The absolute surface available for adsorption has been determined from measurements of the initial rate of dissolution of the powder in dilute hydrofluoric acid. The areas covered by molecules on the surface have been calculated and found to agree with values from other sources.

A general expression has been proposed connecting adsorption potential with amount adsorbed, leading to a new theoretical isothermal.

The restricted applicability of Langmuir's formula to parts of the isothermals is interpreted theoretically.

Special experiments were made on the form of the adsorption isothermal at low adsorptions.

On Gauss' Theorem and the concept of Mass in General Relativity

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§ 1—Introduction

The present communication is concerned with the extension to General Relativity theory of the well-known theorem of Gauss on the Newtonian potential, viz., that the total flux of gravitational force through a simple closed surface is equal to $(-4\pi)x$ the total gravitating mass contained within the surface: and to various questions which arise in connection with this. In the extended theorem, which is found in § 2, the Newtonian concept of "gravitating mass" is naturally replaced by that of the energy-tensor, which does not in general consist solely of the "material" energy-tensor, and need not involve any "matter" at all. This new feature is illustrated in § 3 by an example in which the "gravitating mass" is simply an electrostatic field. In § 4 a theorem of "energy" is obtained which is required later, and which enables us to make precise the concept of the "potential energy" of an infinitesimal particle in a statical field in general relativity; this "potential energy" is shown to be the product of two factors, one depending on the particle alone (which may be called its "potential mass") and the other depending solely on its position. It is shown in § 5 that the definition of "potential mass" introduced in § 4 enables us to express the generalized Gauss' theorem of

§ 2, in the case when the energy-tensor is due to actual masses, by a simple statement practically identical with the original Gauss' theorem of Newtonian theory. Finally in § 6 it is shown that the electrostatical form of Gauss' theorem in Newtonian physics, viz., that the total strength of the tubes of force issuing from a closed surface is equal to the total electric charge within the surface, can also be extended to General Relativity, but that this extension is different in character from the gravitational theorem of § 2.

§ 2—The Extended Gauss' Theorem for the Gravitational Field

We shall first consider the extension to General Relativity of Gauss' theorem

$$\iint_S \frac{\partial V}{\partial \nu}\, dS = 4\pi M, \tag{2.1}$$

where V denotes the gravitational potential, S a simple closed surface of which dS is an element of area, $d\nu$ the element of inwards-drawn normal to dS, and M the total gravitating mass contained within the surface S.

In attempting to generalize this, we must remember that in General Relativity the gravitational force, as measured by any observer, depends not only on the observer's position but also on his velocity and acceleration, being in fact represented by the four-vector

$$g^i = -\left\{\frac{d^2x^i}{d\tau^2} + \sum_{h,k}\begin{Bmatrix} hk \\ i \end{Bmatrix}\frac{dx^h}{d\tau}\frac{dx^k}{d\tau}\right\}, \tag{2.2}$$

where $d\tau$ denotes the element of proper-time of the observer. In speaking of an integral involving gravitational force we must, therefore, specify at every point an observer with respect to whom the force is measured: this can be done in a natural way only for worlds whose metric can be defined by an equation of the type

$$d\tau^2 = U\,dt^2 - \frac{1}{c^2}\sum_{p,q=1}^{3} a_{pq}\,dx^h\,dx^q, \tag{2.3}$$

where U may involve all four co-ordinates and where the coefficients a_{pq} depend only on the co-ordinates (x_1, x_2, x_3).* In this world we can

* The variable t may obviously be replaced by any constant multiple λt, provided U is replaced by U/λ^2; we may take advantage of this to normalize t and U so that at infinity $U \to 1$; for U must in any case tend to a constant value at infinity, since the metric tends to a Galilean metric there.

suppose at every point an observer who is "at rest," *i.e.*, whose co-ordinates (x_1, x_2, x_3) are constant, his co-ordinate t alone varying, and we can define gravitational force to be that measured by these observers. Now for such an observer we have

$$\frac{dt}{d\tau} = U^{-\frac{1}{2}}, \quad \frac{dx^1}{d\tau} = 0, \quad \frac{dx^2}{d\tau} = 0, \quad \frac{dx^3}{d\tau} = 0,$$

and thus equation (2.2) gives for $i = 1, 2, 3$,

$$g^i = -\left\{ {0\,0 \atop i} \right\} \left(\frac{dt}{d\tau}\right)^2 = -\frac{1}{U}\left\{ {0\,0 \atop i} \right\} = \tfrac{1}{2} \sum_k \frac{g^{ik}}{U} \frac{\partial U}{\partial x^k},$$

so

$$g^i = -\frac{c^2}{2} \sum_k \frac{a^{ik}}{U} \frac{\partial U}{\partial x_k} \quad \text{for} \quad i = 1, 2, 3, \tag{2.4}$$

and

$$g^0 = -\left[\frac{d U^{-\frac{1}{2}}}{d\tau} + \left\{ {0\,0 \atop 0} \right\} U^{-1}\right] = 0.$$

It is therefore natural to consider, as a possible generalization of the integral on the left-hand side of (2.1), some constant multiple of the integral

$$\iint \left\{ g^1 \frac{\partial (x^2, x^3)}{\partial (u, v)} + g^2 \frac{\partial (x^3, x^1)}{\partial (u, v)} + g^3 \frac{(x^1, x^2)}{\partial (u, v)} \right\} \sqrt{-g}\, du\, dv, \tag{2.5}$$

where (g^1, g^2, g^3) is the three-vector in the space (x^1, x^2, x^3) (which is the "instantaneous space" of the observer) given by (2.4), which represents the gravitational force as measured by the observer, and where the integration is taken over any simple closed surface S on the space of (x^1, x^2, x^3); u and v are any two parameters which specify the position of points in this surface; and g denotes as usual the determinant of the coefficients of the metric (2.3), so that

$$\sqrt{-g} = c^{-3}\, U^{\frac{1}{2}} \sqrt{a},$$

where a is the determinant of the coefficients of the form $\Sigma a_{pq}\, dx^p\, dx^q$.

To find what the constant multiplier of the integral (2.5) should be, let us calculate the integral (2.5) for the case of the field of a single gravitating mass, for which the metric is given by Schwarzschild's formula

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{1}{c^2}\left(\frac{dR^2}{1 - \frac{\alpha}{R}} + R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2\right),$$

where $\alpha = \frac{2\beta M}{c^2}$, and β denotes the Newtonian constant of gravitation, while M is the mass of the gravitating central body. Here the integral (2.5), taken over a sphere of radius R, becomes

$$-\frac{\beta M}{c^3}\iint \sin\theta\, d\theta\, d\phi \quad \text{or} \quad -\frac{4\pi\beta M}{c^3}.$$

Since we want such a multiple of the integral (2.5) as will give $4\pi\beta M$, we see that the required multiplier must be $-c^3$; and thus we shall study the quantity

$$I = -c^3\iint \left\{g^1\frac{\partial(x^2, x^3)}{\partial(u, v)} + g^2\frac{\partial(x^3, x^1)}{\partial(u, v)} + g^3\frac{\partial(x^1, x^2)}{\partial(u, v)}\right\}\sqrt{-g}\, du\, dv. \quad (2.6)$$

Substituting from (2.4) in (2.6), we have

$$I = \tfrac{1}{2}c^2\iint \sum_k \left\{a^{1k}\frac{\partial U}{\partial x^k}\frac{\partial(x^2, x^3)}{\partial(u, v)} + a^{2k}\frac{\partial U}{\partial x^k}\frac{\partial(x^3, x^1)}{\partial(u, v)} + a^{3k}\frac{\partial U}{\partial x^k}\frac{\partial(x^1, x^2)}{\partial(u, v)}\right\} a^{\frac{1}{2}}\, U^{-\frac{1}{2}}\, du\, dv.$$

Converting this surface-integral into a volume-integral, it becomes

$$I = \tfrac{1}{2}c^2\iiint \sum_{h,k}\frac{\partial}{\partial x^h}\left(\frac{a^{hk} \cdot a^{\frac{1}{2}}}{U^{\frac{1}{2}}}\frac{\partial U}{\partial x^k}\right) dx^1\, dx^2\, dx^3,$$

integrated over the three-dimensional region R contained by the surface S in the space (x^1, x^2, x^3).

Since Beltrami's differential parameter of the second order with respect to the quadratic form $\Sigma a_{pq}\, dx^p\, dx^q$ is defined by the equation

$$\Delta_2 V = \frac{1}{\sqrt{a}}\sum_{h,k}\frac{\partial}{\partial x^h}\left(a^{\frac{1}{2}}\, a^{hk}\frac{\partial V}{\partial x^k}\right),$$

the preceding equation may be written

$$I = c^2\iiint \Delta_2 U^{\frac{1}{2}} \,.\, a^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3. \quad (2.7)$$

Now denoting the contracted Riemann tensor of the metric (2.3) by K_{pq}, we find, on substituting the coefficients of this metric in the ordinary formula for K_{pq}, that

$$K_{00} = -c^2\, U^{\frac{1}{2}} \,.\, \Delta_2 U^{\frac{1}{2}}, \quad \text{so} \quad K_0^{\ 0} = -c^2\, U^{-\frac{1}{2}} \,.\, \Delta_2 U^{\frac{1}{2}}.$$

Thus (2.7) becomes

$$I = -\iiint K_0^{\ 0} \,.\, U^{\frac{1}{2}}\, a^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3. \quad (2.8)$$

But the field-equations of gravitation give

$$K_0{}^0 = -\frac{8\pi\beta}{c^5}(T_0{}^0 - \tfrac{1}{2}T),$$

where β denotes the Newtonian constant of gravitation, and $T_p{}^q$ is the energy-tensor, and thus (2.8) becomes

$$I = \frac{8\pi\beta}{c^5}\iiint (T_0{}^0 - \tfrac{1}{2}T) . U^{\frac{1}{2}} a^{\frac{1}{2}} dx^1 dx^2 dx^3, \qquad (2.9)$$

or

$$-c^3 \iint \left\{ g^1 \frac{\partial (x^2, x^3)}{\partial (u, v)} + g^2 \frac{\partial (x^3, x^1)}{\partial (u, v)} + g^3 \frac{\partial (x^1, x^2)}{\partial (u, v)} \right\} \sqrt{-g}\, du\, dv$$

$$= \frac{8\pi\beta}{c^2}\iiint (T_0{}^0 - \tfrac{1}{2}T)\sqrt{-g}\, dx^1 dx^2 dx^3. \qquad (I)$$

This is the theorem in General Relativity which corresponds to Gauss' theorem in Newtonian potential theory—The left-hand side is the surface-integral of the gravitational force over an arbitrary simple closed surface, and the right-hand side is the volume-integral, taken over the space enclosed by the surface, of a quantity that depends only on the energy-tensor, which in General Relativity plays the part that is taken by matter in Newtonian physics.

§ 3—An Electrostatic Example

The theorem (I) is naturally much wider in its physical significance than the original Gauss' theorem, since the energy-tensor $T_p{}^q$ is not, in general, constituted solely of the "material" energy-tensor, but includes the electrical energy-tensor, etc. Let us take, as an example, a system in which there is no material energy-tensor, namely, an electrostatic system such as a condensor formed of two massless concentric spherical surfaces carrying equal and opposite charges. Let $E_p{}^q$ denote the electric energy-tensor due to the electrostatic field between the surfaces, so that $E_0{}^0$ represents the density of electrostatic energy: the energy-fluxes $E_0{}^q$ are zero, the components of electromagnetic momentum $-E_p{}^0$ are also zero, and the remaining components $E_p{}^q$ ($p, q = 1, 2, 3$) represent the Maxwell stresses. The condenser will not, however, be in equilibrium unless there is also a system of mechanical stresses keeping the surface-elements of the two spherical surfaces in position by antagonizing the mechanical forces exerted on these by the electric field. The simplest arrangement, in theory, is to have stresses represented by an energy-tensor $R_p{}^q$ such that the components $R_p{}^q$ for $p, q = 1, 2, 3$ exactly neutralize the Maxwell stresses $E_p{}^q$, that is, $R_p{}^q + E_p{}^q = 0$ for $p, q = 1, 2, 3$. Since the system is statical, the energy-fluxes $R_0{}^q$ and the momentum components $-R_p{}^0$ will be null. Thus the total energy-tensor $T_p{}^q = E_p{}^q + R_p{}^q$ will have all its components null except $T_0{}^0$; and therefore $T = \sum_p T_p{}^p = T_0{}^0$, and so

$$\iiint (T_0{}^0 - \tfrac{1}{2}T)\sqrt{-g}\,dx^1\,dx^2\,dx^3 = \tfrac{1}{2}\iiint T_0{}^0\sqrt{-g}\,dx^1\,dx^2\,dx^3$$
$$= \tfrac{1}{2} \times \text{(energy of system)}.$$

Now the mass M of the system is $1/c^2 \times$ its energy. Thus

$$\iiint (T_0{}^0 - \tfrac{1}{2}T)\sqrt{-g}\,dx^1\,dx^2\,dx^3 = \tfrac{1}{2}c^2M,$$

and thus the theorem (I) becomes in this case

$$-c^3\iint\left\{g^1\frac{\partial(x^2, x^3)}{\partial(u, v)} + g^2\frac{\partial(x^3, x^1)}{\partial(u, v)} + g^3\frac{\partial(x^1, x^2)}{\partial(u, v)}\right\}\sqrt{-g}\,du\,dv = 4\pi\beta M,$$

the analogy of which with the Newtonian Gauss' theorem is evident.

§ 4—The Integral of Energy for a Particle in a Statical Field

In order to express our extended Gauss' theorem (I), in the case when the energy-tensor is purely "material," in a form more closely analogous to the original Gauss' theorem, we shall need to enquire more closely what is to be understood by "mass" in this connection. Light will be thrown on this question by studying the energy-relations, in a statical gravitational field, of a "test-particle," *i.e.*, a particle so small that it does not sensibly disturb the field, although it is acted on by it.

Let the metric of space-time be specified by

$$d\tau^2 = U\,dt^2 - \frac{1}{c^2}\,dl^2 = U\,dt^2 - \frac{1}{c^2}\sum_{p,q=1}^{3} a_{pq}\,dx^p\,dx^q,$$

where $d\tau$ is the element of proper-time and where U and the a_{pq} are functions of (x^1, x^2, x^3) only. Then if $d\sigma$ denotes the element of time as measured by an observer P at rest in (x^1, x^2, x^3), we have $d\sigma^2 = U\,dt^2$. The kinetic energy of a small particle of proper-mass m, momentarily at P, as measured by this observer, is

$$mc^2\left\{1 - \frac{1}{c^2}\left(\frac{dl}{d\sigma}\right)^2\right\}^{-\frac{1}{2}}, \text{ or } mc^2\left\{1 - \frac{1}{c^2U}\left(\frac{dl}{dt}\right)^2\right\}^{-\frac{1}{2}}, \text{ or } mc^2\,U^{\frac{1}{2}}\frac{dt}{d\tau}. \quad (4.1)$$

Now writing

$$T = \tfrac{1}{2}Ut'^2 - \frac{1}{2c^2}l'^2,$$

where the accents denote differentiations with respect to τ, one of the equations of motion of the particle is

$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial t'}\right) - \frac{\partial T}{\partial t} = 0,$$

which gives

$$\frac{\partial T}{\partial t'} = \text{constant},$$

or

$$U\frac{dt}{d\tau} = \varepsilon, \tag{4.2}$$

where ε denotes a constant. From (4.1) and (4.2), the kinetic energy is

$$mc^2\,\varepsilon\,U^{-\frac{1}{2}}. \tag{4.3}$$

Now at infinity, *i.e.*, where the influence of the gravitational field vanishes and space-time is Galilean, let the velocity of the particle be w. In this region, $U \to 1$, so by (4.1) the kinetic energy tends to $mc^2\left(1 - \frac{w^2}{c^2}\right)^{-\frac{1}{2}}$, and by (4.3) it tends to $mc^2\varepsilon$. Equating these two values, we have

$$\varepsilon = \left(1 - \frac{w^2}{c^2}\right)^{-\frac{1}{2}}. \tag{4.4}$$

From (4.1), (4.3), and (4.4), we have

$$\frac{mc^2}{\left\{1 - \frac{1}{c^2U}\left(\frac{dl}{dt}\right)^2\right\}^{\frac{1}{2}}} = \frac{mc^2}{\left(1 - \frac{w^2}{c^2}\right)^{\frac{1}{2}} . \, U^{\frac{1}{2}}}, \tag{II}$$

and *this is the equation of conservation of energy for the particle.* The expression on the left is the kinetic energy, and the expression on the right may be called the *lost potential energy. The lost potential energy corresponds to that of a particle of mass* $\frac{m}{\left(1 - \frac{w^2}{c^2}\right)^{\frac{1}{2}}}$ *in a field of force* $c^2/U^{\frac{1}{2}}$.

We are now in a position to see the difference between Newtonian dynamics and the dynamics of General Relativity, in the matter of potential energy. In Newtonian dynamics the equation of conservation of energy for a single particle is

$$\text{(kinetic energy)} + \text{(potential energy)} = \mathrm{C}, \tag{4.5}$$

where the potential energy is the product of the mass—a fixed quantity—into a function which depends only on the position of the particle, and where C is a constant which depends on the type of motion, *i.e.*, on the initial circumstances of projection. In General Relativity with statical fields, on the other hand, the equation of conservation of energy for a single small particle of proper-mass m is (as we see from (II)) of the form

$$\text{(kinetic energy)} = \text{(lost potential energy)}, \tag{4.6}$$

where now the lost potential energy is the product of $m\left(1 - \frac{w^2}{c^2}\right)^{-\frac{1}{2}}$ into a function which depends only on the position of the particle: *here* w *denotes the velocity which the particle would have after escaping from the gravitational field and arriving at the Galilean space-time at infinity*; the constant w in (4.6) corresponds to the constant C in (4.5), since it characterizes the various types of motion, but it enters in a wholly different manner into the equation, since in (II) or (4.6) its effect is to modify the effective mass. We shall call $m\left(1 - \frac{w^2}{c^2}\right)^{-\frac{1}{2}}$ the *potential mass*, since it is the coefficient which plays the part of mass in the expression for the potential energy. It is, of course, equal to $1/c^2 \times$ the energy of the particle when it has escaped to infinity out of the influence of the gravitational field.

We may remark that the constant w^2 is not necessarily positive; it will, in fact, be negative if the particle has not sufficient energy to carry it out of the gravitational field into the Galilean field at infinity.

We may remark in passing that equation (II) leads immediately to the formula for the shift to the red of a spectral line which is emitted in a strong gravitational field, when measured by an observer outside the field. For we have only to pass to the limiting case when the material particle becomes a light-quant, so that its kinetic energy is now $h\nu$, where ν is the frequency and h is Planck's constant. Equation (II) now becomes

$$h\nu = \frac{h\nu_0}{\mathrm{U}^{\frac{1}{2}}} \quad \text{or} \quad \nu_0 = \nu \mathrm{U}^{\frac{1}{2}},$$

where ν is the frequency at the place of emission (*e.g.*, on the sun) and ν_0 is the frequency as observed outside the gravitational field (*e.g.*, on the earth). This is the well-known formula for the shift to the red of spectral lines emitted in a strong gravitational field.

§ 5—Interpretation of the Extended Gauss' Theorem for Purely Material Fields

We shall now show that the conception of " potential mass," introduced in § 4, furnishes a physical interpretation of the extended Gauss' theorem (I) of § 2, for purely material statical fields.

When the field is purely material (*i.e.*, there are no electromagnetic phenomena) and incoherent, the energy-tensor is simply

$$T^{pq} = c^2\sigma_0 \frac{d\xi^p}{d\tau}\frac{d\xi^q}{d\tau},$$

where (ξ^0 ..... ξ^3) are the co-ordinates of a particle when its proper-time is τ, and σ_0 is the proper-density of matter, defined by the invariant condition that

$$\int \sigma_0 \sqrt{-g}\, dx^0\, dx^1\, dx^2\, dx^3 \tag{5.1}$$

integrated over any region of space-time, is equal to the sum of the lengths of the world-lines of material particles in this region, each multiplied by the proper-mass of the particle to which it belongs.

Since the field is statical, we shall suppose the material particles to be at rest in the space (x^1, x^2, x^3),* so that $d\xi^p/d\tau = 0$ for $p = 1, 2, 3$. Thus the only non-zero constituent of T^{pq} is

$$T^{00} = c^2\sigma_0\left(\frac{d\xi^0}{d\tau}\right)^2 = \frac{c^2\sigma_0}{U},$$

whence $T_0^{\ 0} = c^2\sigma_0$ and $T = \sum_p T_p^{\ p} = T_0^{\ 0}$. Thus (2.9) becomes

$$I = \frac{4\pi\beta}{c^3}\iiint \sigma_0 U^{\frac{1}{2}} a^{\frac{1}{2}}\, dx^1\, dx^2\, dx^3 = 4\pi\beta \iiint \sigma_0 \sqrt{-g}\, dx^1\, dx^2\, dx^3.$$

Thus if δt denotes a small increment of the variable t, we have

$$I\delta t = 4\pi\beta \iiint \sigma_0 \sqrt{-g}\, \delta t\, dx^1\, dx^2\, dx^3. \tag{5.2}$$

Let Q denote the region of space-time obtained by multiplying the three-dimensional region R (viz., the space inside the surface S in the space of x^1, x^2, x^3), by δt. Then, by (5.1), equation (5.2) is equivalent to the statement that—

> $I\delta t = 4\pi\beta\ \times$ the sum of the lengths of the world-lines of the material particles in the region Q, each multiplied by the proper-mass of the particle to which it belongs.

* This, of course, requires that they should be situated at places for which U has maximum or minimum values.

Dividing both sides of this equation by δt, and making $\delta t \to 0$, we obtain

$I = 4\pi\beta \times$ the sum of the proper-masses of the material particles in the three-dimensional region R, each multiplied by the value of $d\tau/dt$ for the particle in question,

or

$I = 4\pi\beta \times$ the sum of the proper-masses of the material particles in the region R, each multiplied by the value of $U^{\frac{1}{2}}$ at the particle. (5.3)

Now the equation of energy (II) of § 4, for one of these particles at rest in the space (x^1, x^2, x^3), is

$$mc^2 = \frac{mc^2}{\left(1 - \frac{w^2}{c^2}\right)^{\frac{1}{2}} . U^{\frac{1}{2}}}$$

so

$$U^{\frac{1}{2}} = \left(1 - \frac{w^2}{c^2}\right)^{-\frac{1}{2}} \tag{5.4}$$

so (5.3) becomes

$I = 4\pi\beta \times$ the sum of the proper-masses of the material particles in the region R, each multiplied by the value of $(1 - w^2/c^2)^{-\frac{1}{2}}$ belonging to it,

or

$I = 4\pi\beta \times$ the sum of the "potential masses" of the material particles in the region R.

Thus finally we have the result that *when the statical gravitational field is due solely to material particles, the extended Gauss' theorem of* § 2 *takes the form*

$$- c^3 \iint \left\{ g^1 \frac{\partial (x^2, x^3)}{\partial (u, v)} + g^2 \frac{\partial (x^3, x^1)}{\partial (u, v)} + g^3 \frac{\partial (x^1, x^2)}{\partial (u, v)} \right\} \sqrt{-g}\, du\, dv = 4\pi\beta M, \tag{III}$$

where the integration is taken over any simple closed surface in the space of (x^1, x^2, x^3), *and* M *denotes the sum of the "potential masses" of those material particles that are inside this surface.*

The close analogy between this and the original Gauss' theorem of Newtonian potential-theory is obvious. The remarkable feature is that it is the "potential masses," and not the proper masses, of the particles which occur in the right-hand side of the equation.

§ 6—The Extended Gauss' Theorem for the Electric Potential

In Newtonian physics, Gauss' theorem is applicable not only to the gravitational potential but also to the electrostatic potential, taking the form (in suitable units)

$$\iint_S \frac{\partial V}{\partial \nu}\, dS = \text{Total electric charge inside the surface S,}$$

where V denotes the electrostatic potential. This theorem can be extended to elctromagnetic fields of any kind, in gravitational fields of any kind in general relativity, as follows:

Let M be any three-dimensional multipoint in space-time, whose frontier is a closed surface S. Denote by J^p the electric current-vector whose components are $(\rho, \rho v_x, \rho v_y, \rho v_z)$, where ρ is the density of electric charge and (v_x, v_y, v_z) is its velocity. Then the total quantity of electricity belonging to those world-lines of electric charge which intersect the multipoint M is

$$\iiint_M \left\{ J^0 \frac{\partial (x^1, x^2, x^3)}{\partial (p, q, r)} + J^1 \frac{\partial (x^2, x^3, x^0)}{\partial (p, q, r)} + J^2 \frac{\partial (x^3, x^0, x^1)}{\partial (p, q, r)} + J^3 \frac{\partial (x^0, x^1, x^2)}{\partial (p, q, r)} \right\} \sqrt{-g}\, dp\, dq\, dr, \tag{6.1}$$

where (p, q, r) are any parameters specifying position in M.

Now the fundamental equations of the electromagnetic field (Maxwell's equations extended to general relativity) are

$$\frac{1}{\sqrt{-g}} \sum_q \frac{\partial (\sqrt{-g}\, X^{pq})}{\partial x^q} = J^p \qquad (p = 0, 1, 2, 3) \tag{6.2}$$

where

$$X_{rs} = \frac{\partial \phi_r}{\partial x^s} - \frac{\partial \phi_s}{\partial x^r} \tag{6.3}$$

$(\phi_0, \phi_1, \phi_2, \phi_3)$ being the electromagnetic potential-vector, so X_{rs} is the six-vector of which three components represent the electric force and the other three represent the magnetic vector.

Substituting from (6.2) in (6.1), and transforming into a surface-integral, we obtain the result that *the total electric charge, belonging to* *particles whose world-lines intersect a three-dimensional multipoint whose frontier is a simple closed surface* S, *is*

$$\iint_S \left\{ X^{23} \frac{\partial (x^0, x^1)}{\partial (u, v)} + X^{31} \frac{\partial (x^0, x^2)}{\partial (u, v)} + X^{12} \frac{\partial (x^0, x^3)}{\partial (u, v)} + X^{03} \frac{\partial (x^1, x^2)}{\partial (u, v)} + X^{02} \frac{\partial (x^3, x^1)}{\partial (u, v)} + X^{01} \frac{\partial (x^2, x^3)}{\partial (u, v)} \right\} \sqrt{-g}\, du\, dv. \qquad \text{(III)}$$

This is the extension, to the most general electromagnetic field in any gravitational field, of Gauss' theorem on the electrostatic potential. Evidently it differs greatly from the extension of Gauss' theorem on the gravitational potential, which was the subject of §§ 2–5.

Summary

The well-known theorem of Gauss on the Newtonian potential, viz., that the total flux of gravitational force through a simple closed surface is equal to (− 4π) × the total gravitating mass contained within the surface, is extended to General Relativity. In the extended theorem, the Newtonian concept of "gravitating mass" is naturally replaced by that of the energy-tensor, which does not in general consist solely of the "material" energy-tensor, and need not involve any matter at all. It is shown that in order to provide a simple physical interpretation of the formulæ obtained, a new concept must be introduced, to which the name "potential mass" is given. The electrostatical form of Gauss' theorem is also extended to General Relativity.