompositions constructed in (A) and (B).

(B) We insert in (see(2.4))

$$[Q, A'_n(X)] - i\sum_l \partial_l A'_{n/l}(X) = \sum_{Y,Z}\Big\{[Q, \tilde{T}_k(Y)] - i\sum_{l,x_l\in Y} \partial_{x_l}\tilde{T}_{k/l}(Y)\Big\} T_{n-k}(Z, x_n)] + \tag{4.4a}$$

$$+\sum_{Y,Z}\tilde{T}_k(Y)\Big\{[Q, T_{n-k}(Z, x_n)] - i\sum_{l,x_l\in\{Z,x_n\}} \partial_{x_l}T_{n-k/l}(Z, x_n)\Big\} \tag{4.4b}$$

the Cg-operator decompositions in lower orders k, $n-k$ (see (2.7),(2.8))

$$[Q, \tilde{T}_k(Y)] - i\sum_{l,x_l\in Y} \partial_{x_l}\tilde{T}_{k/l}(Y) = \sum_r \tilde{\tau}_{kr}(Y) : \mathcal{O}_r(Y) :, \tag{4.5}$$

$$\tilde{\tau}_{kr}(Y) = 0, \quad \forall r, \tag{4.6}$$

$$[Q, T_{n-k}(Z, x_n)] - i\sum_{l,x_l\in\{Z,x_n\}} \partial_{x_l}T_{n-k/l}(Z, x_n) = \sum_s \tau_{(n-k)s}(Z) : \mathcal{O}_s(Z) :, \tag{4.7}$$

$$\tau_{(n-k)s}(Z) = 0, \quad \forall s. \tag{4.8}$$

For $T_{n-k}(Z, x_n)$ in (4.4a) and $\tilde{T}_k(Y)$ in (4.4b) we insert the natural operator decomposition. Afterwards, we only apply Wick's theorem to (4.4a,b). The resulting operator decomposition

$$(4.4) = (4.4a) + (4.4b) = \sum_j \alpha'^{(3)}_{nj}(X) : \mathcal{O}^{(3)}_j(X) : \tag{4.9}$$

fulfils

$$\alpha'^{(3)}_{nj}(X) = 0, \quad \forall j, \tag{4.10}$$

due to (4.6),(4.8). Again, we omit all truly degenerate terms. (In part (C) we shall see that (4.1) and (4.9) can be obtained from each other by applying the δ-identity (2.6) only. Since the distinction in truly degenerate and δ-degenerate/non-degenerate terms does not depend on the application of (2.6), we omit the same terms here, as we did in part (A).) Moreover, we tranform the δ-degenerate terms of type II in non-degenerate form by means of (2.6). (Note that (2.6) is needed here, instead of the more special (2.12) in (4.1),(4.2).) There are several possibilities to do this transformation for a single term. It does not matter which one we choose. But the permuted terms must be transformed in the same way. Then, the sum over permutations is a totally symmetrical partition of the external legs on the vertices. For example a δ-degenerate term $\delta^{(8)}(x_1 - x_3, x_2 - x_3) : B_1(x_3)B_2(x_3)... :$ goes over in $\sum_{\pi \in S_3} \delta^{(8)}(x_1 - x_3, x_2 - x_3) : B_1(x_{\pi 1})B_2(x_{\pi 2})... :$. We end up with the operator decomposition

$$(4.4)\Big|_{\text{non-deg.}} = \sum_j \alpha'^{(4)}_{nj}(X) : \mathcal{O}^{(4)}_j(X) :, \tag{4.11}$$

which consists of δ-degenerate $: \mathcal{O}^{(4)}_j(X) :$ of type Ia,b and of non-degenerate $: \mathcal{O}^{(4)}_j(X) :$ of all types, and fulfils

$$\alpha'^{(4)}_{nj}(X) = 0, \quad \forall j \tag{4.12}$$

by means of (4.10).

(C) We are going to prove that (4.2) and (4.11) agree

$$: \mathcal{O}^{(2)}_j(X) :=: \mathcal{O}^{(4)}_j(X) :, \forall j, \quad \alpha'^{(2)}_{nj}(X) = \alpha'^{(4)}_{nj}(X) = 0, \forall j, \tag{4.13}$$

after relabeling the index j in a suitable way.

In (A) we *first* applied Wick's theorem to $A'_n(X) = \sum_{Y,Z} \tilde{T}_k(Y)T_{n-k}(Z, x_n)$, where $\tilde{T}_k(Y)$ and $T_{n-k}(Z, x_n)$ are in the natural operator decomposition, and similar for $A'_{n/l}(X)$. *Afterwards*, we commuted with Q, respectively took the divergence ∂_l and applied the Dirac equation (2.5) to obtain (4.1).

Note that the operation of taking the divergence ∂_l (with application of the Dirac equation) commutes with contracting, i.e. applying Wick's theorem. This relies on the fact that $S^{(\pm)}$, which is produced by contracting ψ and $\overline{\psi}$, fulfils the Dirac equation (2.5), too. Moreover, the operation $[Q, .]$ commutes with contracting. The latter was shown in detail in sect. 3.4 of ref. [2]. (The coupling to matter fields causes no complications, since Q commutes with $\psi, \overline{\psi}$.) The non-trivial step in that proof there is the cancellation of the terms, arising by contracting the commutated leg. This is exactely the same cancellation as demonstrated at the end of the proof in subsect. 3(a).

Reversing now in (A) the order of these operations, we obtain the following statement: Inserting the *natural operator decomposition* (with the Dirac equation applied), *instead of the Cg-operator decomposition*, for

$$[Q, \tilde{T}_k(Y)] - i \sum_{l, x_l \in Y} \partial_{x_l} \tilde{T}_{k/l}(Y) \tag{4.14}$$

and for

$$[Q, T_{n-k}(Z, x_n)] - i \sum_{l, x_l \in \{Z, x_n\}} \partial_{x_l} T_{n-k/l}(Z, x_n) \tag{4.15}$$

in (4.4a), respectively in (4.4b), and doing the other steps in exactely the same way as in (B), we obtain (4.1) instead of (4.9). We know by the induction hypothesis (see (2.7)) that the natural operator decomposition (with the Dirac equation applied) and the Cg-operator decomposition can be obtained from each other by applying (2.6) only. The latter property survives contracting between Y and (Z, x_n) in (4.4a,b). Therefore, (4.1) and (4.9) can be obtained from each other by applying (2.6) only.

The following argument is shown for (4.4b). The procedure for (4.4a) is completely analogous. Let us consider a term of (4.4b), in which we inserted for (4.15) different operator decompositions in (A) and (B). The corresponding subdiagram with vertices $\{Z, x_n\}$, belonging to (4.15), must be truly degenerate or δ-degenerate in order $(n-k)$ and it must be of type II, because the natural and the Cg-operator decomposition agree for non-degenerate terms and for terms of type Ia,b. Therefore, the (whole) considered term of (4.4b) is of type II, too. (The proof is finished for the terms in (4.4b) of type Ia,b.) At the beginning of this sect. 4, we realized that there are at most two $\delta^{(4)}$-distributions in direct neighbourhood. Consequently, we only must consider two cases which read (before contracting in (4.4b)):

case (1)

$$\tilde{t}_{kt}(Y) : \mathcal{O}_t(Y) :: B_1(y_1)...B_r(y_r)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n) = \tag{4.16a}$$

$$= \tilde{t}_{kt}(Y) : \mathcal{O}_t(Y) :: B_1(z_1)...B_r(z_r)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n), \tag{4.16b}$$

with $y_1, ...y_r, z_1, ...z_r \in \{x_1, x_2\} \subset \{Z, x_n\}$, $x_l \in [\{Z, x_n\} \setminus \{x_1, x_2\}]$, $2 \leq r \leq 4$,

case (2)

$$\tilde{t}_{kt}(Y) : \mathcal{O}_t(Y) :: B_1(y_1)...B_r(y_r)B_{r+1}(x_l)... : \delta^{(8)}(x_1 - x_3, x_2 - x_3)t_{n-k}(Z, x_n) = \tag{4.17a}$$

$$= \tilde{t}_{kt}(Y) : \mathcal{O}_t(Y) :: B_1(z_1)...B_r(z_r)B_{r+1}(x_l)... : \delta^{(8)}(x_1 - x_3, x_2 - x_3)t_{n-k}(Z, x_n), \tag{4.17b}$$

with $y_1, ...y_r, z_1, ...z_r \in \{x_1, x_2, x_3\} \subset \{Z, x_n\}$, $x_l \in [\{Z, x_n\} \setminus \{x_1, x_2, x_3\}]$, $3 \leq r \leq 5$, where $\tilde{t}_{kt}(Y) : \mathcal{O}_t(Y) :$ (no sum over t) is a term of the natural operator decomposition of $\tilde{T}_k(Y)$. The other factors $(: B_1...B_rB_{r+1}... : \delta^{(4/8)}t_{n-k})$ are a term of the natural operator decomposition of (4.15) (for (4.16a),(4.17a)), or they are the corresponding term in the Cg-operator decomposition of (4.15) (for (4.16b),(4.17b)). These terms belonging to (4.15) are δ-degenerate for $r = 2$ in case (1) ($r = 3$ in case (2)) and truly degenerate for $r = 3, 4$ in case (1) ($r = 4, 5$ in case (2)).

Now we consider the contractions of $: \mathcal{O}_t(Y) :$ with $B_1, ...B_r$. If there remain $s = 3$ or 4 field operators $B_1, ..., B_r$ uncontracted in case (1) ($s = 4$ or 5 in case (2)), the resulting diagram is truly degenerate and, therefore, we omit it.

If all operators $B_1, ..., B_r$ are contracted ($s = 0$), the resulting operator combinations in (4.16a) and (4.16b) ((4.17a) and (4.17b) respectively) agree and the numerical distributions are equal (e.g. $D^{(+)}(x - x_1)\delta(x_1 - x_2) = D^{(+)}(x - x_2)\delta(x_1 - x_2)$, $x \in Y$).

Let us consider *case (1)* with $s = 1$ or 2 field operators uncontracted, e.g. B_1 or B_1B_2. If such a term is δ-degenerate (this is possible for $s = 2$ only), it must be transformed in non-degenerate form by applying (2.12) for (4.16a), respectively (2.6) for (4.16b). Now we consider in each case the sum with the term with x_1, x_2 exchanged. This sum is the same for (4.16a) and (4.16b), namely for $s = 1$

$$\tilde{t}_{kt}(Y) : \mathcal{O}_t(Y)B_1(x_1)B_2(x_1)...B_r(x_1)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n) +$$

$$+ \tilde{t}_{kt}(Y) : \mathcal{O}_t(Y)B_1(x_2)B_2(x_1)...B_r(x_1)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n), \tag{4.18}$$

where all operators $B_2(x_1), ...B_r(x_1)$ are contracted with $\mathcal{O}_t(Y)$ in both terms; for $s = 2$ this sum reads

$$\tilde{t}_{kt}(Y) : \mathcal{O}_t(Y)B_1(x_1)B_2(x_2)B_3(x_1)...B_r(x_1)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n) +$$

$$+ \tilde{t}_{kt}(Y) : \mathcal{O}_t(Y)B_1(x_2)B_2(x_1)B_3(x_1)...B_r(x_1)B_{r+1}(x_l)... : \delta(x_1 - x_2)t_{n-k}(Z, x_n), \tag{4.19}$$

where all operators $B_3(x_1), ...B_r(x_1)$ are contracted with $\mathcal{O}_t(Y)$ in both terms.

We turn to *case (2)* with the operators $B_1, ..., B_s$, $s = 1, 2$ or 3, uncontracted. Again, the δ-degenerate terms must be transformed in non-degenerate form by using (2.12) for (4.17a), respectively (2.6) for (4.17b). Here we consider the sum of all 6 terms generated by permutations of x_1, x_2, x_3. These sums agree for (4.17a) and (4.17b): In both cases we obtain

$$\sum_{\pi \in S_3} :...B_1(x_{\pi 1})...B_s(x_{\pi s})B_{r+1}(x_l)...: \delta^{(8)}(x_1 - x_3, x_2 - x_3).... \tag{4.20}$$

There occur combinations of the various cases which we have discussed. These combinations are no problem, because they concern disjoint groups of vertices, each group having two (case (1)) or three (case (2)) vertices. □

(b) The Cg-identities with matter fields

Going over from $n-1$ to n, the really *new* Cg-identities belong to non-degenerate $: \mathcal{O} :$. Therefore, only such Cg-identities are considered in this section. We recall our convention of denoting numerical distributions of non-degenerate terms which we have introduced in ref. [2]:

$$t^{\alpha 2}_{AB...ab...}(x_1, x_2, \ldots) : A_a(x_1)B_b(x_2)\ldots : \tag{4.21}$$

means a distribution with external field operators (legs) A_a and B_b, a and b are colour indices. The subscripts $\alpha 2$ show that this term belongs to $T^{\alpha}_{n/2}(x_1, x_2, \ldots)$ with Q-vertex at the second argument of the numerical distribution t. An immediate consequence of this notation is the relation

$$t^{\alpha 1}_{AB...ab...}(x_1, x_2, \cdots) = \pm t^{\alpha 2}_{BA...ba...}(x_2, x_1, \cdots), \tag{4.22}$$

where we have a minus sign, if A, B are both ghost or both matter operators and a plus sign in all other cases. The Lorentz indices of the two operators A, B must also be reversed in (4.22). Note that (4.22) particularly holds, if A and B are the same field operators. Moreover, the sum over permutations of the vertices is present in $T_{n(/l)}$. For example,

$$T_n(x_1, ...x_n) = \sum_{l \neq i \neq j \neq k < l \neq j,\, i \neq k} : \overline{\psi}(x_i) t^{\mu\nu}_{\overline{\psi}\psi AA\, ab}(x_i, x_j, x_k, x_l, x_1...\bar{x}_i, \bar{x}_j, \bar{x}_k, \bar{x}_l..., x_n) \cdot$$

$$\cdot \psi(x_j) A_{\mu a}(x_k) A_{\nu b}(x_l) : + ..., \tag{4.23}$$

where the coordinates with bar must be omitted. The matrix multiplication $: \overline{\psi} t(...) \psi :$ concerns the spinor space and the space of the fundamental representation.

The Lorentz structure $T^{\psi\nu}_{1/1} \sim \gamma^\nu$ (1.15) propagates to higher orders in the inductive construction of the $T_{n/l}$. We conclude

$$t^{\nu 1 \ldots}_{\overline{\psi}\psi u \ldots} = \gamma^\nu \tilde{t}^{1 \ldots}_{\overline{\psi}\psi u \ldots}, \quad t^{\nu 2 \ldots}_{\overline{\psi}\psi u \ldots} = \tilde{t}^{2 \ldots}_{\overline{\psi}\psi u \ldots} \gamma^\nu, \tag{4.24}$$

which implies by means of the Dirac equation (2.5)

$$\partial^{x_1}_\nu T^\nu_{n/1}(x_1, ...x_n) =: \overline{\psi}(x_1)(\gamma^\nu \partial^{x_1}_\nu + im) \tilde{t}^{1 \ldots}_{\overline{\psi}\psi u \ldots}(x_1, x_2, ...) \psi(x_2) ... : + ... \tag{4.25}$$

In appendix A the following colour structures are proven

$$t_{\overline{\psi}\psi\, \alpha\beta} = \tau_{\overline{\psi}\psi} \delta_{\alpha\beta}, \quad t_{\overline{\psi}\psi B\, a\, \alpha\beta} = \tau_{\overline{\psi}\psi B} (\lambda_a)_{\alpha\beta}, \tag{4.26}$$

where $B = A, F, u$ and α, β are the indices of the fundamental representation.

The Cg-identities for pure Yang-Mills theories with 2-,3-, and 4-legs were given in refs. [2,3]. Adding the coupling to matter fields, the form of these Cg-identities remains the same, as far as we do not admit *external* ψ or $\overline{\psi}$. Moreover, the presence of *inner* matter lines does not change the symmetry properties of the numerical distributions. Therefore, the forms of the possible anomalies and of the normalization polynomials are unchanged. We see that the Cg-identities without external ψ or $\overline{\psi}$, can be proven as in the case of pure Yang-Mills theories (see refs. [2,4]).

However, there are additional Cg-identities with external legs ψ or $\overline{\psi}$. An operator combination $: \mathcal{O} :$, characterizing a Cg-identity, must fulfil $g_u = g_{\tilde{u}} + 1$ (see (2.10) for the notations). Consequently, only the following new Cg-identities appear (with at most 4-legs), where we use the classification given in sect. [2]:

Type Ia: For the external legs $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2)\partial_\nu u_a(x_3) :$ we get

$$\tau^{\nu}_{\overline{\psi}\psi A} = \tau^{\nu 3}_{\overline{\psi}\psi u}, \tag{4.27}$$

where the factor λ_a is omitted (see (4.26)). For $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2)\partial_\nu u_a(x_3) A_{\mu b}(x_4) :$ we obtain

$$t^{\nu\mu}_{\overline{\psi}\psi AA\,ab} = t^{\nu 3\mu}_{\overline{\psi}\psi uA\,ab}, \tag{4.28}$$

for $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2)\partial_\nu u_a(x_3) F_{\mu\tau\,b}(x_4) :$ we obtain

$$t^{\nu\mu\tau}_{\overline{\psi}\psi AF\,ab} = t^{\nu 3\mu\tau}_{\overline{\psi}\psi uF\,ab} \tag{4.29}$$

and for $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2) u_a(x_3)\partial_\nu F_{\mu\tau\,b}(x_4) :$ we get

$$t^{\nu 4\mu\tau}_{\overline{\psi}\psi uF\,ab} = \frac{1}{2}[g^{\nu\tau} t^{\mu}_{\overline{\psi}\psi u\tilde{u}\,ab} - g^{\nu\mu} t^{\tau}_{\overline{\psi}\psi u\tilde{u}\,ab}]. \tag{4.30}$$

Type Ib: For $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2) u_a(x_3)\frac{1}{2}(\partial_\nu A_{\mu\,b}(x_4) + \partial_\mu A_{\nu\,b}(x_4)) :$ we get

$$t^{\nu 4\mu}_{\overline{\psi}\psi uA\,ab} + (\nu \longleftrightarrow \mu) = 0. \tag{4.31}$$

Type II: For $: \mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2) u_a(x_3) :$ we obtain by means of (4.25) and (4.26)

$$0 = \sum_{l=3}^{n} \partial_\nu^{x_l} \tau^{\nu l}_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_4, ...) + \tag{4.32a}$$

$$+(\gamma^\nu \partial_\nu^{x_1} + im)\bar{\tau}^{1}_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_4, ...) + \bar{\tau}^{2}_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_4, ...)(\overleftarrow{\partial}_\nu^{x_2} \gamma^\nu - im) + \tag{4.32b}$$

$$+\frac{ig}{2}[\delta(x_3 - x_1)\tau_{\overline{\psi}\psi}(x_1, x_2, x_4, ...) - \tau_{\overline{\psi}\psi}(x_1, x_2, x_4, ...)\delta(x_2 - x_3)] + \tag{4.32c}$$

$$+\frac{g}{2}\gamma_\mu \delta(x_1 - x_2) t^{\mu}_{u\tilde{u}}(x_3, x_2, x_4, ...). \tag{4.32d}$$

In *QED* the corresponding Cg-identity has only three terms: The $l = 3$-term of (4.32a) (vertex) and the δ-degenerate terms (4.32c) (self-energy). The δ-distributions in the δ-degenerate terms (4.32c,d) are produced by the divergence $\partial_\nu^{x_l}$ of $\sum_l \partial_l T_{n/l}$, acting on $\overline{\psi}(x_l)\gamma^\nu S^F(x_l - .)$, $S^F(. - x_l)\gamma^\nu \psi(x_l)$ (4.32c) or on $\partial^\nu D^F(x_l - .)$ (4.32d). The 2-legs Cg-identity $\frac{1}{2}[t^{\nu}_{u\tilde{u}} g^{\alpha\mu} - t^{\mu}_{u\tilde{u}} g^{\alpha\nu}] = t^{\alpha 2\nu\mu}_{uF}$ (see ref. [2]) is inserted in (4.32d).

For : $\mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2)u_a(x_3)A_{\mu b}(x_4)$: we get similarly to (4.32)

$$0 = \sum_{l=3}^{n} \partial_\nu^{x_l} t^{\nu l \mu}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, x_5, ...)+ \tag{4.33a}$$

$$+(\gamma^\nu \partial_\nu^{x_1} + im)\bar{t}^{1\mu}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, x_5, ...) + \bar{t}^{2\mu}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, x_5, ...)(\overleftarrow{\partial}_\nu^{x_2} \gamma^\nu - im)+ \tag{4.33b}$$

$$+\frac{ig}{2}[\lambda_a \delta(x_3 - x_1)\lambda_b \tau^\mu_{\overline{\psi}\psi A}(x_1, x_2, x_4, x_5, ...) - \tau^\mu_{\overline{\psi}\psi A}(x_1, x_2, x_4, x_5, ...)\lambda_b \delta(x_2 - x_3)\lambda_a]+ \tag{4.33c}$$

$$-\frac{g}{2}\gamma_\lambda f_{abc}\lambda_c \delta(x_1 - x_2) t^{\mu\lambda}_{Au\tilde{u}}(x_4, x_3, x_2, x_5, ...)+ \tag{4.33d}$$

$$-\frac{ig}{2}[\gamma^\mu \lambda_b \delta(x_4 - x_1)\lambda_a \bar{\tau}^1_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_5, ...) - \bar{\tau}^2_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_5, ...)\lambda_a \delta(x_2 - x_4)\lambda_b \gamma^\mu]+ \tag{4.33e}$$

$$+g f_{abc}\lambda_c \tau^\mu_{\overline{\psi}\psi A}(x_1, x_2, x_3, x_5, ...)\delta(x_3 - x_4). \tag{4.33f}$$

Again, all δ-distributions in (4.33c,d,e,f) are generated by the divergence $\partial_\nu^{x_l}$, acting on $\overline{\psi}(x_l)\gamma^\nu S^F(x_l - .)$, $S^F(. - x_l)\gamma^\nu \psi(x_l)$ (4.33c,e) or on $\partial^\nu D^F(x_l - .)$ (4.33d,f). The 3-legs Cg-identity $t^{\alpha 2\nu\mu\lambda}_{uFA} = \frac{1}{2}[g^{\alpha\mu} t^{\nu\lambda}_{u\tilde{u}A} - g^{\alpha\nu} t^{\mu\lambda}_{u\tilde{u}A}]$ (see ref. [3]) has been used in (4.33d).

For : $\mathcal{O} :=: \overline{\psi}(x_1)...\psi(x_2)u_a(x_3)F_{\mu\tau\, b}(x_4)$: we proceed in a similar way and obtain by means of (4.25), (4.26)

$$0 = \sum_{l=3}^{n} \partial_\nu^{x_l} t^{\nu l \mu\tau}_{\overline{\psi}\psi u F\, ab}(x_1, x_2, x_3, x_4, x_5, ...)+ \tag{4.34a}$$

$$+(\gamma^\nu \partial_\nu^{x_1} + im)\bar{t}^{1\mu\tau}_{\overline{\psi}\psi u F\, ab}(x_1, x_2, x_3, x_4, x_5, ...) + \bar{t}^{2\mu\tau}_{\overline{\psi}\psi u F\, ab}(x_1, x_2, x_3, x_4, x_5, ...)(\overleftarrow{\partial}_\nu^{x_2} \gamma^\nu - im)+ \tag{4.34b}$$

$$+\frac{1}{4}[t^{\mu 4\tau}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, x_5, ...) - (\mu \longleftrightarrow \tau)]+ \tag{4.34c}$$

$$+\frac{ig}{2}[\lambda_a \delta(x_3 - x_1)\lambda_b \tau^{\mu\tau}_{\overline{\psi}\psi F}(x_1, x_2, x_4, x_5, ...) - \tau^{\mu\tau}_{\overline{\psi}\psi F}(x_1, x_2, x_4, x_5, ...)\lambda_b \delta(x_2 - x_3)\lambda_a]+ \tag{4.34d}$$

$$-\frac{g}{2}\gamma_\lambda f_{abc}\lambda_c \delta(x_1 - x_2) t^{\mu\tau\lambda}_{Fu\tilde{u}}(x_4, x_3, x_2, x_5, ...)+ \tag{4.34e}$$

$$+g f_{abc}\lambda_c \tau^{\mu\tau}_{\overline{\psi}\psi F}(x_1, x_2, x_3, x_5, ...)\delta(x_3 - x_4). \tag{4.34f}$$

The δ-distributions in (4.34d,e) are produced by the divergence $\partial_\nu^{x_l}$, acting on $\overline{\psi}(x_l)\gamma^\nu S^F(x_l - .)$, $S^F(. - x_l)\gamma^\nu \psi(x_l)$ (4.34d) or on $\partial^\nu D^F(x_l - .)$ (4.34e). The 3-legs Cg-identity $t^{\alpha 2\nu\mu\lambda\tau}_{uFF} = \frac{1}{2}[g^{\alpha\mu} t^{\nu\lambda\tau}_{u\tilde{u}F} - g^{\alpha\nu} t^{\mu\lambda\tau}_{u\tilde{u}F}]$ (see ref. [3]) is inserted in (4.34e). The δ-distribution in (4.34f) comes from a 4-gluon interaction term (see (3.17-20), (2.13)). In (4.34c,f) the derivative ∂_l of $\sum_l \partial_l T_{n/l}$ acts on a field operator A, which gives $\frac{1}{2}F$ (remember (4.31)). In the other terms (4.34a,b,d,e) ∂_l acts on the numerical distribution.

(c) C-Invariance and Conservation of the Cg-identities in the Splitting

The distribution splitting is the critical step in the proof of gauge invariance. In fact, the axial anomaly appears in this step in QED with pseudovector coupling: The normalization conditions of vector and axial current conservation are not compatible (see ref. [14]).

However, these are *two* different types of gauge invariance, whereas all our Cg-identities express the same gauge invariance (1.16). Therefore, it is not astonishing that all Cg-identities can be fulfilled *simultaneously*, as we are going to prove.

In the process of distribution splitting, only the Cg-identities with $|\mathcal{O}| \geq -1$ can be violated (see (2.15)). These are the identities (4.27), (4.32) and (4.33). The first one is easily preserved in the splitting by *defining* $\tau^{\nu 3}_{\overline{\psi}\psi u}$ by means of (4.27). This is allowed since the corresponding d-distributions fulfil (4.27).

To prove (4.32), (4.33) we make the ansatz (2.14) for the possible anomaly. Since C-invariance essentially restricts these anomalies, we first study *non-abelian C-invariance* [4]: The Dirac equation and its adjoint (2.5) are invariant under C-conjugation, which transforms ψ, $\overline{\psi}$ similarly to QED [15]

$$U_C\psi_\alpha(x)U_C^{-1} = C\overline{\psi}_\alpha(x)^T, \quad U_C\overline{\psi}_\alpha(x)U_C^{-1} = \psi_\alpha(x)^T C, \tag{4.35}$$

where a phase factor has been chosen suitably and the matrix C satisfies

$$\gamma^{\mu T} = -C^{-1}\gamma^\mu C, \quad \mu = 0,1,2,3, \tag{4.36}$$

$$C = -C^T = -C^{-1} = -C^+. \tag{4.37}$$

Then the fermion current transforms as follows

$$U_C j_a^\mu U_C^{-1} = \frac{g}{2}U_C : \overline{\psi}\gamma^\mu\lambda_a\psi : U_C^{-1}$$

$$= \frac{g}{2} : (C^{-1}\psi)_i\gamma^\mu_{ik}\lambda_a C_{kl}\overline{\psi}_l := -\frac{g}{2} : \overline{\psi}\lambda_a^T C^T\gamma^{\mu T}C^{-1}\psi :$$

$$= -\frac{g}{2} : \overline{\psi}\lambda_a^*\gamma^\mu\psi := U_{aa'}j^\mu_{a'}, \tag{4.38}$$

where the definition

$$U_{aa'}\lambda_{a'} \stackrel{\text{def}}{=} -\lambda_a^* = -\lambda_a^T \tag{4.39}$$

of $U_{aa'}$ has been used. The λ_a can be chosen to have the following property under complex conjugation

$$\lambda_a^* = \tau_a\lambda_a, \quad \tau_a = \pm 1, \tag{4.40}$$

where no summation over a is done. Then, $U_{aa'}$ is a simple diagonal matrix

$$U_{aa'} = -\delta_{aa'}\tau_a, \quad a = 1,\dots N^2-1, \tag{4.41}$$

which fulfils $U^{-1} = U^T = U$ and one easily obtains

$$f_{abc} = U_{aa'}U_{bb'}U_{cc'}f_{a'b'c'}, \quad -d_{abc} = U_{aa'}U_{bb'}U_{cc'}d_{a'b'c'} \tag{4.42}$$

by means of (4.39), (3.30) and $\{\lambda_a,\lambda_b\} = \frac{4}{N}\delta_{ab}\mathbf{1} + 2d_{abc}\lambda_c$ (see appendix A of ref. [4]). In order that T_1^ψ (1.5), $T_{1/1}^\psi$ (1.15) and T_1^u (1.4) are C-invariant, $U_C T_{1(/1)} U_C^{-1} = T_{1(/1)}$, the gauge boson and ghost fields must transform according to

$$U_{aa'}A_{a'} = U_C A_a U_C^{-1}, \quad U_{aa'}u_{a'} = U_C u_a U_C^{-1}, \quad U_{aa'}\tilde{u}_{a'} = U_C\tilde{u}_a U_C^{-1}. \tag{4.43}$$

Obviously, these C-transformed fields fulfil the wave equation, too. The restriction of the unitary operator U_C to the gluon-ghost sector, which implements $U_{aa'}$, is explicitly constructed in appendix A of ref. [4]. One easily verifies by means of (4.42), (4.43) that

T_1^A (1.3), $T_{1/1}^A$ (1.12) and $T_{1/1}^u$ (1.13) are C-invariant, too. Moreover, the transformed fields $U_C\psi U_C^{-1}$, $U_C\overline{\psi}U_C^{-1}$ (4.35) and $U_C A U_C^{-1}$, $U_C u U_C^{-1}$, $U_C\tilde{u}U_C^{-1}$ (4.43) fulfil the same (anti)commutation relations as the original ones [15]; otherwise we would have a contradiction to the unitary implementability of these transformations.

We turn to the inductive step. One easily finds that $A'_{n(/l)}$ (2.4) and $R'_{n(/l)}$ are C-invariant and, therefore, also $D_{n(/l)} = R'_{n(/l)} - A'_{n(/l)}$. Assuming $R_{n(/l)}$ to be a splitting solution of $D_{n(/l)}$, we conclude that $U_C R_{n(/l)} U_C^{-1}$ is a splitting solution of $U_C D_{n(/l)} U_C^{-1} = D_{n(/l)}$. Consequently,

$$R^C_{n(/l)} \stackrel{\mathrm{def}}{=} \frac{1}{2}(R_{n(/l)} + U_C R_{n(/l)} U_C^{-1}) \tag{4.44}$$

is a C-invariant splitting solution of $D_{n(/l)}$, and the corresponding $T_{n(/l)}$ is C-invariant, too

$$U_C T_n U_C^{-1} = T_n, \quad U_C T_{n/l} U_C^{-1} = T_{n/l}, \quad l \leq n. \tag{4.45}$$

Let us consider numerical distributions with one external pair $(\overline{\psi}, \psi)$. By means of (4.35), (4.37), (4.43) and (4.45) we obtain

$$T_n(x_1, ...x_n) = \sum_{k\neq i<j\neq k} \Big\{ :\overline{\psi}(x_i) t_{\overline{\psi}\psi B...a...}(x_i, x_j, x_k, ...)\psi(x_j): +$$

$$+ :\overline{\psi}(x_j) t_{\overline{\psi}\psi B...a...}(x_j, x_i, x_k, ...)\psi(x_i): \Big\}: B_a(x_k)... : + ... \tag{4.46a}$$

$$= U_C T_n(x_1, ...x_n) U_C^{-1} = \sum_{k\neq i<j\neq k} U_{ab}\Big\{ :\overline{\psi}(x_j) C t_{\overline{\psi}\psi B...b...}(x_i, x_j, x_k, ...)^T C^{-1}\psi(x_i): +$$

$$+ :\overline{\psi}(x_i) C t_{\overline{\psi}\psi B...b...}(x_j, x_i, x_k, ...)^T C^{-1}\psi(x_j): \Big\}: B_a(x_k)... : + ..., \tag{4.46b}$$

where $B_a, ...$ is a gluon or a ghost operator and the transposition 'T' in (4.46b) concerns the spinor space and the space of the fundamental representation. Considering the numerical distributions belonging to the same operator combination in (4.46a) and (4.46b), we may not *directly* conclude that they are equal, because different operator combinations are not linearly independent if δ-distributions are present. However, the t-distributions in (4.46) are C-invariant, i.e. the mentioned equality holds true, but a somewhat complicated argument is needed to prove this. We avoid it by the simple symmetrization

$$t'_{\overline{\psi}\psi B...a...}(x_i, x_j, x_k, ...) = \frac{1}{2}[t_{\overline{\psi}\psi B...a...}(x_i, x_j, x_k, ...) + U_{ab} C t_{\overline{\psi}\psi B...b...}(x_j, x_i, x_k, ...)^T C^{-1}]. \tag{4.47}$$

Due to (4.45), these t' are numerical distributions of the original T_n. For the t-distributions without external $(\overline{\psi}, \psi)$, we see from (4.43) that C-invariance concerns the colour structure only; the symmetrizations analogous to (4.47) are symmetrizations of the colour tensors. For $T_{n/l}$ we proceed in the same way. Due to the distinction of the Q-vertex, it happens that C-invariance connects different numerical distributions. For example, we obtain for the C-symmetrized $t^{1...}_{\overline{\psi}\psi u...}$, $t^{2...}_{\overline{\psi}\psi u...}$-distributions (see (4.24))

$$\bar{t}'^{1...}_{\overline{\psi}\psi u...a...}(x_1, x_2, x_3, ...) = -U_{ab} C \bar{t}'^{2...}_{\overline{\psi}\psi u...b...}(x_2, x_1, x_3, ...)^T C^{-1}. \tag{4.48}$$

Again the transposition 'T' concerns the spinor space and the space of the fundamental representation.

We return to the ansatz for the possible anomalies in (4.32) and (4.33). Note that the latter Cg-identity appears one step later in the inductive proof. Taking (2.14), Lorentz covariance, the colour structures (A.11), (A.12) and invariance with respect to permutations of the inner vertices into account, we obtain

$$\sum_{l=3}^{n} \partial_\nu^{x_l} \tau^{\nu l}_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_4, ...) + ... =$$

$$= (K_0 + K_1\gamma^\nu\partial_\nu^{x_1} + K_2\gamma^\nu\partial_\nu^{x_2} + K_3\gamma^\nu\partial_\nu^{x_3})\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n), \tag{4.49}$$

$$\sum_{l=3}^{n+1} \partial_\nu^{x_l} t^{\nu l\mu}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, x_5, ...x_{n+1}) + ... =$$

$$= \gamma^\mu(K_4\delta_{ab}\mathbf{1}_{N\times N} + K_5 d_{abc}\lambda_c + K_6 f_{abc}\lambda_c)\delta^{4n}(x_1 - x_{n+1}, ..., x_n - x_{n+1}). \tag{4.50}$$

In these two equations we go over to the C-symmetrized numerical distributions $t'_{...}$, $\tau'_{...}$ (4.47). Since this symmetrization is linear and commutes with taking the divergence, we obtain the C-symmetrized anomalies on the r.h.s.. The latter read by means of (4.35), (4.36), (4.37), (4.39) and (4.42)

$$\sum_{l=3}^{n} \partial_\nu^{x_l} \tau'^{\nu l}_{\overline{\psi}\psi u}(x_1, x_2, x_3, x_4, ...) + ... =$$

$$= \Big\{\frac{1}{2}[K_1 + K_2]\gamma^\nu(\partial_\nu^{x_1} + \partial_\nu^{x_2}) + K_3\gamma^\nu\partial_\nu^{x_3}\Big\}\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n), \tag{4.51}$$

$$\sum_{l=3}^{n+1} \partial_\nu^{x_l} t'^{\nu l\mu}_{\overline{\psi}\psi u A\, ab}(x_1, x_2, x_3, x_4, ...x_{n+1}) + ... = \gamma^\mu K_6 f_{abc}\lambda_c\delta^{4n}(x_1 - x_{n+1}, ..., x_n - x_{n+1}). \tag{4.52}$$

In contrast to the $:\overline{\psi}\psi uA:$-Cg-identity in QED (see ref. [16]), the anomaly (4.50) is not completely removed by the C-symmetrization (4.47). The non-abelian Cg-identity (4.33) contains completely new terms. We already have met this phenomenom in second order: The term (3.27), which has the C-invariant colour structure $f_{abc}\lambda_c$ (4.52), is absent in QED.

Now we are going to remove the possible anomalies (4.51) and (4.52) by means of *finite* renormalizations of the $t'_{...}$, $\tau'_{...}$-distributions. For simplicity we omit the primes. Taking the singular order of the numerical distributions [15], Lorentz covariance, the colour structures (A.8), (A.11) and invariance with respect to permutations of the inner vertices into account, the normalization polynomials of $\tau^{\nu}_{\overline{\psi}\psi A} = \tau^{\nu 3}_{\overline{\psi}\psi u}$ (see (4.27)), $\tau^{\nu l}_{\overline{\psi}\psi u}$ $(l = 4, ...n)$, $\bar{\tau}^{1}_{\overline{\psi}\psi u}$, $\bar{\tau}^{2}_{\overline{\psi}\psi u}$ and $\tau_{\overline{\psi}\psi}$ must have the following forms

$$N^{\nu}_{\overline{\psi}\psi A}(x_1, ...x_n) = N^{\nu 3}_{\overline{\psi}\psi u}(x_1, ...x_n) = C_3\gamma^\nu\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n), \tag{4.53}$$

$$N^{\nu l}_{\overline{\psi}\psi u}(x_1, ...x_n) = C_0\gamma^\nu\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n), \quad l = 4, ...n, \tag{4.54}$$

$$\bar{N}^{1}_{\overline{\psi}\psi u}(x_1, ...x_n) = C_1\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n),$$

$$\bar{N}^{2}_{\overline{\psi}\psi u}(x_1, ...x_n) = C_2\delta^{4(n-1)}(x_1 - x_n, ..., x_{n-1} - x_n), \tag{4.55}$$

$$N_{\overline{\psi}\psi}(x_1, ...x_{n-1}) = [C_4 + C_5\gamma^\nu\partial_\nu^{x_1} + C_6\gamma^\nu\partial_\nu^{x_2}]\delta^{4(n-2)}(x_1 - x_{n-1}, ..., x_{n-2} - x_{n-1}). \tag{4.56}$$

The 4-legs distributions $t^{l}_{\overline{\psi}\psi uA}$ $(l=3,4,...n)$, $\bar{t}^{1}_{\overline{\psi}\psi uA}$ and $\bar{t}^{2}_{\overline{\psi}\psi uA}$ have singular order [15] $\omega=-1$ and, therefore, have no freedom of normalization. Note that $t_{u\tilde{u}}$ and $t_{Au\tilde{u}}$ could be renormalized, too. However, this causes a chain reaction of other renormalizations in order to maintain the Cg-identities already proven (see refs. [2,3]). We shall see that the removal of the anomalies (4.51), (4.52) is possible without renormalizing $t_{u\tilde{u}}$, $t_{Au\tilde{u}}$. C-invariance restricts the normalization constants in $\bar{N}^{1}_{\overline{\psi}\psi u}$, $\bar{N}^{2}_{\overline{\psi}\psi u}$ (4.55) and $N_{\overline{\psi}\psi}$ (4.56)

$$C_1=C_2,\quad C_5=-C_6. \tag{4.57}$$

In the step from $n-1$ to n we must find constants C_0, $C_1=C_2$, C_3, C_4, $C_5=-C_6$ fulfilling (see (4.32))

$$-\Big\{\frac{1}{2}[K_1+K_2]\gamma^\nu(\partial_\nu^{x_1}+\partial_\nu^{x_2})+K_3\gamma^\nu\partial_\nu^{x_3}\Big\}\delta^{4(n-1)}(x_1-x_n,...,x_{n-1}-x_n)=$$

$$=\sum_{l=3}^{n}\partial_\nu^{x_l}N^{\nu l}_{\overline{\psi}\psi u}(x_1,x_2,x_3,x_4,...)+$$

$$+(\gamma^\nu\partial_\nu^{x_1}+im)\bar{N}^{1}_{\overline{\psi}\psi u}(x_1,x_2,x_3,x_4,...)+\bar{N}^{2}_{\overline{\psi}\psi u}(x_1,x_2,x_3,x_4,...)(\overleftarrow{\partial}_\nu^{x_2}\gamma^\nu-im)+$$

$$+\frac{ig}{2}[\delta(x_3-x_1)N_{\overline{\psi}\psi}(x_1,x_2,x_4,...)-N_{\overline{\psi}\psi}(x_1,x_2,x_4,...)\delta(x_2-x_3)], \tag{4.58}$$

for arbitrarily given K_1, K_2, K_3, in order to remove the anomaly (4.51). This equation (4.58) is equivalent to the linear system

$$-\frac{1}{2}[K_1+K_2]=-C_0+C_1,$$

$$-K_3=C_3-C_0+igC_5, \tag{4.59}$$

which has a 2-dimensional set of solutions for C_0, C_1, C_3, C_5. Having performed a renormalization fulfilling (4.59), the Cg-identity (4.32) holds true and there remains a C-invariant and gauge invariant freedom of normalization: (4.53), (4.54), (4.55) and (4.56) with C_k replaced by C'_k $(k=0,1,...6)$, the latter fulfilling (see (4.57), (4.59))

$$C'_1=C'_2,\quad C'_5=-C'_6$$

$$-C'_0+C'_1=0,\quad C'_3-C'_0+igC'_5=0. \tag{4.60}$$

Note that C'_4 is restricted neither by C-invariance nor by gauge invariance.

The freedom (4.60) is used in the inductive step from n to $n+1$, where the anomaly in (4.52) must be removed. We have to solve the equation (see (4.33))

$$-\gamma^\mu K_6 f_{abc}\lambda_c\delta^{4n}(x_1-x_{n+1},...,x_n-x_{n+1})=$$

$$=\frac{ig}{2}[\lambda_a\delta(x_3-x_1)\lambda_b N'^{\mu}_{\overline{\psi}\psi A}(x_1,x_2,x_4,x_5,...)-N'^{\mu}_{\overline{\psi}\psi A}(x_1,x_2,x_4,x_5,...)\lambda_b\delta(x_2-x_3)\lambda_a]+$$

$$-\frac{ig}{2}[\gamma^\mu\lambda_b\delta(x_4-x_1)\lambda_a\bar{N}'^{1}_{\overline{\psi}\psi u}(x_1,x_2,x_3,x_5,...)-\bar{N}'^{2}_{\overline{\psi}\psi u}(x_1,x_2,x_3,x_5,...)\lambda_a\delta(x_2-x_4)\lambda_b\gamma^\mu]+$$

$$+gf_{abc}\lambda_c N'^{\mu}_{\overline{\psi}\psi A}(x_1,x_2,x_3,x_5,...)\delta(x_3-x_4), \tag{4.61}$$

for an arbitrarily given K_6. Inserting (4.53), (4.55) we obtain

$$-K_6 f_{abc}\lambda_c = \frac{ig}{2}(C_3' + C_1')[\lambda_a, \lambda_b] + gC_3' f_{abc}\lambda_c = -gC_1' f_{abc}\lambda_c, \tag{4.62}$$

where (3.30) has been used. The linear system (4.60), (4.62) has a one dimensional set of solutions for C_0', C_1', C_3', C_5'. Therefore, the anomaly (4.52) can be removed.

For $n = 3$ the story is simpler, since $\partial_\nu^{x_3}\delta(x_1 - x_3, x_2 - x_3) = -(\partial_\nu^{x_1} + \partial_\nu^{x_2})\delta(x_1 - x_3, x_2 - x_3)$ and $N^{\nu l}_{\overline{\psi}\psi u}$ $(l = 4, \ldots n)$ does not appear, but the results are the same.

For the Cg-identities without external $\overline{\psi}$, ψ, we found the following rule in trying to remove the possible anomaly of a certain Cg-identity: Every renormalization of the δ-degenerate terms which preserves all other Cg-identities, can be absorbed in a renormalization of the non-degenerate terms. In other words: We do not gain an additional freedom of normalization from the δ-degenerate terms to remove the anomaly (see refs. [3,4]). The above calculation shows that this rule holds true also for (4.32c), and one easily verifies it for renormalizations of $t_{u\tilde{u}}$ in (4.32d). However, this rule is broken in (4.33): There is no freedom of normalization of the non-degenerate terms (4.33a,b), but the possible anomaly (4.52) can be removed by renormalizing the δ-degenerate terms (4.33c,d,e,f). This is a new feature, which can be understood by the fact that Q commutes with $\overline{\psi}$ and ψ (1.20).

In order to complete the proof of the Cg-identities we still have to show that the Cg-identities of type Ia,b belonging to δ-degenerate : $\mathcal{O}$: can be maintained in the process of distribution splitting. This holds generally true for all Cg-identities of type Ia,b, as we realized in sect. 2 of ref. [3]: The Cg-identities of type Ia are *identifications* of numerical distributions of the theory with one Q-vertex with numerical distributions of the normal theory, both in the natural operator decomposition (see e.g. (4.27-30)). They can easily be preserved in the process of distribution splitting, because we are free to normalize the extended theory properly. The Cg-identities of type Ib concern the *Lorentz structure* only (see e.g. (4.31)). They hold true in the natural operator decomposition, if the Lorentz structure is preserved in the process of distribution splitting, which is always assumed.

Remark: The Cg-identities of type Ia,b belonging to δ-degenerate : $\mathcal{O}$: can be proven directly, without using the Cg-identities for A', R' proven in sect. 4(a). The corresponding terms have one non-basic field operator. Therefore, the δ-distribution which is responsible for the δ-degeneration originates from a 4-gluon interaction term (see (1.17), (2.13), (3.19-20)). The corresponding diagram is obtained by replacing in fig.1 $\Delta(x_1 - x_2)$ by $\delta(x_1 - x_2)$. The non-basic field operator is an element of $\{B_3, \ldots B_r\}$. Analogously to sect. 3(a) the considered Cg-identities can be reduced to the corresponding Cg-identities for the subdiagram with vertices $x_2, \ldots x_n$.

5. Discrete Symmetries and Pseudo-Unitarity

(a) Compatibility of P-, T-, C-Invariance and Pseudo-Unitarity

First we study the behaviour of the free field operators with respect to parity P and time reversal T (see refs. [15,17,18]). Since the free Dirac equation (2.5) is diagonal in the colour space, we can adopt the transformations of the matter fields from the abelian case

$$U_P\psi_\alpha(x)U_P^{-1} = i\gamma^0\psi_\alpha(x_P), \quad U_P\overline{\psi}_\alpha(x)U_P^{-1} = -i\overline{\psi}_\alpha(x_P)\gamma^0, \tag{5.1}$$

$$V_T \psi_\alpha(x) V_T^{-1} = \gamma^5 C \psi_\alpha(x_T), \quad V_T \overline{\psi}_\alpha(x) V_T^{-1} = \overline{\psi}_\alpha(x_T) C^{-1} \gamma^5, \tag{5.2}$$

where $x_P \stackrel{\text{def}}{=} (x^0, -\mathbf{x})$, $x_T \stackrel{\text{def}}{=} (-x^0, \mathbf{x})$ and some phase factors have been chosen in a suitable way (see sect. 4.4 of ref.[15]). U_P is unitary, whereas V_T is antiunitary, otherwise the (anti)commutation relations of the free field operators would not be invariant with respect to these transformations [15,17,18]. Then, the matter current j_a^μ (1.6) transforms according to (remember (4.39))

$$U_P j_a^\mu(x) U_P^{-1} = j_{\mu a}(x_P), \quad V_T j_a^\mu(x) V_T^{-1} = -U_{ab} j_{\mu b}(x_T). \tag{5.3}$$

Note that it is not a printing mistake that μ is once an upper and once a lower index. We want T_1 to be P- and T-invariant

$$U_P T_1(x) U_P^{-1} = T_1(x_P), \quad V_T T_1(x) V_T^{-1} = \tilde{T}_1(x_T) = -T_1(x_T), \tag{5.4}$$

where $\tilde{T}_{1(/1)}$ is the first order T-distribution of the inverse S-matrix. Then, due to $UQU^{-1} = Q$ for $U = U_P, V_T$, gauge invariance implies

$$i\partial^\nu_{x_T} \tilde{T}_{1/1\,\nu}(x_T) = [Q, \tilde{T}_1(x_T)] = V_T [Q, T_1(x)] V_T^{-1} =$$

$$= V_T i\partial^x_\nu T^\nu_{1/1}(x) V_T^{-1} = -i(-\partial^\nu_{x_T}) V_T T^\nu_{1/1}(x) V_T^{-1}$$

and a similar statement for parity. Therefore, P- and T-invariance means for $T_{1/1}$

$$U_P T^\nu_{1/1}(x) U_P^{-1} = T_{1/1\,\nu}(x_P), \quad V_T T^\nu_{1/1}(x) V_T^{-1} = \tilde{T}_{1/1\,\nu}(x_T) = -T_{1/1\,\nu}(x_T). \tag{5.5}$$

Inserting T_1^ψ (1.5), $T^\psi_{1/1}$ (1.15) and T_1^u (1.4) in (5.4), (5.5), we see that the gluon and ghost fields must transform according to

$$U_P A_a^\mu(x) U_P^{-1} = A_{\mu a}(x_P), \quad U_P u_a(x) U_P^{-1} = u_a(x_P), \quad U_P \tilde{u}_a(x) U_P^{-1} = \tilde{u}_a(x_P), \tag{5.6}$$

$$V_T A_a^\mu(x) V_T^{-1} = -U_{ab} A_{\mu b}(x_T), \quad V_T u_a(x) V_T^{-1} = -U_{ab} u_b(x_T), \quad V_T \tilde{u}_a(x) V_T^{-1} = -U_{ab} \tilde{u}_b(x_T). \tag{5.7}$$

One easily verifies that T_1^A (1.3), $T^A_{1/1}$ (1.12) and $T^u_{1/1}$ (1.13) are P- and T-invariant, too. We leave it to the reader (or refer to [15,17,18]) to check that the Dirac equation, respectively wave equation and the (anti)commutation relations of the free field operators are invariant with respect to the transformations (5.1), (5.2), (5.6) and (5.7).

Next we consider pseudo-unitarity. The conjugation 'K' was introduced in sect. 5 of ref. [4] and in [6]. It is related to taking the adjoint '+' and transforms the free field operators in the following way

$$A^\mu(x)^K = A^\mu(x), \quad u(x)^K = u(x), \quad \tilde{u}(x)^K = -\tilde{u}(x),$$

$$\psi(x)^K = \psi(x)^+, \quad \overline{\psi}(x)^K = \overline{\psi}(x)^+. \tag{5.8}$$

Obviously 'K' agrees with taking the adjoint on the physical subspace, which is defined by excluding scalar and longitudinal gluons and the ghost fields u, $\tilde{u}$ (see sect. 5 of ref. [4]). In contrast to the C-, P- and T-transformations, the conjugation 'K' cannot be unitarily or antiunitarily implemented in Fock space, since

$$(B_1 B_2)^K = B_2^K B_1^K, \tag{5.9}$$

where B_1, B_2 are arbitrary free field operators. It is easy to check that T_1 and $T_{1/1}$ are pseudo-unitary

$$T_1(x)^K = \tilde{T}_1(x) = -T_1(x), \quad T_{1/1}(x)^K = \tilde{T}_{1/1}(x) = -T_{1/1}(x). \tag{5.10}$$

We turn to the inductive step in the construction of the $T_{n(/l)}$. By means of the fact that $A'_{n(/l)}$, $R'_{n(/l)}$ are sums of tensor products of lower $T_{k(/r)}$, $\tilde{T}_{k(/r)}$, $k < n$ (see(2.4)), one obtains the P-, T-invariance and pseudo-unitarity of $A'_{n(/l)}$, $R'_{n(/l)}$ and $D_{n(/l)} = R'_{n(/l)} - A'_{n(/l)}$ (see ref. [15]). The discrete symmetries P, T, C and pseudo-unitarity can be violated (by local terms) in the causal splitting $D_{n(/l)} = R_{n(/l)} - A_{n(/l)}$ only. Analogously to (4.44), one obtains a P-, respectively T-invariant or pseudo-unitary splitting solution by symmetrizing an arbitrary splitting solution (see ref. [15]). For example, the symmetrization

$$R^u_{n(/l)} \stackrel{\text{def}}{=} \frac{1}{2}(R_{n(/l)} - R^K_{n(/l)}) \tag{5.11}$$

yields a pseudo-unitary splitting solution. Note that these symmetrizations only change the normalization of $R_{n(/l)}$. The question is, whether these symmetrizations are compatible. Since

$$U_P^2 = U_V, \quad V_T^2 = U_V, \quad V_T U_P = U_V U_P V_T, \tag{5.12}$$

where U_V is a so-called valency operator, $U_V^2 = \mathbf{1}$, the group generated by U_C, U_P, V_T has more than 8 elements. Due to $U_C^2 = \mathbf{1}$, $[U_C, U_P] = 0$, $[U_C, V_T] = 0$, it has 16 elements. By symmetrizing an arbitrary splitting solution with respect to this group, we obtain a C- and P- and T-invariant splitting solution (see sect. 4.4 of ref. [15]).

Now we add the symmetrization (5.11) with respect to pseudo-unitarity. To prove

$$(:\mathcal{O}:)^K =: \mathcal{O}^K :, \tag{5.13}$$

for an arbitrary operator combination $\mathcal{O}$, one needs (5.9) and that 'K' transforms a creation operator in an annihilation operator and vice versa. Because $B \to UBU^{-1}$ (for $U = U_C, U_P, V_T$) transforms a creation operator (annihilation operator respectively) in a creation (annihilation) operator, we obtain

$$U : \mathcal{O} : U^{-1} =: U\mathcal{O}U^{-1} :, \quad U = U_C, U_P, V_T. \tag{5.14}$$

By means of (4.35), (4.43), (5.1), (5.2), (5.6), (5.7) and (5.8) one easily verifies

$$UB^K U^{-1} = (UBU^{-1})^K, \ U = U_C, U_P, V_T, \ B = A, F, u, \tilde{u}, \overline{\psi}, \psi. \tag{5.15}$$

Together with (5.13) and (5.14) we conclude

$$U(:\mathcal{O}:)^K U^{-1} =: U\mathcal{O}^K U^{-1} :=: (U\mathcal{O}U^{-1})^K := (U : \mathcal{O} : U^{-1})^K. \tag{5.16}$$

This implies for $U = U_C$ (with $R_{n(/l)} = \sum_j r_j : \mathcal{O}_j :$)

$$U_C R^K_{n(/l)} U_C^{-1} = \sum_j r_j^* U_C(:\mathcal{O}_j:)^K U_C^{-1} = \sum_j r_j^* (U_C : \mathcal{O}_j : U_C^{-1})^K = (U_C R_{n(/l)} U_C^{-1})^K, \tag{5.17}$$

where $*$ is the complex conjugation. This means that the C-transformation $R_{n(/l)} \to U_C R_{n(/l)} U_C^{-1}$ (see (4.44)) commutes with $R_{n(/l)} \to -R^K_{n(/l)}$ (see (5.11)). Therefore, through

$$R^{Cu}_{n(/l)} \stackrel{\text{def}}{=} \frac{1}{4}[R_{n(/l)} - R^K_{n(/l)} + U_C R_{n(/l)} U_C^{-1} - U_C R^K_{n(/l)} U_C^{-1}], \tag{5.18}$$

we obtain a C-invariant and pseudo-unitary splitting solution $R^{Cu}_{n(/l)}$. Similarly one concludes from (5.16) that the P- and T-transformations (on the space of the retarded distributions $R_{n(/l)}$) commute with $R_{n(/l)} \to -R^K_{n(/l)}$ (see sect. 4.4 of ref. [15] for the detailed form of the P- and T-transformations). Moreover note that $R_{n(/l)} \to -R^K_{n(/l)}$ is an involution. Consequently, the group G, generated by $R_{n(/l)} \to -R^K_{n(/l)}$ and by these P-, T-, C-transformations (on the space of the $R_{n(/l)}$'s), has 32 elements. By symmetrizing an arbitrary splitting solution with respect to this group

$$R^s_{n(/l)} = \frac{1}{32} \sum_{g \in G} R^g_{n(/l)} \tag{5.19}$$

and constructing

$$T'_n \stackrel{\text{def}}{=} R^s_n - R'_n, \quad T^s_n(x_1, ..., x_n) = \frac{1}{n!} \sum_\pi T'_n(x_{\pi 1}, ..., x_{\pi n}),$$

$$T'_{n/l} \stackrel{\text{def}}{=} R^s_{n/l} - R'_{n/l}, \quad T^s_{n/l}(x_1, ..., x_n) = \frac{1}{n!} \sum_\pi T'_{n/\pi^{-1}l}(x_{\pi 1}, ..., x_{\pi n}), \tag{5.20}$$

we obtain a $T_{n(/l)}$-distribution, which fulfils all these symmetries

$$U_C T^s_{n(/l)}(X) U_C^{-1} = T^s_{n(/l)}(X), \quad U_P T^s_n(X_P) U_P^{-1} = T^s_n(X), \quad U_P T^s_{n/l\,\nu}(X_P) U_P^{-1} = T^{s\,\nu}_{n/l}(X),$$

$$V_T \tilde{T}^s_n(X_T) V_T^{-1} = T^s_n(X), \quad V_T \tilde{T}^s_{n/l\,\nu}(X_T) V_T^{-1} = T^{s\,\nu}_{n/l}(X), \quad \tilde{T}^s_{n(/l)}(X)^K = T^s_{n(/l)}(X), \tag{5.21}$$

where $X \stackrel{\text{def}}{=} (x_1, ...x_n)$, $X_P \stackrel{\text{def}}{=} (x_{1P}, ...x_{nP})$, $X_T \stackrel{\text{def}}{=} (x_{1T}, ...x_{nT})$ and $\tilde{T}^s_{n(/l)}$ is the n-th order T-distribution of $S^{-1}(g)$. The latter means

$$\tilde{T}^s_n(X) = -T^s_n(X) - \sum_{Y \cup Z = X,\, Y \neq \emptyset,\, Z \neq \emptyset} T_{n-k}(Y) \tilde{T}_k(Z) = -T^s_n(X) - R'_n(X) - R''_n(X) \tag{5.22}$$

and analogously for $\tilde{T}^s_{n/l}$. Thereby, R'_n is the usual R'_n-distribution of the causal construction and R''_n is a similar object

$$R''_n(X) \stackrel{\text{def}}{=} \sum_{U \cup V = \{x_1, ...x_{n-1}\},\, U \neq \emptyset} T_{n-k}(U) \tilde{T}_k(V, x_n). \tag{5.23}$$

Note that R'_n and R''_n are given by the induction hypothesis and that their sum $R'_n(X) + R''_n(X) = \sum T_{n-k}(Y) \tilde{T}_k(Z)$ is symmetrical with respect to permutations of $x_1, ..., x_n$. Therefore, the latter holds true also for $\tilde{T}^s_n(X)$.

Moreover, if we start with a Lorentz covariant and SU(N)-invariant $R_{n(/l)}$ in (5.19), the resulting $T^s_{n(/l)}$ is Lorentz covariant and SU(N)-invariant, too.

We turn to the *numerical* distributions t^s in the natural operator decomposition of $T^s_{n(/l)}$. We assume them to be Lorentz covariant, SU(N)-invariant and to respect the permutation symmetry (5.20). (Since we have chosen the natural operator decomposition, these properties could be violated by local terms only. For example, a single numerical distribution could have a non-covariant normalization term which drops out in the sum over all diagrams.) Similarly to C-invariance (see (4.45-47)), it requires some work to prove the P-, T-, C-invariance and the pseudo-unitarity of the t^s-distributions belonging to $T^s_{n(/l)}$. We avoid it by symmetrizing them with respect to G

$$t^{s\prime} \stackrel{\mathrm{def}}{=} \frac{1}{32} \sum_{g \in G} t^s_g, \tag{5.24}$$

analogously to (4.47). Thereby the transformation $t \longrightarrow t_g$, $g \in G$ is defined in the following way: It is induced from the operator transformation

$$T(X) \longrightarrow g(T)(X) = U_C T(X) U_C^{-1},\ U_P T(X_P) U_P^{-1},\ V_T \tilde{T}(X_T) V_T^{-1},\ \tilde{T}(X)^K, \ldots \tag{5.25}$$

(see (5.21)). Applying these transformations to $T(X) = \sum_j t_j(X) : \mathcal{O}_j(X) :$, we define $g(T)(X) = \sum_j t_{j\,g}(X) : \mathcal{O}_j(X) :$, e.g. for $T^\nu_{n/l}(X) = \sum_j t^{\nu l}_j(X) : \mathcal{O}_j(X) :$ and the P-transformation

$$g(T^\nu_{n/l})(X) = \sum_i t^l_{i\nu}(X_P) U_P : \mathcal{O}_i(X_P) : U_P^{-1} \stackrel{\mathrm{def}}{=} \sum_j t^{\nu l}_{j\,g}(X) : \mathcal{O}_j(X) :,$$

more precisely

$$t^{\nu l}_{j\,g}(X) \stackrel{\mathrm{def}}{=} t^l_{i\nu}(X_P) \quad \text{for} \quad : \mathcal{O}_j(X) := U_P : \mathcal{O}_i(X_P) : U_P^{-1}. \tag{5.26}$$

Note $\{U_P : \mathcal{O}_i(X_P) : U_P^{-1} | i\} = \{: \mathcal{O}_j(X) : | j\}$ and that $t_j(X)$ and $t_{j\,g}(X)$ belong to the *same* operator combination $: \mathcal{O}_j(X) :$ in $T(X)$, respectively $g(T)(X)$. For g being the C-conjugation, we refer the reader to (4.46-48). Let us illustrate this transformation by another example: By identifying the numerical distributions in $T^\nu_{n/l}(X)$ and in $V_T \tilde{T}_{n/l\,\nu}(X_T) V_T^{-1}$ belonging to the operator combination $: \overline{\psi}(x_1) \ldots \psi(x_2) u_a(x_3) A_{\mu b}(x_4) :$ we get

$$t^{\nu l \mu}_{\overline{\psi}\psi u A\, ab\, g}(X) = (-U_{ac})(-U_{bd}) \gamma^5 C \tilde{t}^l_{\overline{\psi}\psi u A\, cd\, \nu \mu}(X_T)^* \gamma^5 C^{-1},$$

where $\tilde{t}^{(l)}$ is a numerical distribution of $\tilde{T}_{n(/l)}$. Because of (5.22), the latter can be defined by

$$\tilde{t} \stackrel{\mathrm{def}}{=} -t - r' - r'', \tag{5.27}$$

with r', r'' being the numerical distributions in the natural operator decomposition of $R'_{n(/l)}$, $R''_{n(/l)}$. (Remark: In sect. 4(a) we have proven the Cg-identities for r'. Analogously one proves the Cg-identities for r''. With (5.27) we conclude that the Cg-identities for t imply the Cg-identities for $\tilde{t}$.) Due to (5.21), the $t^{s\prime}$-distributions (5.24) are still numerical distributions of $T^s_{n(/l)}$.

(b) Compatibility with the Cg-Identities

The last step is to prove the *compatibility of the discrete symmetries and pseudo-unitarity with the Cg-identities.* For this purpose we consider finite renormalizations $T_{n(/l)} \rightarrow T_{n(/l)} + N_{n(/l)}$, $N_{n(/l)} = \sum_j n_j^{(l)} : \mathcal{O}_j :$ and a possible anomaly $\mathcal{A}(X) = \sum_j a_j : \mathcal{O}_j :$ violating the operator gauge invariance (1.16)

$$\sum_l \partial_l T_{n/l}(X) + i[Q, T_n(X)] = \mathcal{A}(X). \tag{5.28}$$

Due to (5.22), $\tilde{T}_{n(/l)}$ must be renormalized, too

$$\tilde{T}_{n(/l)} \rightarrow \tilde{T}_{n(/l)} + \tilde{N}_{n(/l)} = \tilde{T}_{n(/l)} - N_{n(/l)}, \tag{5.29}$$

and taking additionally the gauge invariance of $R'_{n(/l)}$ and $R''_{n(/l)}$ into account we conclude

$$\sum_l \partial_l \tilde{T}_{n/l}(X) + i[Q, \tilde{T}_n(X)] = \tilde{\mathcal{A}}(X) = -\mathcal{A}(X). \tag{5.30}$$

Analogously to (5.25), (5.26), the operator transformations $N \rightarrow g(N)$, $\mathcal{A} \rightarrow g(\mathcal{A})$, $g \in G$, induce transformations $n_j^{(l)} \rightarrow n_{j\,g}^{(l)}$, $a_j \rightarrow a_{j\,g}$ of the corresponding numerical distributions.

By means of

$$Q^K = Q, \quad U_C Q U_C^{-1} = Q, \quad U_P Q U_P^{-1} = Q, \quad V_T Q V_T^{-1} = Q, \tag{5.31}$$

we obtain from (5.28)

$$\sum_l \partial_l g(T_{n/l}) + i[Q, g(T_n)] = (-1)^{T(g)} g(\mathcal{A}), \quad g \in G, \tag{5.32}$$

where $T(g)$ is the number of time reversals in g. Next we consider a Cg-identity belonging to (5.28)

$$\sum_l \partial_l t_l + t = a, \tag{5.33}$$

with t being a sum of terms, possibly containing δ-degenerate terms. According to (5.25-26), the numerical distributions t_l, t, a are replaced by $t_{l\,g}, t_g, a_g$ in the step from (5.28) to (5.32). Therefore, performing the operator decomposition in (5.32), the Cg-identity (5.33) changes into

$$\sum_l \partial_l t_{l\,g} + t_g = (-1)^{T(g)} a_g. \tag{5.34}$$

(We have pointed out several times that such an operator decomposition is not unique. (5.34) can be deduced *rigorously* from (5.33). But for this purpose some properties of the transformations $t_{(l)} \rightarrow t_{(l)g}$, $a \rightarrow a_g$ must be worked out. Note that these transformations are not linear or antilinear in general.) Similarly to (5.33), (5.34), one obtains the following statement: If n_l, n remove the anomaly in (5.33), i.e. they fulfil

$$\sum_l \partial_l n_l + n = -a, \tag{5.35}$$

the transformed normalization polynomials remove $(-1)^{T(g)}a_g$

$$\sum_l \partial_l n_{l\,g} + n_g = -(-1)^{T(g)} a_g. \qquad (5.36)$$

Now we make the ansatz for the possible anomalies with the symmetrical $t^{s\prime}$-distributions (5.24)

$$\sum_l \partial_l t_l^{s\prime} + t^{s\prime} = a^s. \qquad (5.37)$$

By means of (5.34) we conclude

$$a^s = (-1)^{T(g)} a_g^s. \qquad (5.38)$$

In the case of C-invariance this has given us a restriction of the possible anomalies (4.51), (4.52). We have proven that there exist normalization polynomials n_l^s, n^s of $t_l^{s\prime}$, $t^{s\prime}$ which remove the anomaly a^s, i.e. they fulfil (5.35). The question is whether $t_l^{s\prime} + n_l^s$, $t^{s\prime} + n^s$ are still invariant with respect to G: $(t_{(l)}^{s\prime} + n_{(l)}^s)_g = t_{(l)}^{s\prime} + n_{(l)}^s$? Generally, this is not true. But we can symmetrize the normalization polynomials with respect to G

$$n_l^{s\prime} \stackrel{\text{def}}{=} \frac{1}{32} \sum_{g\in G} n_{l\,g}^s, \qquad n^{s\prime} \stackrel{\text{def}}{=} \frac{1}{32} \sum_{g\in G} n_g^s. \qquad (5.39)$$

We conclude by means of (5.36), (5.38) that $n_l^{s\prime}$, $n^{s\prime}$ remove the anomaly a^s, too. Moreover, $t_l^{s\prime} + n_l^{s\prime}$, $t^{s\prime} + n^{s\prime}$ are still invariant with respect to G.

Summing up, we have proven that the numerical distributions in the natural operator decomposition of T_n, $T_{n/l}$ can be normalized in such a way that, simultaneously, they are Lorentz covariant, SU(N)-invariant, P-, T-, C-invariant, pseudo-unitary and fulfil the Cg-identities.

Appendix A: Colour Structure in the Fundamental Representation

In this appendix we study the structure of the colour tensor of an arbitrary diagram with external field operators

$$\text{case 1}: \quad :\mathcal{O}_1 :=: \overline{\psi}_\alpha(x_1)\psi_\beta(x_2):, \qquad (A.1)$$

$$\text{case 2}: \quad :\mathcal{O}_2 :=: \overline{\psi}_\alpha(x_1)\psi_\beta(x_2)u_a(x_3):, \qquad (A.2)$$

$$\text{case 3}: \quad :\mathcal{O}_3 :=: \overline{\psi}_\alpha(x_1)\psi_\beta(x_2)u_a(x_3)A_b^\mu(x_4):, \qquad (A.3)$$

These tensors are SU(N)-scalars (A.1), -vectors (A.2), -tensors of second rank (A.3) in the fundamental representation. They are used in subsects. 4(b,c) for the determination of

- the colour structure of the numerical distributions $t_{\overline{\psi}\psi}$, $t_{\overline{\psi}\psi B}$ (4.26),
- the normalization polynomials of $t_{\overline{\psi}\psi}$, $t_{\overline{\psi}\psi A}$, $t_{\overline{\psi}\psi u}^l$ (4.53-56),
- the ansatz for the possible anomalies of the Cg-identities characterized by $:\mathcal{O}_2:$ and $:\mathcal{O}_3:$ (4.49-50).

All diagrams with external legs : $\mathcal{O}_1$:, $\mathcal{O}_2$: or : $\mathcal{O}_3$: have an open matter line going through the diagram from ψ to $\overline{\psi}$. For every vertex on this line we have a matrix λ_{a_i} $(i = 1, 2, ...l)$ (see (1.5), (1.15)). These matrizes get multiplied: : $\overline{\psi}_\alpha(x_1)(\lambda_{a_1}\lambda_{a_2}...\lambda_{a_l})_{\alpha\beta}\psi_\beta(x_2)$:, since the propagators $S^{(\pm)}_{\alpha\beta}$, $S^{ret/av}_{\alpha\beta}$, $S^F_{\alpha\beta}$ are $\sim \delta_{\alpha\beta}$. By applying several times

$$\lambda_a\lambda_b = \frac{2}{N}\delta_{ab}\mathbf{1}_{N\times N} + d_{abc}\lambda_c + if_{abc}\lambda_c, \tag{A.4}$$

this matrix product can be reduced to

$$(\lambda_{a_1}\lambda_{a_2}...\lambda_{a_l}) = F^1_{a_1a_2...a_l}\mathbf{1}_{N\times N} + F^2_{a_1a_2...a_lc}\lambda_c, \tag{A.5}$$

where $F^1_{a_1a_2...a_l}$ and $F^2_{a_1a_2...a_lc}$ are invariant SU(N)-tensors of rank $r = l,\ l+1$ respectively. (A basis for the latter tensors (up to rank $r = 5$) was given in appendix A of ref. [4]. We shall repeatedly use these bases.) The other vertices and the other inner lines of the diagram (which may include matter loops) contribute another invariant SU(N)-tensor $F_{b_1b_2...b_s}$. The total colour structure of the diagram is

$$F_{b_1b_2...b_s}\lambda_{a_1}\lambda_{a_2}...\lambda_{a_l} = F_{b_1b_2...b_s}F^1_{a_1a_2...a_l}\mathbf{1}_{N\times N} + F_{b_1b_2...b_s}F^2_{a_1a_2...a_lc}\lambda_c. \tag{A.6}$$

There are contractions between the indices $b_1, b_2, ...b_s$ and $a_1, a_2, ...a_l$:

In *case 1* all these indices must be contracted $(s = l)$

$$F_{a_1a_2...a_l}F^1_{a_1a_2...a_l}\mathbf{1}_{N\times N} + F_{a_1a_2...a_l}F^2_{a_1a_2...a_lc}\lambda_c. \tag{A.7}$$

Since $F_{a_1a_2...a_l}F^1_{a_1a_2...a_l} = C \in \mathbf{C}$ and since there exists no SU(N)-vector $F_{a_1a_2...a_l}F^2_{a_1a_2...a_lc}$, we obtain

$$(A.7) = C\mathbf{1}_{N\times N}. \tag{A.8}$$

In *case 2* one index $(= a)$ is not contracted. There are two possibilities: (1) $a \in \{b_1, ...b_s\}$, $(s = l+1)$

$$F_{aa_1a_2...a_l}F^1_{a_1a_2...a_l}\mathbf{1}_{N\times N} + F_{aa_1a_2...a_l}F^2_{a_1a_2...a_lc}\lambda_c. \tag{A.9}$$

(2) $a \in \{a_1, ...a_l\}$, $(s = l-1)$

$$F_{a_1a_2...a_{l-1}}F^1_{a_1a_2...a...a_{l-1}}\mathbf{1}_{N\times N} + F_{a_1a_2...a_{l-1}}F^2_{a_1a_2...a...a_{l-1}c}\lambda_c. \tag{A.10}$$

In both cases the term $(FF^1)_a\mathbf{1}_{N\times N}$ vanishes because there exists no SU(N)-vector $(FF^1)_a$. All SU(N)-tensors of second rank are multiples of δ_{ac}. Therefore, $(FF^2)_{ac}$ must have this form for (1) and (2). We obtain

$$(A.9), (A.10) = C\lambda_a, \quad C \in \mathbf{C}. \tag{A.11}$$

In *case 3* two indices $(= a, b)$ are not contracted in (A.6). Then, $(FF^1)_{ab}$ is an SU(N)-tensor of second rank: $(FF^1)_{ab} = C^1\delta_{ab}$, $C^1 \in \mathbf{C}$ (see above), whereas $(FF^2)_{abc}$ is an SU(N)-tensor of rank 3. Therefore, the latter must have the form $(FF^2)_{abc} = C^2d_{abc} + C^3f_{abc}$, $C^2, C^3 \in \mathbf{C}$. Consequently, we obtain the total colour structure

$$C^1\delta_{ab}\mathbf{1}_{N\times N} + C^2d_{abc}\lambda_c + C^3f_{abc}\lambda_c. \tag{A.12}$$

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Figure Captions

Fig.1. The two field operators $B_1(x_1)B_2(x_1)$ can not be partitioned on two *different* vertices by means of (2.6). Therefore, terms of (1.16) having the structure of fig.1, are *truly* degenerate.

Fig.2. Four external legs can be partitioned on two vertices by means of (2.6). Therefore, such a term in the natural operator decomposition of (1.16) is *truly* degenerate.

Fig.3. Five external legs and three vertices - a *truly* degenerate term in the natural operator decomposition of (1.16).

Figs.4,5 and 6. Combinations of figs.2 and 3 with fig.1. Not all external legs can be partitioned on *different* vertices. Therefore, such terms in the natural operator decomposition of (1.16) are *truly* degenerate.

Fig.7. In the natural operator decomposition of (1.16) some non-degenerate, some δ-degenerate and all truly degenerate terms (apart from the terms of figs.2 and 3) have the structure of fig.7, where $\Delta \neq \delta^{(4)}$. The Cg-identities for these terms are proven in subsect. 3(a) by means of the Cg-identities for the two (arbitrary) subdiagrams with vertices X_1, respectively X_2.

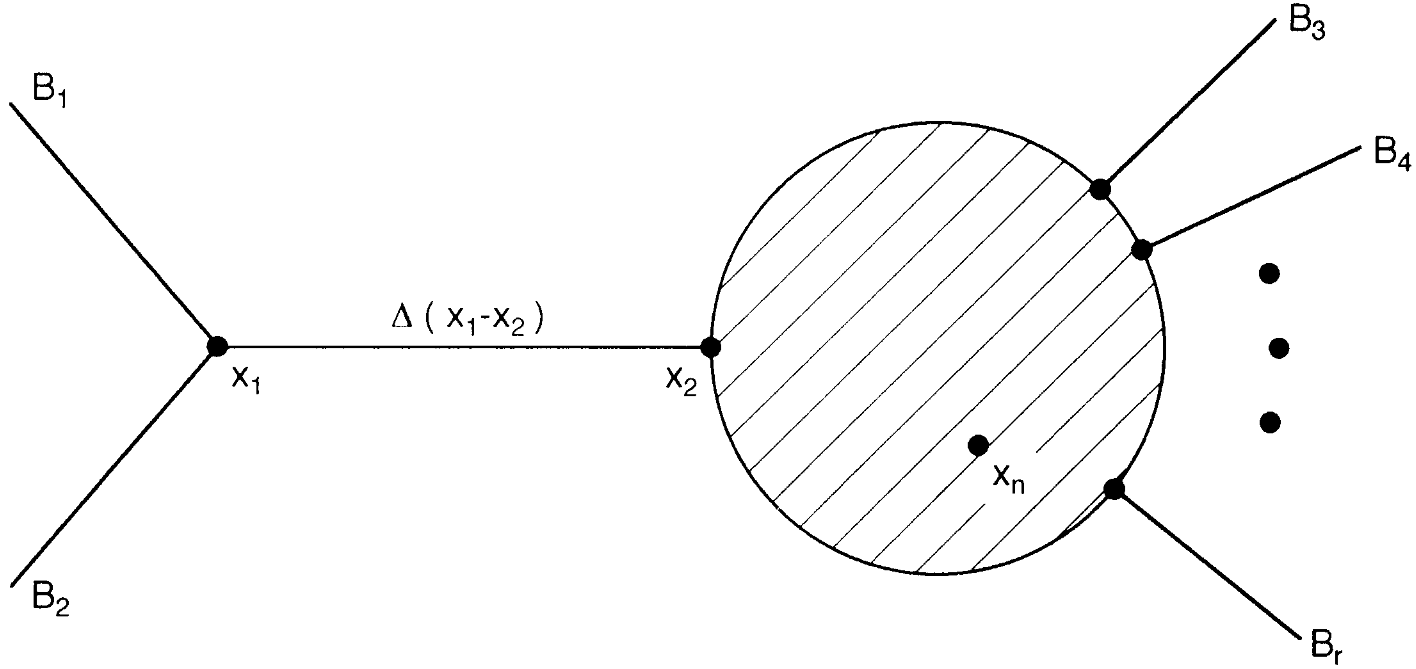

Fig. 1

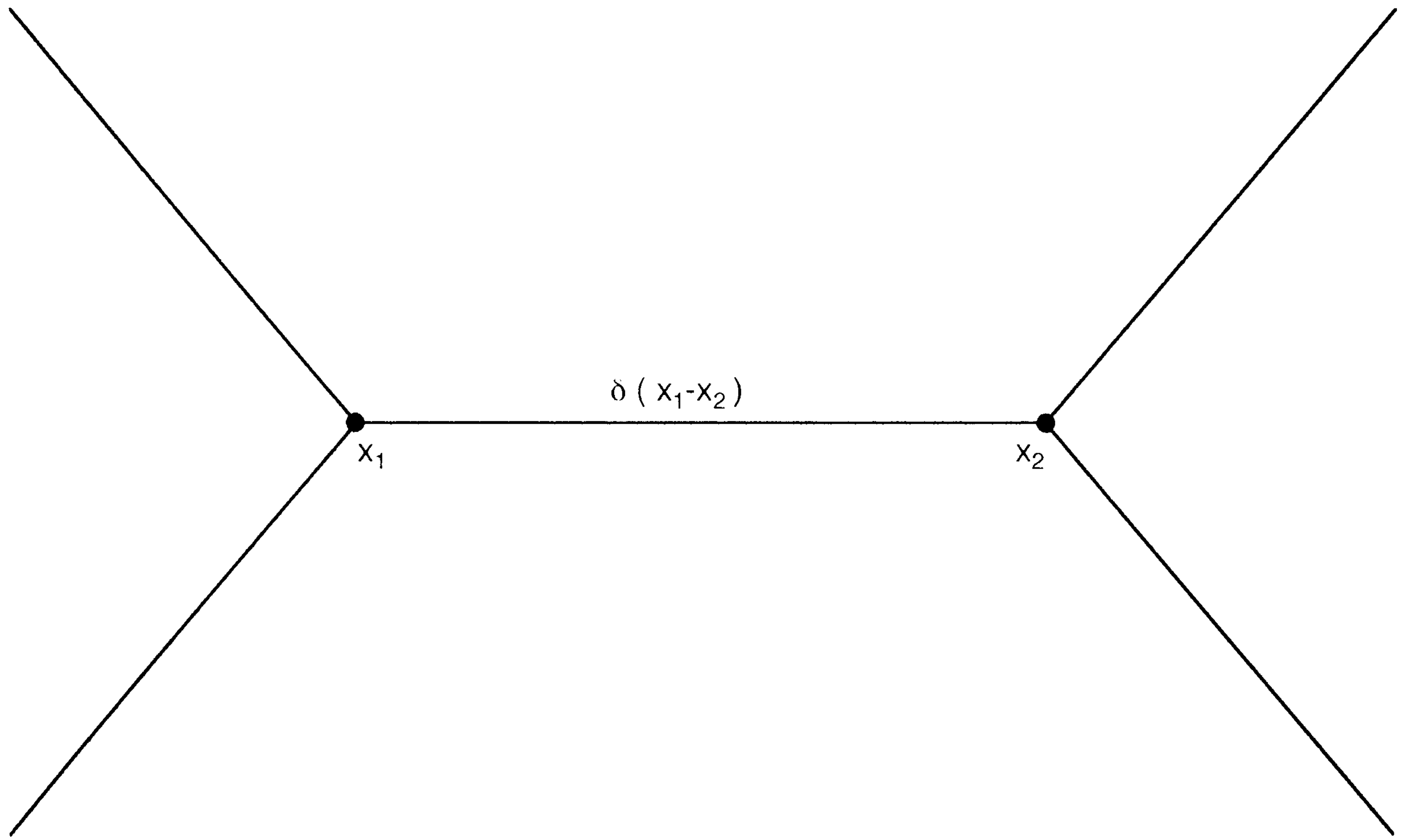

Fig. 2

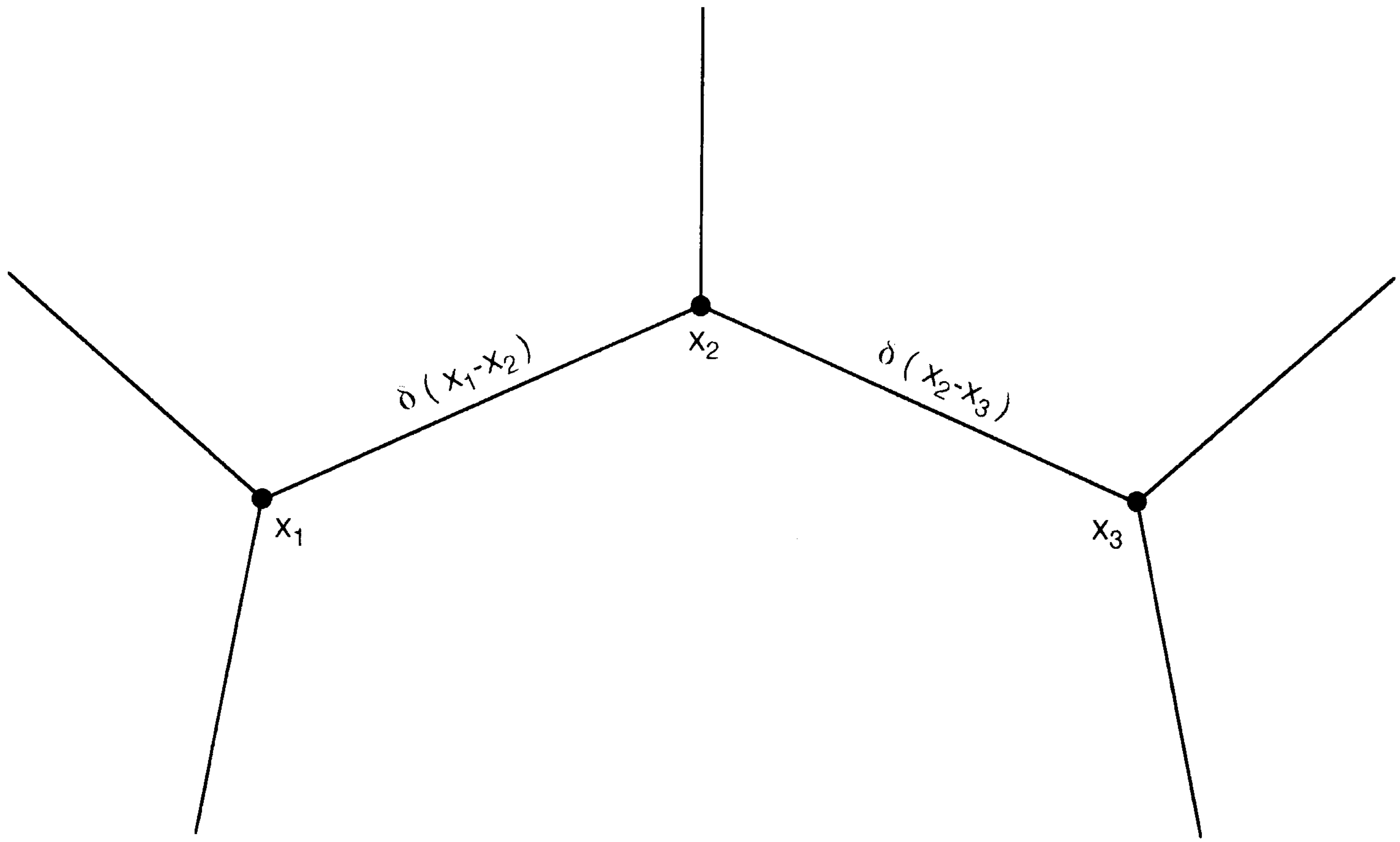

Fig. 3

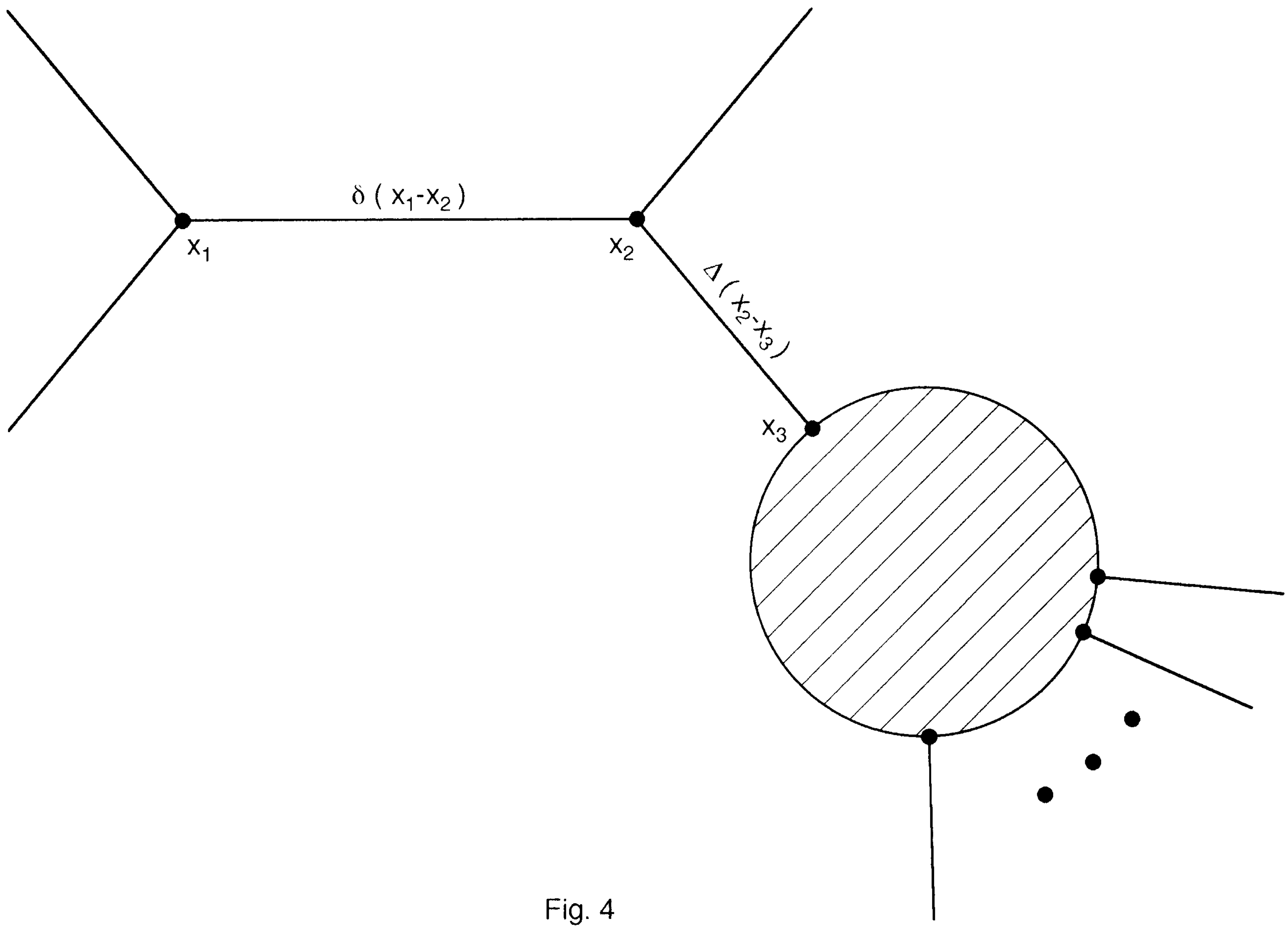

Fig. 4

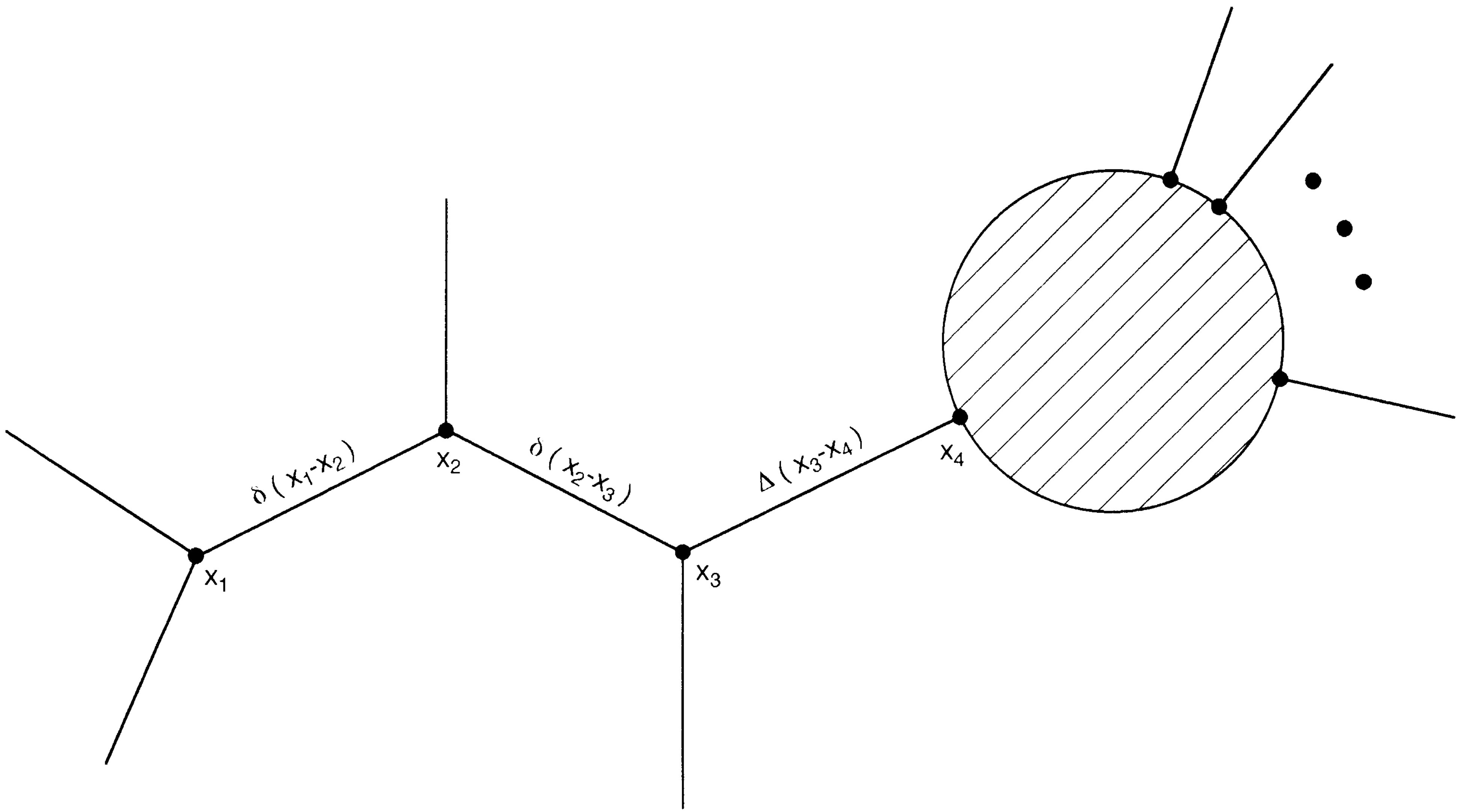

Fig. 5

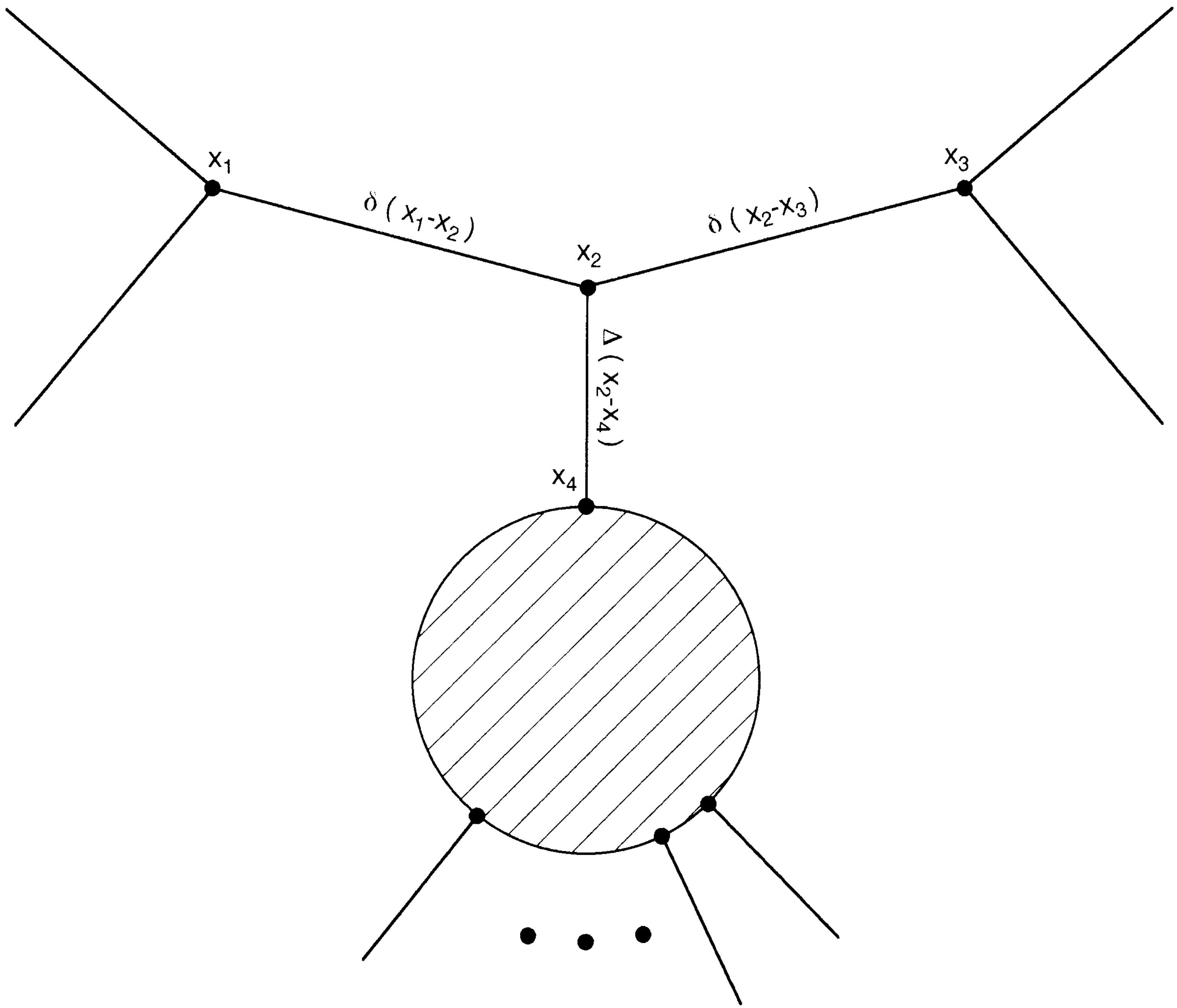

Fig. 6

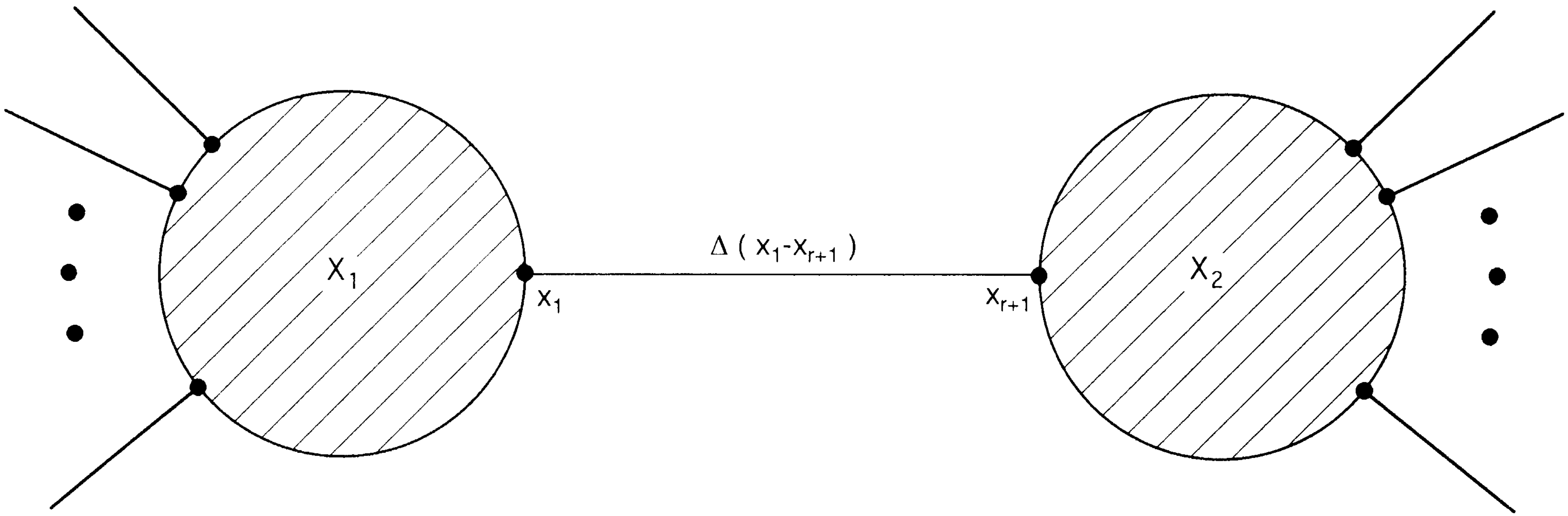

Fig. 7