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ON GENERAL ROUTING PROBLEMS

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ON GENERAL ROUTING PROBLEMS

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ABSTRACT

In two recent papers by C.S. Orloff [*Networks* 4(1974)35-64,147-162] general routing problems for one or more vehicles on a graph $G = (N,A)$ are introduced and discussed. The single vehicle problem is to find an optimal tour on G, containing required subsets $Q \subseteq N$ and $R \subseteq A$. We show that a proposed conversion of required nodes to required arcs is not allowed and that the problem remains polynomial complete if $Q = \emptyset$, which throws some doubt on the effectiveness of such conversions. Furthermore, the proposed transformations from M vehicle to single vehicle problems are shown to be incorrect; correct transformations are presented as well.

NOTE

This paper is registered at the Mathematisch Centrum as BW 42 and at the Graduate School of Management as WP/74/13.

KEY WORDS AND PHRASES: *general routing problem, travelling salesman, Chinese postman, rural postman, transformation, polynomial completeness.*

1. INTRODUCTION

In two recent papers by C.S. Orloff [10;11] general routing problems for one or more vehicles are introduced and discussed. Both papers seem to describe excellent strategies to attack real world vehicle routing problems. Unfortunately, some of the proposed transformations are not correct.

In section 2 below we comment on some aspects of the single vehicle problem. A rather fundamental theorem, presented in [10], turns out to be erroneous, and a result from complexity theory throws some doubt on the effectiveness of the suggested approach.

In section 3 the transformations from M vehicle to single vehicle problems, proposed in [11], are shown to be incorrect. Correct transformations are presented as well.

2. THE GENERAL ROUTING PROBLEM

Given a graph $G = (N,A)$ with non-negative weights on the arcs, the *general routing problem* (GRP) is to find a minimum weight tour containing a subset $Q \subseteq N$ of *required nodes* and a subset $R \subseteq A$ of *required arcs*. This problem specializes to

- the *travelling salesman problem* (TSP) if $Q = N$, $R = \emptyset$,
- the *Chinese postman problem* (CPP) if $Q = \emptyset$, $R = A$, and
- the *rural postman problem* (RPP) if $Q = \emptyset$.

A polynomial bounded algorithm for the TSP would imply the existence of efficient algorithms for a large number of other *polynomial complete* problems and through them for every problem solvable by polynomial-depth backtrack search [6]. However, the CPP can be solved in $O(|N|^3)$ steps [3; 4; 8,Ch.6]. Partly in view of this fact, it is recommended in [10] that as far as possible required nodes should be converted to required arcs.

If G is undirected and the weight function on its arcs satisfies the triangle inequality, a basic method proposed to do this is to replace required nodes i, j, k where

$$(k,\ell) \in A \text{ if and only if } \ell = i \text{ or } \ell = j$$

by one required arc (i,j) with weight $w((i,j)) = w((i,k)) + w((k,j))$, representing the chain i-k-j [10,Theorem 5]. This transformation rule plays an essential role in the examples, presented in [10]. However, it is not correct,

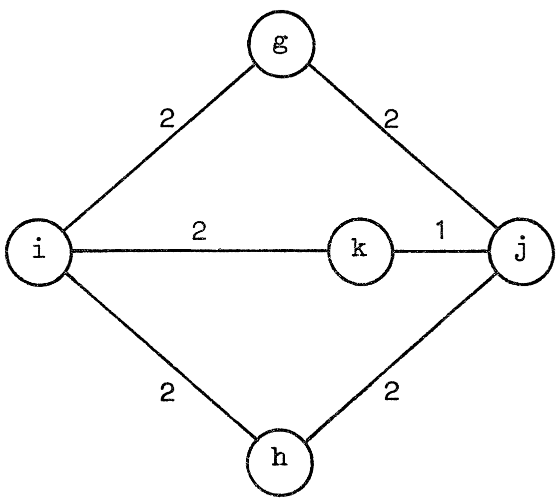

Figure 1 Graph for counterexample.

as shown by the following counterexample. The GRP on the graph $G = (N,A)$, given in Figure 1, with $Q = N$ and $R = \emptyset$, has an optimal tour k-j-g-i-h-j-k with weight 10. One application of the transformation rule leads to $Q = \{g,h\}$, $R = \{(i,j)\}$, and a weight 11 for the "optimal" tour. Repeated application of the same rule increases the minimum weight to 14. Similarly, the solution to example A found in [10] is non-optimal.

Moreover, conversion of required nodes to required arcs will not necessarily lead to an easier problem. The RPP is as difficult as the TSP, as indicated by the following theorem.

Theorem: The RPP *is polynomial complete both in the case that* G *is undirected and in the case that* G *is directed.*

Proof: Consider the following problems.

UNDIRECTED (DIRECTED) HAMILTON CIRCUIT

Does a given undirected (directed) graph $H = (V,E)$ have a *hamilton circuit* (i.e. an undirected (directed) cycle visiting each node exactly once)?

UNDIRECTED (DIRECTED) RURAL POSTMAN

Does a given undirected (directed) graph $G = (N,A)$ with a weight function $w: A \to \mathbb{N} \cup \{0\}$ have a *postman's tour* (i.e. an undirected (directed) cycle traversing each arc in a given subset $R \subseteq A$ at least once) of weight $\leq k$?

We shall show that

UNDIRECTED HAMILTON CIRCUIT $\propto$ UNDIRECTED RURAL POSTMAN,

DIRECTED HAMILTON CIRCUIT $\propto$ DIRECTED RURAL POSTMAN,

where $L \propto M$ means that any instance of L can be reduced to an instance of M in a polynomial number of steps. The theorem then follows from the polynomial completeness of UNDIRECTED (DIRECTED) HAMILTON CIRCUIT, which is established in [6], and the solvability of UNDIRECTED (DIRECTED) RURAL POSTMAN by polynomial-depth backtrack search, which is obvious.

Given an undirected graph $H = (V,E)$ with $|V| = v \geq 3$, we define an UNDIRECTED RURAL POSTMAN problem as follows.

$N = V \times \{0,1\}$,

$A = \{(\langle s,0\rangle,\langle s,1\rangle) | s \in V\} \cup \{(\langle s,0\rangle,\langle t,0\rangle) | (s,t) \in E\}$,

$R = \{(\langle s,0\rangle,\langle s,1\rangle) | s \in V\}$,

$w((\langle s,0\rangle,\langle s,1\rangle)) = 0,$
$w((\langle s,0\rangle,\langle t,0\rangle)) = 1,$
$k = v.$

We claim that H has a hamilton circuit if and only if G has a postman's tour of weight $\leq k$.

If H has a hamilton circuit, then G has a postman's tour, traversing all arcs in R exactly twice and v arcs in A-R exactly once; hence its total weight is equal to $v = k$.

If G has a postman's tour of weight $\leq k$, then such a tour traverses at most k times an arc from A-R. Since no two arcs from R are incident to the same node and $|R| = v = k$, it traverses k arcs from A-R exactly once. It follows that the tour corresponds to a hamilton circuit on H.

The second reduction is similar. Given a directed graph $H = (V,E)$, we define a DIRECTED RURAL POSTMAN problem with

$N = V \times \{0,1\},$
$A = \{(\langle s,0\rangle,\langle s,1\rangle),(\langle s,1\rangle,\langle s,0\rangle) \mid s \in V\} \cup \{(\langle s,0\rangle,\langle t,0\rangle) \mid (s,t) \in E\},$
$R = \{(\langle s,0\rangle,\langle s,1\rangle) \mid s \in V\},$
$w((\langle s,0\rangle,\langle s,1\rangle)) = w((\langle s,1\rangle,\langle s,0\rangle)) = 0,$
$w((\langle s,0\rangle,\langle t,0\rangle)) = 1,$
$k = v.$

H has a hamilton circuit if and only if G has a postman's tour of weight $\leq k$. The proof is left to the reader. (Q.E.D.)

This theorem indicates that the RPP ($R \subseteq A$) is fundamentally more difficult than the CPP ($R = A$). A similar result is the following. Given an undirected graph $G = (N,A)$ with weights on the arcs, the STEINER TREE problem of finding a minimum weight subtree containing a subset $Q \subseteq N$ is polynomial complete [6], whereas the SPANNING TREE problem ($Q = N$) can be solved efficiently [7; 12; 2; 8,Ch.7].

3. THE M-VEHICLE GENERAL ROUTING PROBLEM

Given a graph $G = (O \cup N \cup \{D\}, A)$ with non-negative weights on the arcs, where

- $O = \{O(i) | i = 1,\ldots,M\}$ is the set of tour origins,
- D is the common tour destination,
- A is assumed to contain a set $B = \{(D,O(i)) | i = 1,\ldots,M\}$ of zero-weight arcs,

the M-*vehicle general routing problem* (M-GRP) is to find a set of M cycles

- of minimum total weight,
- containing required subsets $Q \subseteq N$ and $R \subseteq A$,
- such that the i-th cycle traverses the arc $(D,O(i))$ exactly once and contains no other arc from B.

This last point is more precise than the requirements in [11]:

a) each cycle contains one and only one origin;

b) each cycle contains a (destination, origin) arc;

permitting, presumably unintentionally, a set of one or more cycles traversing the same arc $(D,O(i))$ more than once.

If G is directed, then arc $(D,O(i))$ is assumed to be directed from D to $O(i)$ for all i. We also assume that there exists at least one feasible M-GRP solution on G.

In [11], it is claimed that the M-GRP can be transformed into a (single vehicle) GRP, both in the case that G is directed or mixed and in the case that G is undirected.

If G is a directed or mixed graph, the following procedure is proposed in [11].

- Add a dummy destination DD and a zero-weight directed arc (D,DD) to G.
- Add a zero-weight directed arc $(DD,O(i))$ to G and to R, for $i = 1,\ldots,M$.
- Call the new graph T. Solve the GRP on T. The solution will consist of exactly M cycles DD-O(i)-...-D-DD $(i = 1,\ldots,M)$ and hence provides a feasible and optimal M-GRP solution on G.

Consider the 2-GRP on the graph G, given in Figure 2, with $Q = N$ and $R = \emptyset$. The optimal solution consists of the cycles

D-O(1)-N(1)-D & D-O(2)-N(2)-N(3)-D

with total weight 13. An optimal GRP solution on T is

DD-O(1)-N(1)-D-DD-O(2)-N(2)-D-DD-O(2)-N(3)-D-DD

with weight 12. It corresponds to three cycles of the type, described above, and hence does not provide a feasible 2-GRP solution on G.

We suggest the following procedure.

- Change the weight of each arc in B into a large constant λ. Add B to R.
- Call the new graph U. Solve the GRP on U. The solution will consist of exactly M cycles D-O(i)-...-D (i = 1,...,M) and thus provides a feasible and optimal M-GRP solution on G.

The latter statement is easily proved by noting that any GRP solution on U has to contain B while any feasible solution traversing an arc from B more than once can be made too costly through appropriate choice of λ. The optimal GRP solution on U in the example is

D-O(1)-N(1)-D-O(2)-N(2)-N(3)-D

with weight $2\lambda+13$. It corresponds to the optimal 2-GRP solution on G.

If G is an undirected graph, the following procedure is proposed in [11].

- Add B to R.
- Let $I = 2M - |\{(s,t)|(s,t) \in R, D \in \{s,t\}\}|$. If $I > 0$, then for i = 1,...,I:
  - add a dummy destination DD(i) to G;
  - add a zero-weight arc (DD(i),D) to G and to R;
  - for each $N(j) \in N$, add an arc (DD(i),N(j)) to G with a weight equal to the weight of a shortest path between D and N(j).
- Call the new graph T. Solve the GRP on T with the extra condition that between any two successive visits to D the tour traverses at most once an arc from B. The solution will consist of exactly M cycles D-O(i)-...-D (i = 1,...,M) and provides a feasible and optimal M-GRP solution on G.

The extra condition eliminates GRP solutions on T of the form

...-D-O(i)-...-O(j)-D-...

which traverse an arc (D,O(j)) in the wrong direction. Such a condition is rather artificial but seems unavoidable, since in fact we are imposing a direction on arcs in an undirected GRP.

The counterexample previously given is easily adapted for the undirected case. Consider the 2-GRP on the graph G, given in Figure 3, with Q = N and R = ∅. The optimal solution consists of the cycles

D-O(1)-N(1)-D & D-O(2)-N(2)-N(3)-N(4)-N(5)-D

with weight 17. An optimal GRP solution on T is

D-O(1)-N(1)-DD(1)-D-O(2)-N(2)-N(3)-DD(2)-D-O(2)-N(4)-N(5)-DD(2)-D

with weight 16. It uses three vehicles and does not provide a feasible 2-GRP solution on G.

We suggest the following procedure.

- Change the weight of each arc in B into a large constant λ. Add B to R.
- Call the new graph U. Solve the GRP on U with the extra condition that between any two successive visits to D the tour traverses at most once an arc from B. The solution will consist of exactly M cycles D-O(i)-...-D (i = 1,...,M) and provides a feasible and optimal M-GRP solution on G.

This procedure is similar to the one for the directed case; the proof of its correctness is analogous. The optimal GRP solution on U in the example is

D-O(1)-N(1)-D-O(2)-N(2)-N(3)-N(4)-N(5)-D

with weight $2\lambda+17$. It corresponds to the optimal 2-GRP solution on G.

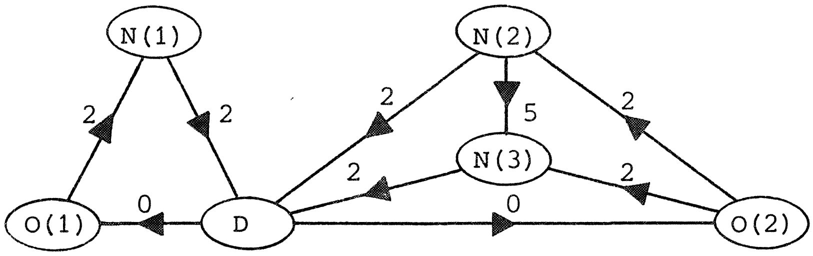

Figure 2 Directed graph for counterexample.

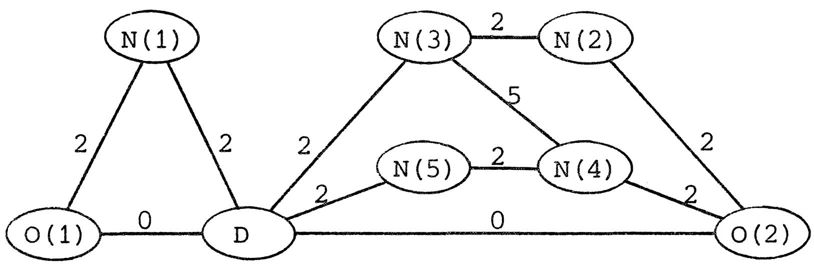

Figure 3 Undirected graph for counterexample.

4. CONCLUSION

Although [10] and [11] contain some serious inaccuracies as reported in sections 2 and 3, we do believe Orloff's approach to be basically sound and useful. He himself rightly remarks that further research on this problem area is required. In this context we may refer to the survey in [5,Ch.9], the recent ingenious approach to the multisalesmen problem in [1] and some of our experiences with vehicle routing through a travelling salesman approach [9].

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