Rolf Hünlich; Joachim Naumann

On general boundary value problems and duality in linear elasticity. I

Aplikace matematiky, Vol. 23 (1978), No. 3, 208–230

Persistent URL: http://dml.cz/dmlcz/103746

ON GENERAL BOUNDARY VALUE PROBLEMS AND DUALITY IN LINEAR ELASTICITY, I

ROLF HÜNLICH and JOACHIM NAUMANN

(Received December 28, 1976)

INTRODUCTION

Let us consider a deformable body under the action of prescribed body forces. Its equilibrium state is governed by the well-known (geometrically linear) conditions of equilibrium and strain-displacement relations. These conditions and relations have to be completed by two different types of other relations:

1° *the constitutive law* (i.e. a relation between the stress tensor and the strain tensor in the interior of the body);

2° *the boundary conditions* (i.e. a system of relations between the stress vector and the displacement vector along the edge of the body; these relations describe the interaction of the body with its neighbourhood).

It is the purpose of the present paper to give a detailed discussion of the boundary conditions. The point of view that we are going to develop, consists in considering the relation between the stress vector and the displacement vector along the boundary as an independent, self-consistent "law of interaction" which can be expressed in terms of a subgradient relation. This relation will include the known classical, unilateral and bilateral boundary conditions as special cases. However, the main intention of our discussion is the establishing of the complete equivalence of the boundary value problem (in its generalized setting), the principle of virtual work and the principle of minimum potential energy on the one hand, and in bringing more light into the relationship between the principle of minimum potential energy and its dual problem (principle of minimum complementary energy) on the other one. Although our approach can be extended to more general types of constitutive laws, in the present paper we restrict ourselves for the sake of simplicity to Hooke's law (with the standard assumptions on the elastic coefficients).

A very comprehensive discussion of mechanical systems (static case and quasi-static evolution case) in the context of variational statements, convex functionals and duality with respect to paired topological vector spaces may be found in Moreau [10]–[13] and Nayroles [14]–[16]. In these papers a great variety of constitutive laws and phenomena (e.g. friction) are expressed in terms of a subgradient relation. Special constitutive laws of this type are also studied in Lené [9]. Let us finally refer to [5] where nonlinear problems involving Hencky type laws are studied.

General classical boundary conditions (in linear elasticity) are discussed in Hlaváček [6] and Hlaváček, Nečas [7]. A profound investigation of boundary conditions involving unilateral constraints, in particular the Signorini problem, may be found in Fichera [3], [4] (cf. also Duvaut, Lions [2]). Boundary conditions of friction type are studied in the book of Duvaut, Lions [2].

The present Part I of our paper is arranged as follows. In Section 1 we summarize first of all some known facts concerning traces of Sobolev space functions. Then we introduce the concept of the trace of a stress tensor and prove some auxiliary results which are also of interest by themselves. In particular, the advantage of this concept is that the traces of the stress tensors belong to the dual of the space of traces of the displacement vectors. Finally, we discuss in this section some properties of convex functionals. The following section presents the (generalized) formulation of our boundary value problem (Problem I) and several equivalent versions and special cases. Section 3 is devoted to a detailed discussion of a number of examples of our abstract formulation of the boundary conditions. In Section 4 we introduce the principle of virtual displacements (Problem II) and the principle of minimum potential energy (Problem III) and make clear the relationships between all problems stated.

In Part II of our paper we prove first of all some existence theorems for Problem III. Then we present a detailed discussion of the dual problem to Problem III and of the relationships between the both problems.

1. NOTATION. PRELIMINARIES

1° Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ. We suppose that Ω belongs to the class $C^{0,1}$ [1]). Then the unit outer normal $n = \{n_1, n_2, n_3\}$ exists

[1]) The bounded domain $\Omega \subset \mathbb{R}^3$ is said to belong to the class $C^{k,\mu}$ ($k = 0, 1, 2 \ldots; 0 < \mu \leqq 1$) if (i) to each $x \in \Gamma$ there exists an open ball B_x centred at x such that the intersection $B_x \cap \Gamma$ can be described by a $C^{k,\mu}$-function, and (ii) $B_x \cap \Gamma$ divides B_x into an exterior and an interior part with respect to Ω; cf. [17] for further details.

a. e. on Γ (with respect to the surface measure), and its components are measurable and bounded (cf. [17]).

We introduce the Hilbert spaces

$$\mathscr{H} = [L^2(\Omega)]^3, \quad \mathscr{V} = [W_2^1(\Omega)]^3$$

with the scalar products and norms

$$(u, v) = \int_\Omega u_i v_i \, dx \ ^{2)}, \quad |u| = (u, u)^{1/2},$$

$$((u, v)) = \int_\Omega u_i v_i \, dx + \int_\Omega u_{i,j} v_{i,j} \, dx \ ^{2)}, \quad \|u\| = ((u, u))^{1/2},$$

respectively ($W_2^1(\Omega)$ denotes the usual Sobolev space; cf. [17]).

Setting

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad u \in \mathscr{V},$$

$$\mathscr{R} = \{u \in \mathscr{V} : \int_\Omega \varepsilon_{ij}(u) \, \varepsilon_{ij}(u) \, dx = 0\}$$

we have (cf. [2], [7]).

Lemma 1.1. *Let $\Omega \in C^{0,1}$. Then*:

(i) *There exists a positive constant c_1 such that*

$$\int_\Omega \varepsilon_{ij}(u) \, \varepsilon_{ij}(u) \, dx + |u|^2 \geqq c_1 \|u\|^2 \quad \forall u \in \mathscr{V}.$$

(ii) (*Korn's inequality*). *Let $\mathscr{V}_0$ be any closed subspace of $\mathscr{V}$ with $\mathscr{V}_0 \cap \mathscr{R} = \{0\}$. Then there exists a positive constant c_3 such that*

$$\int_\Omega \varepsilon_{ij}(u) \, \varepsilon_{ij}(u) \, dx \geqq c_2 \|u\|^2 \quad \forall u \in \mathscr{V}_0.$$

(iii) $\mathscr{R} = \{u \in \mathscr{V} : u = a + b \times x, a, b = \text{const}, x \in \Omega\}$.

The functions in $\mathscr{R}$ are called rigid displacements. Maintaining the assumption $\Omega \in C^{0,1}$ we introduce further the Hilbert spaces

$$H = [L^2(\Gamma)]^3, \quad V = [W_2(\Gamma)]^3 \ ^{3)}.$$

Let $(h, g)_H = \int_\Gamma h_i g_i \, dS$ denote the scalar product on H, while let

$$\|h\|_V = \left(\sum_{i=1}^3 \|h_i\|^2_{W_2^{1/2}(\Gamma)}\right)^{1/2}$$

be the norm on V. The imbedding $V \subset H$ is compact and dense (cf. [17]).

---

[2]) Throughout the whole paper, unless otherwise stated, Latin subscripts take the values 1, 2, 3. Further, we use the convention that a repeated subscript means summation over 1, 2, 3, and the notation $u_i = \partial u / \partial x_i$.

[3]) Let us refer to [17] for the definition, norm etc. of the spaces $L^p(\Gamma)$ $(1 \leqq p < +\infty)$ and $W_p^s(\Gamma)$ $(s > 0$, real).

Further, let V^* denote the dual of V, $\| \ \|_{V^*}$ the dual norm on V^* and $\langle h^*, h\rangle_V$ the dual pairing between $h^* \in V^*$ and $h \in V$. Identifying H with its dual one obtains the continuous and dense imbedding $H \subset V^*$, and in the case $h \in H$ and $g \in V$ the dual pairing between h and g coincides with their scalar product in H.

For the elements in $\mathscr{V}$ one can introduce the concept of trace (cf. [17]).

Lemma 1.2. *Let $\Omega \in C^{0,1}$. Then:*

(i) *There exists a uniquely determined mapping $\gamma \in \mathscr{L}(\mathscr{V}, V)$ such that*

$$\gamma(u) = u|_\Gamma \quad \forall u \in [C^\infty(\bar{\Omega})]^3 .$$

(ii) *For each $h \in V$ there exists a $u \in \mathscr{V}$ such that*

$$\gamma(u) = h\,, \quad \|u\| \leqq c\|h\|_V$$

$(c = \text{const} > 0)$.

2° Now we introduce the spaces of tensor fields

$$\mathbf{S} = \{\tau : \tau_{ij} \in L^2(\Omega),\ \tau_{ij} = \tau_{ji}\}\,,$$
$$\mathbf{T} = \{\tau \in \mathbf{S} : \tau_{ij,j} \in L^2(\Omega)\}\,.$$

It is easily verified that $\mathbf{S}$ and $\mathbf{T}$ are Hilbert spaces with respect to the scalar products

$$(\sigma, \tau)_{\mathbf{S}} = \int_\Omega \sigma_{ij}\tau_{ij}\,\mathrm{d}x\,, \quad \sigma, \tau \in \mathbf{S}\,,$$
$$(\sigma, \tau)_{\mathbf{T}} = \int_\Omega (\sigma_{ij}\tau_{ij} + \sigma_{ij,j}\tau_{ik,k})\,\mathrm{d}x\,, \quad \sigma, \tau \in \mathbf{T}\,,$$

respectively. Using the arguments of [17, Théorème 2.3.1] it can be shown that $[C^\infty(\bar{\Omega})]^9$ is dense in $\mathbf{T}$. For the elements in $\mathbf{T}$ we have the following concept of trace.

Lemma 1.3. *Let $\Omega \in C^{0,1}$. Then:*

(i) *There exists a uniquely determined mapping $\pi \in \mathscr{L}(\mathbf{T}, V^*)$ such that*

$$(\pi(\tau))_i = \tau_{ij}|_\Gamma\, n_j \quad \forall \tau \in [C^\infty(\bar{\Omega})]^9 \text{ [4]}\,.$$

(ii) (*Generalized Green's formula*) *For any $\tau \in \mathbf{T}$ and any $u \in \mathscr{V}$ it holds*

$$\int_\Omega \tau_{ij}u_{i,j}\,\mathrm{d}x + \int_\Omega \tau_{ij,j}u_i\,\mathrm{d}x = \langle \pi(\tau), \gamma(u)\rangle_V\,.$$

(iii) *For each $h^* \in V^*$ there exists a $\tau \in \mathbf{T}$ such that*

$$\pi(\tau) = h^*\,, \quad \|\tau\|_{\mathbf{T}} \leqq c\|h^*\|_{V^*}$$

$(c = \text{const} > 0)$.

---

[4]) More precisely, it holds

$$\langle \pi(\tau), h\rangle_V = \int_\Gamma \tau_{ij}n_jh_i\,\mathrm{d}S \quad \forall \tau \in [C^\infty(\bar{\Omega})]^9\,, \quad \forall h \in V\,.$$

However, we use the above notation for the sake of simplicity.

Proof. The assertions (i) and (ii) are readily obtained when starting from the (classical) Green formula and using the density of $[C^\infty(\bar{\Omega})]^9$ in $\boldsymbol{T}$.

Let us prove (iii). To this end, let $h^* \in V^*$ be arbitrarily given. We have then $|\langle h^*, \gamma(\varrho)\rangle_V| \leqq c_1|\varrho|$ for all $\varrho \in \mathscr{R}$. With regard to the fact that $\mathscr{R}$ is a closed subspace of $\mathscr{H}$, the Hahn-Banach theorem yields the existence of an element $f \in \mathscr{H}$ such that

$$-\langle h^*, \gamma(\varrho)\rangle_V = (f, \varrho) \quad \forall \varrho \in \mathscr{R}\,, \quad |f| \leqq c_1 \|h^*\|_{V^*}\,.$$

Hence, by virtue of Lemma 1.1 (ii) one obtains exactly one $u \in \mathscr{V} \ominus \mathscr{R}$ such that

$$\int_\Omega \varepsilon_{ij}(u)\, \varepsilon_{ij}(v)\, \mathrm{d}x = (f, v) + \langle h^*, \gamma(v)\rangle_V \quad \forall v \in \mathscr{V}\,. \tag{1.1}$$

Set $\tau_{ij} = \varepsilon_{ij}(u)$. Then $\tau_{ij} \in L^2(\Omega)$, and $\|\tau\|_{\boldsymbol{S}} \leqq c_2\|h^*\|_{V^*}$ which is immediately seen when setting $v = u$ in (1.1). On the other hand, (1.1) implies $\tau_{ij,j} = -f_i$ (in the sense of $\mathscr{D}'(\Omega)$). Therefore $\tau \in \boldsymbol{T}$ and $\|\tau\|_{\boldsymbol{T}} \leqq c_3\|h^*\|_{V^*}$. Finally, using the generalized Green formula we find

$$(f, v) + \langle h^*, \gamma(v)\rangle_V = \int_\Omega \varepsilon_{ij}(u)\, \varepsilon_{ij}(v)\, \mathrm{d}x =$$
$$= \int_\Omega \tau_{ij} v_{i,j}\, \mathrm{d}x = \langle \pi(\tau), \gamma(v)\rangle_V + \int_\Omega f_i v_i\, \mathrm{d}x$$

for all $v \in \mathscr{V}$. By Lemma 1.2 (ii), $h^* = \pi(\tau)$.

3° For the discussion of the mixed boundary conditions (see Section 3) the following lemma will be useful.

Lemma 1.4. *Let $\Omega \in C^{0,1}$ and suppose that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup N$ where Γ_1, Γ_2 are disjoint open subsets of Γ and N has measure zero.*

(i) *Let $p \in [L^2(\Gamma_2)]^3$. Then for each $\varepsilon > 0$ there exists an $h_\varepsilon \in V$ such that*

$$h_\varepsilon = 0 \quad \text{a. e. on} \quad \Gamma_1\,, \quad \int_{\Gamma_2} |p - h_\varepsilon|^2\, \mathrm{d}S \leqq \varepsilon\,.$$

(ii) *Let $p \in [L^2(\Gamma_2)]^3$ such that*

$$\int_{\Gamma_2} p_i h_i\, \mathrm{d}S = 0 \quad \forall h \in V \text{ with } h = 0 \text{ a. e. on } \Gamma_1\,.$$

Then $p = 0$ a. e. on Γ_2.

(iii) *Let $h^* \in V^*$ admit the decomposition*

$$h^* = h_1^* + j^*(p)\ ^{5})$$

where

$$h_1^* \in V^*\,, \quad \langle h_1^*, h\rangle_V = 0 \quad \forall h \in V \text{ with } h = 0 \text{ a. e. on } \Gamma_1\,,$$
$$p \in H \text{ with } p = 0 \text{ a. e. on } \Gamma_1\,.$$

Then this decomposition is uniquely determined.

[5]) For the sake of clarity, we indicate here explicitly the adjoint $j^* : H \to V^*$ of the injection $j : V \to H$ (recall that H is identified with its dual).

Proof. Assertion (i) can be proved by using a system of local charts and a standard argument (cf. [17, Théorème 2.4.9]).

Assertion (ii) is an immediate consequence of (i).

To prove (iii), let $h^* \in V^*$ have two decompositions

$$h^* = h_1^* + j^*(p) = k_1^* + j^*(q)$$

where h_1^*, p and k_1^*, q satisfy the corresponding conditions in (iii). Then we have

$$\int_{\Gamma_2} (p_i - q_i)\, h_i \, \mathrm{d}S = \langle j^*(p - q), h \rangle_V = 0$$

for all $h \in V$ with $h = 0$ a. e. on Γ_1. Hence by (ii), $p = q$ a. e. on Γ_2, therefore $p = q$ and $h_1^* = k_1^*$.

4^0 Let $h \in H$. Then we have the decomposition

$$h = h_n n + h_t\,, \quad h_n = h_i n_i\,, \quad h_t = h - h_n n\,.$$

Obviously, $h_n \in L^2(\Gamma)$, $h_t \in H_t$ where

$$H_t = \{k \in H : k_i n_i = 0 \text{ a. e. on } \Gamma\}\,.$$

It is readily seen that $L^2(\Gamma) \times H_t$ is a Hilbert space with respect to the scalar product $(.,.)_{L^2(\Gamma)} + (.,.)_H$. Thus, the mapping $h \mapsto \{h_n, h_t\}$ is an isometry from H onto $L^2(\Gamma) \times H_t$; in particular, it holds

$$(h, k)_H = \int_\Gamma h_n k_n \, \mathrm{d}S + \int_\Gamma h_{ti} k_{ti} \, \mathrm{d}S \qquad \forall h, k \in H\,.$$

Under stronger assumptions upon the boundary Γ we have a similar situation with respct to V. In order to make this precise we note first of all

Lemma 1.5. *Let $\Omega \in C^{1,1}$. Then the mapping $h \mapsto hn_i$ is linear and continuous from $W_2^{1/2}(\Gamma)$ into itself.*

Proof. Let $\{S_r, a_r\}$ $(r = 1, \ldots, m)$ be any system of local charts for Γ in the sense of [17]. The hypothesis $\Omega \in C^{1,1}$ implies that each component n_{ri} of the unit outer normal (with respect to the local chart under consideration) is Lipschitzian. Our assertion follows now from [17, Lemme 2.5.5].

Let $\Omega \in C^{1,1}$. We have then $h_n \in W_2^{1/2}(\Gamma)$ and $h_t \in V_t$ for any $h \in V$, where

$$V_t = \{k \in V : k_i n_i = 0 \text{ a. e. on } \Gamma\}\,.$$

The norm $\|\cdot\|_{W_2^{1/2}(\Gamma)} + \|\cdot\|_V$ turns $W_2^{1/2}(\Gamma) \times V_t$ into a Banach space. Then the mapping $h \mapsto \{h_n, h_t\}$ is an algebraic and topological isomorphism from V onto $W_2^{1/2}(\Gamma) \times V_t$. Indeed, the continuity of this mapping follows from Lemma 1.5, while its bijectivity is seen at once. The assertion is now a consequence of the Open Mapping Theorem (cf. [8]).

Passing to the dual spaces one obtains that the mapping $h^* \mapsto \{h_n^*, h_t^*\}$ where h_n^* and h_t^* are uniquely determined by

$$\langle h^*, h \rangle_V = \langle h_n^*, h_n \rangle_{W_2^{1/2}(\Gamma)} + \langle h_t^*, h_t \rangle_{V_t} \ ^{6)} \quad \forall h \in V,$$

is an algebraic and topological isomorphism from V^* onto $W_2^{-1/2}(\Gamma) \times V_t^*$.

Let us finally note that V_t is continuously and densely imbedded into H_t. Thus, identifying H_t with its dual one obtains the continuous and dense imbedding $H_t \subset V_t^*$.

Collecting the above results we get

Lemma 1.6. *Let $\Omega \in C^{1,1}$. Then:*

(i) *There exists uniquely determined mappings*

$$\gamma_n \in \mathscr{L}(\mathscr{V}, W_2^{1/2}(\Gamma)), \quad \gamma_t \in \mathscr{L}(\mathscr{V}, V_t)$$

such that

$$\gamma_n(u) = u_i|_\Gamma n_i, \quad \gamma_{tj}(u) = u_j|_\Gamma - \gamma_n(u) n_j$$

for all $u \in [C^\infty(\bar{\Omega})]^3$, and

$$\gamma(u) = \gamma_n(u) n + \gamma_t(u) \qquad \forall u \in \mathscr{V}.$$

For each pair $\{h, k\} \in W_2^{1/2}(\Gamma) \times V_t$ there exists a $u \in \mathscr{V}$ such that

$$\gamma_n(u) = h, \quad \gamma_t(u) = k,$$

$$\|u\| \leqq c_1(\|h\|_{W_2^{1/2}(\Gamma)} + \|k\|_V)$$

($c_1 = \text{const} > 0$).

(ii) *There exists uniquely determined mappings*

$$\pi_n \in \mathscr{L}(\mathbf{T}, W_2^{-1/2}(\Gamma)), \quad \pi_t \in \mathscr{L}(\mathbf{T}, V_t^*)$$

such that

$$\pi_n(\tau) = \tau_{ij}|_\Gamma n_i n_j, \quad \pi_{ti}(\tau) = \tau_{ij}|_\Gamma n_j - \pi_n(\tau) n_i$$

for all $\tau \in [C^\infty(\bar{\Omega})]^9$, and

$$\langle \pi(\tau), h \rangle_V = \langle \pi_n(\tau), h_n \rangle_{W_2^{1/2}(\Gamma)} + \langle \pi_t(\tau), h_t \rangle_{V_t^*}$$

for all $\tau \in \mathbf{T}$ and all $h \in V$. For each pair $\{h^*, k^*\} \in W_2^{-1/2}(\Gamma) \times V_t^*$ there exists a $\tau \in \mathbf{T}$ such that

$$\pi_n(\tau) = h^*, \quad \pi_t(\tau) = k^*,$$

$$\|\tau\|_T \leqq c_2(\|h^*\|_{W_2^{-1/2}(\Gamma)} + \|k^*\|_{V_t^*})$$

($c_2 = \text{const} > 0$).

(iii) *For any $\tau \in \mathbf{T}$ and any $u \in \mathscr{V}$ it holds*

$$\int_\Omega \tau_{ij} u_{i,j} \, dx + \int_\Omega \tau_{ij,j} u_i \, dx =$$
$$= \langle \pi_n(\tau), \gamma_n(u) \rangle_{W_2^{1/2}(\Gamma)} + \langle \pi_t(\tau), \gamma_t(u) \rangle_{V_t}.$$

[6]) The parentheses on the right hand side mean the dual pairings between the respective spaces. Further, $W_2^{-1/2}(\Gamma) =$ dual of $W_2^{1/2}(\Gamma)$, $V_t^* =$ dual of V_t.

5° Let $\varphi : V \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous functional. We denote by $D(\varphi)$ its effective domain, i.e.

$$D(\varphi) = \{h \in V : \varphi(h) < +\infty\}.$$

Next, for any proper functional $\varphi : V \to (-\infty, +\infty]$ we introduce the conjugate functional

$$\varphi^*(h^*) = \sup_{h \in V} [\langle h^*, h \rangle_V - \varphi(h)]$$

with the effective domain

$$D(\varphi^*) = \{h^* \in V^* : \varphi^*(h^*) < +\infty\}.$$

Obviously,

$$\varphi(h) + \varphi^*(h^*) \geqq \langle h^*, h \rangle_V \quad \forall h \in V, \quad \forall h^* \in V^*. \tag{1.2}$$

The functional φ^* is proper, convex and lower semi-continuous if φ does (cf. [8], [10]). Further, by a result of Brondsted and Rockafellar (cf. [1]),

$$\exists h_0 \in V, \quad \exists h_0^* \in V^* : \varphi(h_0) + \varphi^*(h_0^*) = \langle h_0^*, h_0 \rangle_V. \tag{1.3}$$

We introduce the following

Definition. *Let $\Omega \in C^{1,1}$. The proper functional $\varphi : V \to (-\infty, +\infty]$ is said to be decomposable if there exist functionals $\varphi_n : W_2^{1/2}(\Gamma) \to (-\infty, +\infty]$ and $\varphi_t : V_t \to \to (-\infty, +\infty]$ such that*

$$\varphi(h) = \varphi_n(h_n) + \varphi_t(h_t) \quad \forall h \in V$$

where $h = h_n n + h_t$.

Obviously, the functionals φ_n and φ_t are proper. Let us note two properties of proper, decomposable functionals.

Lemma 1.7. *Let $\Omega \in C^{1,1}$ and let $\varphi : V \to (-\infty, +\infty]$ be a proper, decomposable functional.*

Then the following two conditions are equivalent:

(i) *φ is convex (resp. lower semi-continuous),*

(ii) *both φ_n and φ_t are convex (resp. lower semi-continuous).*

Proof. (i) $\Rightarrow$ (ii). The convexity of both φ_n and φ_t is easily deduced from the convexity of φ.

Let $\{h^s\} \subset W_2^{1/2}(\Gamma)$ $(s = 1, 2, \ldots)$ be any sequence such that $h^s \to h$ strongly in $W_2^{1/2}(\Gamma)$ as $s \to \infty$. Fix $k_0 \in V_t$ with $\psi_t(k_0) < +\infty$, and set $\bar{h}^s = h^s n + k_0$, $\bar{h} = hn + k_0$. By Lemma 1.5, $\bar{h}^s \to \bar{h}$ strongly in V as $s \to \infty$, and therefore

$$\varphi_n(h) = \varphi(\bar{h}) - \varphi_t(k_0) \leqq \liminf \varphi(\bar{h}^s) - \varphi_t(k_0) = \liminf \varphi_n(h^s).$$

The lower semi-continuity of φ_t is deduced from the lower semi-continuity of φ by an analogous argument.

(ii) $\Rightarrow$ (i). The convexity of φ is obvious, while the lower semi-continuity is obtained easily when taking into account the algebraic and topological isomorphy between V and $W_2^{1/2}(\Gamma) \times V_t$.

Lemma 1.8. *Let $\Omega \in C^{1,1}$, and let $\varphi : V \to (-\infty, +\infty]$ be a proper, decomposable functional.*

Then

$$\varphi^*(h^*) = \varphi_n^*(h_n^*) + \varphi_t^*(h_t^*) \qquad \forall h^* \in V^*$$

where $\langle h^*, h\rangle_V = \langle h_n^*, h_n\rangle_{W_2^{1/2}(\Gamma)} + \langle h_t^*, h_t\rangle_{Vt}$ for all $h \in V$.

Proof. In virtue of the algebraic and topological isomorphy between V and $W_2^{1/2}(\Gamma) \times V_t$,

$$\begin{aligned}\varphi^*(h^*) &= \sup_{h \in V} [\langle h^*, h\rangle_V - \varphi(h)] \\ &= \sup_{h \in W_2^{1/2}(\Gamma)} [\langle h_n^*, h\rangle_{W_2^{1/2}(\Gamma)} - \varphi_n(h)] + \\ &\quad + \sup_{k \in V_t} [\langle h_t^*, k\rangle_{V_t^*} - \varphi_t(k)] \\ &= \varphi_n^*(h_n^*) + \varphi_t^*(h_t^*)\end{aligned}$$

for any $h^* \in V^*$.

2. SETTING OF THE BOUNDARY VALUE PROBLEM

1° In what follows, we consider the following situation. Suppose that a deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ which is assumed to belong to the class $C^{0,1}$. We look for the displacement vector $u \in \mathscr{V}$ and the stress tensor $\sigma \in \boldsymbol{T}$ in the equilibrium state of the body under the action of the given body force $f \in \mathscr{H}$.

To make this situation precise, let us assume that the strain-displacement relations

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

hold a.e. in Ω, and that the conditions of equilibrium

$$\sigma_{ij,j} + f_i = 0$$

are satisfied a.e. in Ω. Further, we suppose that in Ω the stress-strain relations (Hooke's law)

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}$$

hold where the elastic coefficients a_{ijkl} are assumed to satisfy the following conditions:

$$a_{ijkl} \textit{ is measurable and bounded on } \Omega\,,$$

$$a_{ijkl} = a_{jikl} = a_{klij} \quad \textit{for a.a.} \quad x \in \Omega\,,$$
$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq a_0\varepsilon_{ij}\varepsilon_{ij} \quad \textit{for all symmetric}$$
$$\textit{tensors } \varepsilon_{ij} \textit{ and all } x \in \Omega\,; \quad a_0 = \text{const} > 0\,.$$

To complete the formulation of our problem we introduce a general relation between the stress vector and the displacement vector along the boundary Γ. This relation will include a number of known classical boundary conditions as well as conditions involving unilateral or bilateral constraints. On the other hand, the general setting that we are going to introduce enables us to present a transparent approach to the dual problems.

In all what follows, let φ denote a proper, convex and lower semi-continuous functional from V into $(-\infty, +\infty]$, φ^* its conjugate functional.

Definition. *The displacement vector $u \in \mathscr{V}$ and the stress tensor $\sigma \in \boldsymbol{T}$ are said to satisfy the boundary conditions (associated with the pair $\{\varphi, \varphi^*\}$) if*

$$\varphi(\gamma(u)) + \varphi^*(-\pi(\sigma)) + \langle \pi(\sigma), \gamma(u) \rangle_V = 0\,.$$

In view of (1.3) and the surjectivity of both γ and π (cf. Lemmas 1.2 and 1.3), the set of all $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ which satisfy the boundary condition (associated with the pair $\{\varphi, \varphi^*\}$) is non-void.

Now we state the following

Problem I. *Find $u \in \mathscr{V}$ and $\sigma \in \boldsymbol{T}$ such that*

(2.1) $$\sigma_{ij,j} + f_i = 0 \quad \textit{a.e. in } \Omega\,,$$

(2.2) $$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(u) \quad \textit{a.e. in } \Omega\,,$$

(2.3) $$\varphi(\gamma(u)) + \varphi^*(-\pi(\sigma)) + \langle \pi(\sigma), \gamma(u) \rangle_V = 0\,.$$

Let us note two equivalent formulations of the boundary condition (2.3). Firstly, $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ satisfies (2.3) if and only if

(2.3′) $$\varphi(h) - \varphi(\gamma(u)) + \langle \pi(\sigma), h - \gamma(u) \rangle_V \geqq 0 \;\; \forall h \in D(\varphi)$$

or, equivalently,

$$\gamma(u) \in D(\varphi)\,, \quad -\pi(\sigma) \in \partial\varphi(\gamma(u))\,. \text{ [7]}$$

Secondly, $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ satisfies (2.3) if and only if

(2.3″) $$\varphi^*(h^*) - \varphi^*(-\pi(\sigma)) - \langle h^* + \pi(\sigma), \gamma(u) \rangle_V \geqq 0 \;\; \forall h^* \in D(\varphi^*)$$

[7]) Let X be a normed linear space, $\langle .,. \rangle$ the dual pairing between the dual X^* and X. The subdifferential mapping $\partial\Phi$ of a proper functional $\Phi : X \to (-\infty, +\infty]$ is defined to be $\partial\Phi(x) = \{x^* \in X^* : \Phi(y) \geqq \Phi(x) + \langle x^*, y - x \rangle \; \forall y \in X\}$.

or, equivalently,

$$-\pi(\sigma) \in D(\varphi^*), \quad \gamma(u) \in \partial\varphi^*(-\pi(\sigma)) \ ^8)$$

(cf. e.g. [8]).

2° The boundary condition (2.3) may be equivalently replaced by other relations when imposing certain additional conditions upon φ.

Lemma 2.1. *Let $\Omega \in C^{0,1}$. Suppose that φ is positively homogeneous, i.e.*

$$\varphi(0) = 0, \quad \varphi(th) = t\,\varphi(h) \quad \forall t > 0, \quad \forall h \in V.$$

Then $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ satisfies (2.3) if and only if

$$\varphi(\gamma(u)) + \langle \pi(\sigma), \gamma(u) \rangle_V = 0, \quad -\pi(\sigma) \in \partial\varphi(0). \tag{2.3$_1$}$$

Proof. First of all, it holds

$$D(\varphi^*) = \partial\varphi(0), \quad \varphi^*(h^*) = 0 \quad \forall h^* \in D(\varphi^*)$$

(cf. [8, §§ 4.1, 4.2]). Let $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ satisfy (2.3). Since $-\pi(\sigma) \in D(\varphi^*)$, we have $\varphi^*(-\pi(\sigma)) = 0$ and (2.3) turns into (2.3$_1$).

Conversely, let $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ fulfil the conditions (2.3$_1$). The first one means $\gamma(u) \in D(\varphi)$, while the second one is equivalent to $\varphi(h) + \langle \pi(\sigma), h \rangle_V \geqq 0$ for all $\gamma \in V$. Hence $\{u, \sigma\}$ satisfies (2.3').

Lemma 2.2. *Let $\Omega \in C^{0,1}$. Suppose we are given a proper, convex functional $\psi : H \to (-\infty, +\infty]$ such that $D(\psi) = \{h \in H : \psi(h) < +\infty\}$ is open. Further, let ψ be continuous at some point of $D(\psi)$.*

Let ψ^* denote the conjugate functional of ψ:

$$\psi^*(h) = \sup_{g \in H} [(h, g)_H - \psi(g)].$$

Set $\varphi = \psi \circ j$ [9]). Then $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ satisfies (2.3) if and only if

$$\pi(\sigma) = j^*(p) \quad \text{with} \quad p \in H,$$

$$\psi(j\,\gamma(u)) + \psi^*(-p) + (p, j\,\gamma(u))_H = 0.$$

Proof. Observing that $\mathrm{Im}(j) \cap D(\psi) \neq \emptyset$ and that j^* is injective we conclude from [8, § 3.4, Theorem 3] that

$$\left\{ \begin{aligned} D(\varphi^*) &= \{h^* \in V^* : h^* = j^*(g),\ g \in D(\psi^*)\}, \\ \varphi^*(h^*) &= \psi^*((j^*)^{-1} h^*) \quad \forall h^* \in D(\varphi^*). \end{aligned} \right. \tag{2.4}$$

The assertion is now seen at once.

[8]) Here we have identified $\gamma(u)$ with its canonical image in V^{**}.
[9]) Cf. footnote 5.

Lemma 2.3. *Let $\Omega \in C^{1,1}$ and let φ be decomposable (cf. Section 1.5°).*

Then $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ satisfies (2.3) if and only if

$$(2.3_2)\qquad \begin{cases} \varphi_n(\gamma_n(u)) + \varphi_n^*(-\pi_n(\sigma)) + \langle \pi_n(\sigma), \gamma_n(u)\rangle_{W_2^{1/2}(\Gamma)} = 0\,, \\ \varphi_t(\gamma_t(u)) + \varphi_t^*(-\pi_t(\sigma)) + \langle \pi_t(\sigma), \gamma_t(u)\rangle_{V_t} = 0\,. \end{cases}$$

Proof. Let $\{u, \sigma\} \in \mathscr{V} \times \mathbf{T}$ satisfy (2.3'). Given arbitrary $h \in W_2^{1/2}(\Gamma)$ and $k \in V_t$, we set $\bar{h} = hn + k$. Lemma 1.5 implies $\bar{h} \in V$, and by Lemma 1.6 (ii)

$$\varphi_n(h) - \varphi_n(\gamma_n(u)) + \langle \pi_n(\sigma), h - \gamma_n(u)\rangle_{W_2^{1/2}(\Gamma)}$$
$$+ \varphi_t(k) - \varphi_t(\gamma_t(u)) + \langle \pi_t(\sigma), k - \gamma_t(u)\rangle_{V_t} \geqq 0\,.$$

Setting $k = \gamma_t(u)$ (resp. $h = \gamma_n(u)$) in the last inequality one finds

$$\varphi_n(h) - \varphi_n(\gamma_n(u)) + \langle \pi_n(\sigma), h - \gamma_n(u)\rangle_{W_2^{1/2}(\Gamma)} \geqq 0\,,$$
$$\varphi_t(k) - \varphi_t(\gamma_t(u)) + \langle \pi_t(\sigma), k - \gamma_t(u)\rangle_{V_t} \geqq 0$$

for all $h \in W_2^{1/2}(\Gamma)$ and all $k \in V_t$, respectively. This system is equivalent to (2.3_2).

The converse assertion is obvious.

3. EXAMPLES

We now illustrate that (2.3) includes a number of classical boundary conditions as well as conditions involving unilateral or bilateral constraints along the boundary. Existence theorems which apply to the examples discussed below may be found in Part II, Section 5.

1° Let us begin with considering some classical boundary conditions. In Examples 1–4 the domain Ω is assumed to belong to the class $C^{0,1}$.

Example 1 (*displacement boundary condition*). Let $u_0 \in V$ be given. We consider the condition

$$(3.1)\qquad \gamma(u) = u_0\,.$$

Let φ denote the indicator function of the closed convex set $\{u_0\}$, i.e.

$$\varphi(h) = \begin{cases} 0 & \text{for} \quad h = u_0 \quad \text{a.e. on } \Gamma\,, \\ +\infty & \text{for} \quad h \in V, \quad h \neq u_0\,. \end{cases}$$

The functional φ is proper, convex and lower semi-continuous. It is easy to see that

$$\varphi^*(h^*) = \langle h^*, u_0\rangle_V \quad \forall h^* \in V^*\,.$$

The equivalence of (2.3) and (3.1) is immediate.

Example 2 (*traction boundary condition*). Given $g^* \in V^*$ we subject the stress tensor σ to the condition

$$(3.2)\qquad \pi(\sigma) = g^*\,.$$

Set

$$\varphi(h) = -\langle g^*, h\rangle_V \quad \forall h \in V.$$

Then

$$\varphi^*(h^*) = \begin{cases} 0 & \textit{for} \quad h^* = -g^*, \\ +\infty & \textit{otherwise}. \end{cases}$$

As above, the equivalence of (2.3) and (3.2) is seen at once.

Example 3 (*generalized support condition*). Let $A : H \to H$ be a monotone gradient mapping (cf. [5]). Then u and σ are required to satisfy the condition

$$\pi(\sigma) = -j^* \circ A \circ j(\gamma(u)) \tag{3.3}$$

(recall that j denotes the injection from V into H).

The mapping A maps strongly convergent sequences into weakly convergent sequences; further, A is the Gateaux derivative of the convex, continuous functional

$$\psi(h) = \int_0^1 (A(th), h)_H \, \mathrm{d}t, \quad h \in H$$

(cf. [5]). Therefore, (3.3) is equivalent to each of the following conditions:

$$\psi(jh) - \psi(j\,\gamma(u)) + \langle \pi(\sigma), h - \gamma(u)\rangle_V \geqq 0 \quad \forall h \in V, \tag{3.3$_1$}$$

$$\begin{cases} \pi(\sigma) \in \operatorname{Im}(j^*), \\ \psi(j\,\gamma(u)) + \varphi^*(-(j^*)^{-1}\,\pi(\sigma)) + \langle \pi(\sigma), \gamma(u)\rangle_V = 0. \end{cases} \tag{3.3$_2$}$$

Set $\varphi = \psi \circ j$. By (2.4),

$$\varphi^*(h^*) = \begin{cases} \psi^*((j^*)^{-1}\,h^*) & \textit{for} \quad (j^*)^{-1}\,h^* \in D(\psi^*), \\ +\infty & \textit{otherwise}, \end{cases}$$

and the equivalence of (2.3) and (3.3) is an immediate consequence of Lemma 2.2.

Suppose additionally that A is bijective. Then A^{-1} is also a monotone gradient mapping, and it holds

$$\psi^*(h) = \int_0^1 (A^{-1}(th), h)_H \, \mathrm{d}t - \int_0^1 (A(tA^{-1}(0)), A^{-1}(0))_H \, \mathrm{d}t$$

for any $h \in H$ (cf. [5]).

As a special case of (3.3) we consider the elastic support condition

$$\pi(\sigma) = j^*(p) \quad \textit{with} \quad p = -a\,\gamma(u) + g \quad \textit{a.e. on } \Gamma$$

where $a \in L^\infty(\Gamma)$, $a \geqq a_0 = \text{const} > 0$ a.e. on Γ and $g \in H$. In the present case, the functionals φ and φ^* take the form

$$\varphi(h) = \frac{1}{2}\int_\Gamma a|h|^2 \, \mathrm{d}S - \int_\Gamma g_i h_i \, \mathrm{d}S, \; h \in V,$$

$$\varphi^*(h^*) = \begin{cases} \frac{1}{2}\int_\Gamma \frac{1}{a}|g + q|^2 \, dS & \text{for} \quad h^* = j^*(q), \quad q \in H, \\ +\infty & \text{otherwise}. \end{cases}$$

Example 4 (*mixed boundary conditions*). Let Γ_s ($s = 1, 2, 3$) be mutually disjoint open subsets of Γ such that $\Gamma \setminus \bigcup_{s=1}^{3} \Gamma_s$ has measure zero.

Suppose we are given the following data:

$$u_0 \in V,$$

$$g \in [L^2(\Gamma_2 \cup \Gamma_3)]^3, \quad a \in L^\infty(\Gamma_2 \cup \Gamma_3) \quad \textit{where}$$

$$a = 0 \quad \textit{a.e. on } \Gamma_2, \quad a \geqq a_0 = \text{const} > 0 \quad \textit{a.e. on } \Gamma_3.$$

Let us consider the following boundary conditions:

$$\gamma(u) = u_0 \quad \textit{a.e. on } \Gamma_1, \tag{3.4}$$

$$\langle \pi(\sigma), h \rangle_V = \int_{\Gamma_2 \cup \Gamma_3} (-a\,\gamma_i(u) + g_i)\, h_i \, d\mathbf{S} \tag{3.5}$$

$$\forall h \in V \quad \textit{with} \quad h = 0 \quad \textit{a.e. on } \Gamma_1.$$

Analogously as above, we introduce the functionals

$$\varphi_1(h) = \begin{cases} 0 & \textit{for} \quad h = u_0 \quad \textit{a.e. on } \Gamma_1, \\ +\infty & \textit{for} \quad h \in V, \quad h \neq u_0 \quad \textit{a.e. on } \Gamma_1, \end{cases}$$

$$\psi(h) = \frac{1}{2}\int_{\Gamma_3} a|h|^2 \, dS - \int_{\Gamma_2 \cup \Gamma_3} g_i h_i \, dS \quad \textit{for} \quad h \in H$$

and

$$\varphi = \varphi_1 + \varphi_2 \quad \textit{where} \quad \varphi_2 = \psi \circ j.$$

The conjugate functionals of φ_1 and ψ can be calculated (with minor changes) as above:

$$\varphi_1^*(h^*) = \begin{cases} \langle h^*, u_0 \rangle_V & \textit{for } h^* \in V^* \textit{ such that } \langle h^*, h \rangle_V = 0 \\ & \textit{for any } h \in V \textit{ with } h = 0 \textit{ a.e. on } \Gamma_1, \\ +\infty & \textit{otherwise}; \end{cases}$$

$$\psi^*(q) = \begin{cases} \frac{1}{2}\int_{\Gamma_3} \frac{1}{a}|q + g|^2 \, d\mathbf{S} & \textit{for } q \in H \textit{ with } q = 0 \\ & \textit{a.e. on } \Gamma_1, \; q = -g \\ & \textit{a.e. on } \Gamma_2, \\ +\infty & \textit{otherwise}. \end{cases}$$

Further, (2.4) implies $\varphi_2^*(h^*) = \psi^*(q)$ for $h^* \in V^*$ such that $h^* = j^*(q)$ with $q \in D(\psi^*)$.

In order to calculate explicitly the functional φ^* we note first of all that φ_2 is defined and continuous on the whole V. Then

$$\varphi^*(h^*) = \inf_{\substack{h_1^*, h_2^* \in V^* \\ h^*_1 + h^*_2 = h^*}} [\varphi_1^*(h_1^*) + \varphi_2^*(h_2^*)]$$

(cf. [8], [10], [18]). Let now $h_1^* \in D(\varphi_1^*)$ and $h_2^* \in D(\varphi_2^*)$ be arbitrary. Setting $h^* = h_1^* + h_2^*$ it holds $\varphi^*(h^*) < +\infty$; i.e. $h^* \in D(\varphi^*)$. Conversely, let $h^* \in D(\varphi^*)$ be arbitrary. Then there exists a decomposition $h^* = h_1^* + h_2^*$ with $h_1^* \in D(\varphi_1^*)$ and $h_2^* \in D(\varphi_2^*)$ (i.e. $h_2^* = j^*(q)$ with $q \in D(\psi^*)$) such that $\varphi^*(h^*) = \varphi_1^*(h_1^*) + \varphi_2^*(h_2^*)$. By Lemma 1.4 (iii), the elements h_1^* and h_2^* are uniquely determined by h^*, and therefore

$$\varphi^*(h^*) = \begin{cases} \langle h, u_0 \rangle_V + \dfrac{1}{2} \displaystyle\int_{\Gamma_3} \dfrac{1}{a} |g + q|^2 \, dS & \text{for} \quad h^* \in D(\varphi^*) , \\ +\infty & \text{otherwise} , \end{cases}$$

where

$$\begin{aligned} D(\varphi^*) = \{ h^* \in V^* : h^* = h_1^* + j^*(q) \ & \text{where } h_1^* \in V^* \text{ such that} \\ & \langle h_1^*, h \rangle_V = 0 \text{ for any } h \in V \text{ with } h = 0 \text{ a.e. on } \Gamma_1, \\ & q \in H \text{ such that } q = 0 \text{ a.e. on } \Gamma_1, \ q = -g \text{ a.e. on } \Gamma_2 \} . \end{aligned}$$

It is now readily verified that with the above choice of φ the boundary conditions (3.4), (3.5) can be equivalently written in the form (2.3).

Example 5. Let $\Omega \in C^{1,1}$, and let $k_0 \in V_t$ and $h_0^* \in W_2^{-1/2}(\Gamma)$ be given. We consider the boundary conditions

$$\gamma_t(u) = k_0 , \quad \pi_n(\sigma) = h_0^* . \tag{3.6}$$

Let us introduce the functionals

$$\varphi_n(h) = -\langle h_0^*, h \rangle_{W_2^{1/2}} (\Gamma) \quad \text{for} \quad h \in W_2^{1/2}(\Gamma) ,$$

$$\varphi_t(k) = \begin{cases} 0 & \text{if} \quad k = k_0 , \\ +\infty & \text{if} \quad k \in V_t , \quad k \neq k_0 \end{cases}$$

and define

$$\varphi(h) = \varphi_n(h_n) + \varphi_t(h_t) \quad \text{for} \quad h \in V$$

where $h = h_n n + h_t$. The functional φ is proper, convex, lower semi-continuous and decomposable. It holds

$$\varphi_n^*(h^*) = \begin{cases} 0 & \text{if} \quad h^* = -h_0^* , \\ +\infty & \text{otherwise} , \end{cases}$$

$$\varphi_t^*(k^*) = \langle k^*, k_0 \rangle_{V_t} \quad \text{for} \quad k^* \in V_t^* .$$

By Lemma 1.8,

$$\varphi^*(h^*) = \begin{cases} \langle h_t^*, k_0 \rangle_{V_t} & \text{if} \quad h_n^* = -h_o^* , \\ +\infty & \text{otherwise} , \end{cases}$$

where $\langle h^*, h \rangle_V = \langle h_n^*, h_n \rangle \sigma_{W_2^{1/2}(\Gamma)} + \langle h_t^*, h_t \rangle_{V_t}$ for any $h \in V$.

The equivalence of (3.6) with (2.3_2) is obvious.

Preserving the assumption $\Omega \in C^{1,1}$, we can verify analogously that the boundary conditions

$$(3.7) \qquad \gamma_n(u) = h_0 , \quad \pi_t(\sigma) = k_0^*$$

where $h_0 \in W_2^{1/2}(\Gamma)$, $k_0^* \in V_t^*$ (e.g., $h_0 = 0$, $k_0^* = 0$ in the case of a contact support) are also included in (2.3_2) as a special case when setting

$$\varphi_n(h) = \begin{cases} 0 & \text{if } h = h_0 , \\ +\infty & \text{if } h \in W_2^{1/2}(\Gamma) , \quad h \neq h_0 , \end{cases}$$

$$\varphi_t(k) = -\langle k_0^*, k \rangle_{V_t} . \quad \text{for} \quad k \in V_t .$$

Finally, when choosing appropriately the functionals φ_n, φ_t and using similar arguments as those of Example 4, the general mixed boundary conditions discussed in [6], [7] can also be expressed in the form (2.3_2). Further, let us note that our approach does not require $\sigma \in [W_2^1(\Omega)]^9$ (cf. [6]). The disadvantage of this requirement consists in the fact that Problem I and Problem II are no more equivalent (cf. Theorem 4.1 below). Moreover, this requirement seems to be less convenient with regard to the general duality (cf. Part II, Section 6).

2° We pass to a discussion of some boundary conditions that involve unilateral and bilateral constraints.

Example 6 (*Signorini problem*; *cf.* [3], [4]). Let $k_0^* \in V_t^*$ be given. We consider the boundary conditions

$$(3.8) \qquad \begin{cases} \gamma_n(u) \leqq 0 \quad \text{a.e. on } \Gamma , \quad \langle \pi_n(\sigma), \gamma_n(u) \rangle_{W_2^{1/2}(\Gamma)} = 0 , \\ \langle \pi_n(\sigma), h \rangle_{W_2^{1/2}(\Gamma)} \geqq 0 \ \forall h \in W_2^{1/2}(\Gamma) , \quad h \leqq 0 \quad \text{a.e. on } \Gamma , \end{cases}$$

$$(3.9) \qquad \pi_t(\sigma) = k_0^* .$$

Let us set $\varphi(h) = \varphi_n(h_n) + \varphi_t(h_t)$ for any $h \in V$, where

$$\varphi_n(h) = \begin{cases} 0 & \text{for} \quad h \in W_2^{1/2}(\Gamma) , \quad h \leqq 0 \quad \text{a.e. on } \Gamma , \\ +\infty & \text{for} \quad h \in W_2^{1/2}(\Gamma) , \quad h > 0 \quad \text{on a subset} \\ & \qquad \text{of positive measure,} \end{cases}$$

$$\varphi_t(k) = -\langle k_0^*, k \rangle_{V_t} \quad \text{for} \quad k \in V_t .$$

The conjugate functional of φ_n and φ_t, respectively, becomes

$$\varphi_n^*(h^*) = \begin{cases} 0 & \text{for } h^* \in W_2^{-1/2}(\Gamma) \text{ with } \langle h^*, h \rangle_{W_2^{1/2}(\Gamma)} \leqq 0 \\ & \text{for any } h \in W_2^{1/2}(\Gamma) \text{ with } h \leqq 0 \text{ a.e. on } \Gamma , \\ +\infty & \text{otherwise ;} \end{cases}$$

$$\varphi_t^*(k^*) = \begin{cases} 0 & \text{for} \quad k^* = -k_0^* , \\ +\infty & \text{otherwise .} \end{cases}$$

It is easily seen that the boundary conditions (3.8), (3.9) are a special case of (2.3_2) with the above choice of φ_n and φ_t.

Further, the boundary condition (3.9) may be obviously replaced by

$$\gamma_t(u) = k_0 \tag{3.9'}$$

where $k_0 \in V_t$ is given (for the choice of the functional φ_t we refer to Example 5).

Example 7 (friction along any tangential direction; cf. [2]). Let $h_0^* \in W_2^{-1/2}(\Gamma)$ and $k \in L^\infty(\Gamma)$ be given where $k > 0$ a.e. on Γ. Let us introduce the conditions

$$\pi_n(\sigma) = h_0^*, \tag{3.10}$$

$$\begin{cases} \pi_t(\sigma) = j_t^*(p) \quad with \quad p \in H_t \text{ [10]} \quad and \\ |p| \leqq k \quad a.e.\ on\ \Gamma, \quad where: \\ |p| < k \Rightarrow \gamma_t(u) = 0, \\ |p| = k \Rightarrow \exists \lambda \geqq 0 : \gamma_t(u) = -\lambda p. \end{cases} \tag{3.11}$$

Note that (3.11) is equivalent to

$$\begin{aligned} &\pi_t(\sigma) = j_t^*(p) \quad with \quad p \in H_t \quad and \\ &|p| \leqq k, \quad k|\gamma_t(u)| + p_i\, \gamma_{ti}(u) = 0 \quad a.e.\ on\ \Gamma. \end{aligned} \tag{3.11_1}$$

We define

$$\psi_t(q) = \int_\Gamma k|q|\, \mathrm{d}S \quad for \quad q \in H_t.$$

The functional ψ_t is convex, continuous and positively homogeneous on H_t. A simple calculation yields

$$\psi_t^*(q) = \begin{cases} 0 & for \quad q \in H_t \quad with \quad |q| \leqq k \quad a.e.\ on\ \Gamma, \\ +\infty & otherwise. \end{cases}$$

The equivalence of (3.11_1) to

$$\begin{cases} \pi_t(\sigma) \in \operatorname{Im}(j_t^*), \\ \psi_t(j_t\, \gamma_t(u)) + \psi_t^*(-(j_t^*)^{-1}\, \pi_t(\sigma)) + \langle \pi_t(\sigma), \gamma_t(u) \rangle_{V_t} = 0 \end{cases} \tag{3.11_2}$$

is easily verified.

Finally, setting $\varphi_t = \psi_t \circ j_t$ we have

$$\varphi_t^*(k^*) = \begin{cases} 0 & for \quad k^* \in V_t^* \quad with \quad k^* = j_t^*(q), \\ & q \in H_t, \quad |q| \leqq k \quad a.e.\ on\ \Gamma, \\ +\infty & otherwise \end{cases}$$

(cf. (2.4)) and (3.11_2) is obviously equivalent to the second condition in (2.3_2).

---

[10]) We denote by j_t, j_n the injections from V_t into H_t and from $W_2^{1/2}(\Gamma)$ into $L^2(\Gamma)$ respectively, and by j_t^*, j_n^* the adjoint mappings. Recall that j_t^* and j_n^* are injections, too.

As in the preceding example, (3.10) may be replaced by the condition

$$\gamma_n(u) = h_0 \tag{3.10'}$$

where $h_0 \in W_2^{1/2}(\Gamma)$ is given (let us refer to Example 5 for the choice of the functional φ_n).

Example 8 (*friction along the normal direction*; *cf.* [2]). Suppose we are given $k_0^* \in V_t^*$ and $k_i \in L^\infty(\Gamma)$ $(i = 1, 2)$ where $k_1 \leqq 0 \leqq k_2$ a. e. on Γ. We consider the boundary conditions

$$\begin{cases} \pi_n(\sigma) = j_n^*(p) \quad \textit{with} \quad p \in L^2(\Gamma) \quad \textit{and} \\ k_1 \leqq p \leqq k_2 \quad \textit{a.e. on } \Gamma\,, \quad \textit{where:} \\ k_1 < p < k_2 \Rightarrow \gamma_n(u) = 0\,, \\ p = k_1 \Rightarrow \gamma_n(u) \geqq 0\,, \\ p = k_2 \Rightarrow \gamma_n(u) \leqq 0\,, \end{cases} \tag{3.12}$$

$$\pi_t(\sigma) = k_0^*\,. \tag{3.13}$$

A straightforward calculation shows that (3.12) is equivalent to

$$\begin{cases} \pi_n(\sigma) = j_n^*(p) \quad \textit{with} \quad p \in L^2(\Gamma) \quad \textit{and} \quad k_1 \leqq p \leqq k_2\,, \\ -k_1\,\gamma_n(u)^+ + k_2\,\gamma_n(u)^- + p\,\gamma_n(u) = 0 \quad \textit{a.e. on } \Gamma\,. \end{cases} \tag{3.12$_1$}$$

Setting

$$\varphi_n(q) = \int_\Gamma (-k_1 q^+ + k_2 q^-)\,\mathrm{d}S \quad \textit{for} \quad q \in L^2(\Gamma)\,,$$

we have

$$\psi_n^*(q) = \begin{cases} 0 & \textit{if} \quad q \in L^2(\Gamma) \quad \textit{with} \quad k_1 \leqq -q \leqq k_2 \quad \textit{a.e. on } \Gamma\,, \\ +\infty & \textit{otherwise}\,, \end{cases}$$

and the system of conditions (3.12$_1$) is equivalent to the conditions

$$\begin{cases} \pi_n(\sigma) \in \mathrm{Im}(j_n^*)\,, \\ \psi_n(j_n\,\gamma_n(u)) + \psi_n^*(-(j_n^*)^{-1}\,\pi_n(\sigma)) + \langle \pi_n(\sigma), \gamma_n(u) \rangle_{W_2^{1/2}(\Gamma)} = 0\,. \end{cases} \tag{3.12$_2$}$$

Analogously as in the preceding example, set $\varphi_n = \psi_n \circ j_n$. Then

$$\varphi_n^*(h^*) = \begin{cases} 0 & \textit{for} \quad h^* \in W_2^{-1/2}(\Gamma) \quad \textit{with} \quad h^* = j_n^*(q)\,, \\ & q \in L^2(\Gamma)\,, \quad k_1 \leqq -q \leqq k_2 \quad \textit{a.e. on } \Gamma\,, \\ +\infty & \textit{otherwise}\,, \end{cases}$$

and the equivalence of (3.12$_2$) to the first relation in (2.3$_2$) is evident.

As in Example 6, the boundary condition (3.13) may be replaced by

$$\gamma_t(u) = k_0 \tag{3.13'}$$

where $k_0 \in V_t$ is given.

In the end let us note that we may consider mixed boundary conditions of all foregoing types (e.g. classical, Signorini and friction type conditions on the respective parts of the boundary) when adopting the arguments of Example 4. However, we omit the details.

3° We conclude this section by considering the relationships between some of our above examples.

Lemma 3.1. *Let $\Omega \in C^{0,1}$, and let $h_0^* \in V^*$ be fixed. Further, let $\varphi_0 : V \to \mathbb{R}$ be a convex, lower semi-continuous functional that fulfils the following additional conditions:*

$$\varphi_0(h) \geqq 0 \quad \forall h \in V,$$
$$D_0 = \{h \in V : \varphi_0(h) = 0\} \neq \emptyset .$$

Set

$$\varphi_m(h) = m \ \varphi_0(h) - \langle h_0^*, h\rangle_V \quad \text{for} \quad h \in V, \quad m = 1, 2, \ldots .$$

Then it holds:

(i) $\lim_{m \to \infty} \varphi_m(h) = \psi(h)$ *for any* $h \in V$, *where*

$$\varphi(h) = \begin{cases} -\langle h_0^*, h\rangle_V & \text{if} \quad h \in D_0 , \\ +\infty & \text{otherwise} . \end{cases}$$

(ii) *Let $\{u_m, \sigma_m\} \in \mathscr{V} \times \mathbf{T}$ $(m = 1, 2, \ldots)$ satisfy the boundary conditions (2.3) associated with the pair $\{\varphi_m, \varphi_m^*\}$, and let $\{u_m, \sigma_m\} \to \{u, \sigma\}$ in $\mathscr{V} \times \mathbf{T}$ as $m \to \infty$. Then the pair $\{u, \sigma\}$ satisfies the boundary condition associated with the pair $\{\varphi, \varphi^*\}$.*

Proof. Assertion (i) is immediate (note also that D_0 is convex).

For proving (ii) we conclude first of all from the inequality

$$\varphi_m(h) - \varphi_m(\gamma(u_m)) + \langle \pi(\sigma_m), h - \gamma(u_m)\rangle_V \geqq 0 \tag{3.14}$$

which holds for any $h \in V$ and $m = 1, 2, \ldots$ that

$$m \ \varphi_0(\gamma(u_m)) \leqq c\|h - \gamma(u_m)\|_V \ \forall h \in D_0 , \quad m = 1, 2, \ldots \tag{3.15}$$

($c = \text{const} > 0$). Therefore $\varphi_0(\gamma(u)) \leqq \liminf \varphi_0(\gamma(u_m)) = 0$, i.e. $\gamma(u) \in D_0 = D(\varphi)$. Next, inserting $h = \gamma(u)$ in (3.15) one obtains $\lim \varphi_m(\gamma(u_m)) = -\langle h_0^*, \gamma(u)\rangle_V = \varphi(\gamma(u))$. Letting now $m \to \infty$ in (3.14) we get (2.3′).

Let $\Omega \in C^{1,1}$. Set for $h \in V$ and $m = 1, 2, \ldots$

$$\varphi_m(h) = m \int_\Gamma |h_t| \, \mathrm{d}S - \langle h_0^*, h_n\rangle_{W_2^{1/2}(\Gamma)} ,$$

$$\varphi_m(h) = m \int_\Gamma |h_n| \, \mathrm{d}S - \langle k_0^*, h_t\rangle_{V_t} ,$$

$$\varphi_m(h) = m \int_\Gamma h_n^+ \, \mathrm{d}S - \langle k_0^*, h_t\rangle_{V_t}$$

where $h_0^* \in W_2^{-1/2}(\Gamma)$, $k_0^* \in V_t^*$ are fixed. The first functional corresponds to the boundary conditions (3.10), (3.11) with $k = m$, while the second and third one correspond to (3.12), (3.13) with $-k_1 = k_2 = m$ and $-k_1 = m$, $k_2 = 0$, respectively. Passing to the limit $m \to \infty$ we get the functionals which correspond to the boundary, conditions (3.6) with $k_0 = 0$, (3.7) with $h_0 = 0$ and (3.8), (3.9), respectively.

4. VARIATIONAL FORMULATION

1° In this section, we present two equivalent formulations of Problem I (cf. Section 2.1°).

Let us introduce the proper, convex and lower semi-continuous functional

$$F(v) = \tfrac{1}{2}a(v, v) + \varphi(\gamma(v)) - (f, v), \quad v \in \mathscr{V}$$

where

$$a(u, v) = \int_\Omega a_{ijkl}\, \varepsilon_{ij}(u)\, \varepsilon_{kl}(v)\, dx, \quad u, v \in \mathscr{V}.$$

Further, we define

$$\mathscr{V}_{ad} = \{v \in \mathscr{V} : \gamma(v) \in D(\varphi)\}.$$

The functions in $\mathscr{V}_{ad}$ will be called "*geometrically admissible displacement fields*"; for $v \in \mathscr{V}_{ad}$ the expression $F(v)$ represents the "*potential energy*" of the body considered.

Let $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ satisfy (2.1), (2.2). We obtain by the generalized Green formula (cf. Lemma 1.3 (ii))

$$a(u, v) = \langle \pi(\sigma), \gamma(v) \rangle_V + (f, v) \ \forall v \in \mathscr{V}. \tag{4.1}$$

Now we state the following two problems:

Problem II (*principle of virtual displacements*).
Find $u \in \mathscr{V}_{ad}$ *such that*

$$a(u, v - u) + \varphi(\gamma(v)) - \varphi(\gamma(u)) \geqq (f, v - u) \ \forall v \in \mathscr{V}_{ad}. \tag{4.2}$$

Problem III (*principle of minimum potential energy*).
Find $u \in \mathscr{V}_{ad}$ *such that*

$$F(v) \geqq F(u) \quad \forall v \in \mathscr{V}_{ad}. \tag{4.3}$$

The relationships between the problems stated is explained by

Theorem 4.1.

(i) *If* $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ *is a solution to Problem I then* u *is a solution to Problem II.*

(ii) *Let* $u \in \mathscr{V}_{ad}$ *be a solution to Problem II. Set* $\sigma_{ij} = a_{ijkl}u_{k,l}$ *a.e. in* Ω. *Then* $\{u, \sigma\}$ *belongs to* $\mathscr{V} \times \boldsymbol{T}$ *and it is a solution to Problem I.*

(iii) *The function $u \in \mathscr{V}_{ad}$ is a solution to Problem II if and only if it is a solution to Problem III.*

Proof. (i) Let $\{u, \sigma\} \in \mathscr{V} \times \boldsymbol{T}$ be a solution to Problem I. Then $u \in \mathscr{V}_{ad}$ and

$$\varphi(\gamma(v)) - \varphi(\gamma(u)) \geqq -\langle \pi(\sigma), \gamma(v) - \gamma(u) \rangle_V \quad \forall v \in \mathscr{V}_{ad}$$

(cf. (2.3')). Replacing v in (4.1) by $v - u$ and adding the result obtained to the above inequality we get (4.2).

(ii) Let $u \in \mathscr{V}_{ad}$ be a solution to Problem II. Inserting $v = u \pm \psi$ in (4.2), $\psi \in \in [\mathscr{D}(\Omega)]^3$ being arbitrary, one obtains

$$a(u, \psi) = (f, \psi) .$$

Hence, setting $\sigma_{ij} = a_{ijkl} u_{k,l}$ a.e. in Ω we have

$$\int_\Omega \sigma_{ij} \psi_{i,j} \, dx = \int_\Omega f_i \psi_i \, dx \quad \forall \psi \in [\mathscr{D}(\Omega)]^3 .$$

Thus $\sigma \in \boldsymbol{T}$ and $\sigma_{ij,j} + f_i = 0$ a.e. in Ω (and the relation (4.1) holds).

Replacing v in (4.1) by $v - u$ we conclude from (4.2) that

$$\varphi(\gamma(v)) - \varphi(\gamma(u)) + \langle \pi(\sigma), \gamma(v) - \gamma(u) \rangle_V \geqq 0$$

for any $v \in \mathscr{V}_{ad}$. The mapping γ being surjective, this inequality is equivalent to that in (2.3').

(iii) Let $u \in \mathscr{V}_{ad}$ be a solution to Problem II. Then

$$\begin{aligned} F(v) &\geqq F(u) + \tfrac{1}{2} a(v - u, v - u) \\ &\geqq F(u) \end{aligned}$$

for any $v \in \mathscr{V}_{ad}$, i.e. u is a solution to Problem III.

Let, conversely, $u \in \mathscr{V}_{ad}$ be a solution to Problem III. Let $v \in \mathscr{V}_{ad}$ be arbitrary. Then $(1 - t) u + tv \in \mathscr{V}_{ad}$ for any $t \in (0, 1)$ and

$$\begin{aligned} F(u) &\leqq F((1 - t) u + tv) \\ &\leqq F(u) + \tfrac{1}{2} t^2 a(v - u, v - u) \\ &\quad + t[a(u, v - u) + \varphi(\gamma(v)) - \varphi(\gamma(u)) - (f, v - u)] , \end{aligned}$$

or, equivalently,

$$a(u, v - u) + \varphi(\gamma(v)) - \varphi(\gamma(u)) + \tfrac{1}{2} t \, a(v - u, v - u) \geqq (f, v - u) .$$

Letting $t \to 0$ we obtain (4.2).

2° We conclude this section with the following simple observation.

Let $\{u_1, \sigma_1\}, \{u_2, \sigma_2\} \in \mathscr{V} \times \boldsymbol{T}$ be two solutions to Problem I. Then

$$u_1 - u_2 \in \mathscr{R} , \quad \varepsilon_{ij}(u_1) = \varepsilon_{ij}(u_2) , \quad \sigma_{1ij} = \sigma_{2ij} .$$

Indeed, by Theorem 4.1 both u_1 and u_2 satisfy (4.2). This yields

$$0 \geqq a(u_1 - u_2, u_1 - u_2)$$
$$\geqq a_0 \int_\Omega \varepsilon_{ij}(u_1 - u_2)\, \varepsilon_{ij}(u_1 - u_2)\, dx .$$

Consequently, $\varepsilon_{ij}(u_1 - u_2) = 0$ a.e. in Ω, and the assertion is obvious.

References

[1] *A. Brondsted* and *R. T. Rockafellar:* On the subdifferentiability of convex functions. Proc. Amer. Math. Soc., 16 (1965), 605—611.

[2] *G. Duvaut* et *J. L. Lions:* Les inéquations en mécanique et en physique. Dunod, Paris 1972.

[3] *G. Fichera:* Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Memorie (Cl. Sci. fis., mat. e nat.), serie 8, vol. 7 (1964), 91—140.

[4] *G. Fichera:* Boundary value problems of elasticity with unilateral constraints. In: Handbuch der Physik (Herausg.: S. Flügge), Band VI a/2, Springer, 1972.

[5] *H. Gajewski, K. Gröger* und *K. Zacharias:* Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie-Verlag, Berlin 1974.

[6] *I. Hlaváček:* Variational principles in the linear theory of elasticity for general boundary conditions. Apl. Mat., 12 (1967), 425—447.

[7] *I. Hlaváček* and *J. Nečas:* On inequalities of Korn's type. I: Boundary value problems for elliptic systems of partial differential equations. II: Applications to linear elasticity. Arch. Rat. Mech. Anal., 36 (1970), 305—311, 312—334.

[8] *A. D. Ioffe* and *V. M. Tikhomirov:* The theory of extremum problems (Russian). Moscow, 1974.

[9] *F. Lené:* Sur les matériaux élastiques à énergie de déformation non quadratique. J. Méc., 13 (1974), 499—534.

[10] *J. J. Moreau:* Fonctionelles convexes. Collège de France, 1966—1967.

[11] *J. J. Moreau:* On unilateral constraints, friction and plasticity. In: New variational techniques in mathematical physics. C. I. M. E., Ed. Cremonese, Roma 1974, 173—322.

[12] *J. J. Moreau:* La convexité en statique. In: Analyse convexe et ses applications (ed. by J. P. Aubin), Lecture Notes Econ. and Math. Systems, No. 102 (1974), 141—167.

[13] *J. J. Moreau:* La notion de sur-potentiel et les liasions unilaterales en élastostatique. 12th Intern. Congr. Appl. Mech.

[14] *B. Nayroles:* Quelques applications variationnelles de la théorie des fonctions duales à la mécanique de solides. J. Méc., 10 (1971), 263—289.

[15] *B. Nayroles:* Duality and convexity in solid equilibrium problems. Laboratoire Méc. et d'Acoustique, C. N. R. S., Marseille 1974.

[16] *B. Nayroles:* Point de vue algebrique. Convexité et integrandes convexes en mécanique des solides. In: New variational techniques in mathematical physics. C. I. M. E., Ed. Cremonese, Roma 1974, 325—404.

[17] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[18] *R. T. Rockafellar:* Extension of Fenchel's duality theorem for convex functions. Duke Math. J., 33 (1966), 81—90.

Souhrn

OBECNÉ OKRAJOVÉ ÚLOHY A DUALITA V LINEÁRNÍ TEORII PRUŽNOSTI, I

ROLF HÜNLICH, JOACHIM NAUMANN

Rovnovážný stav pružného tělesa, na něž působí vnější síly, je popsán všeobecně známými podmínkami rovnováhy, vztahy mezi posunutím a deformacemi, konstitutivními rovnicemi lineární teorie pružnosti a okrajovými podmínkami. V článku se podrobně studují okrajové podmínky, přičemž východiskem je obecný vztah mezi vektory, napětí a posunutí na hranici, jenž může být vyjádřen v termínech subgradientního vztahu. Ukazuje se, že tento vztah zahrnuje jako speciální případy všechny známé klasické, oboustranné i jednostranné okrajové podmínky. Dále je v článku ustanoven princip virtuálních posunutí a princip minima potenciální energie a je dokázáno, že tyto principy jsou ekvivalentní výchozí okrajové úloze.

Author's addresses: Dr. *Rolf Hünlich*, Dr. *Joachim Naumann*, Sektion Mathematik der Humboldt-Universität zu Berlin, PF 1297, 1086 Berlin.