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On general systems of partial differential operators with constant coefficients

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§1. Introduction.

In the investigation of solutions of partial differential equations with constant coefficients, it is very useful to study the algebraic variety defined by the corresponding polynomials. In this direction, L. Gårding characterized the hyperbolic equations (see [**1**]) and L. Hörmander the hypoelliptic equations (see [**2**]). The algebraic-geometric stand-point clarifies the situation and makes it possible to study the equations without any preliminary reduction to canonical forms. In this connection, real points or real parts of complex points of the variety play an essential role. A deep result of this kind is the theorem of C. Lech (see [**6**]) which made it possible for Hörmander to characterize the hypoelliptic systems (see [**4**]).

In the present paper, we investigate points at infinity of the variety and construct null solutions corresponding to each characteristic direction for the most general system of partial differential operators with constant coefficients and prove a fundamental theorem which asserts the equivalence of the various definitions of regularity of solutions in the case of constant coefficients. These facts were, so far, known only when the matrix (1) in §2 was square. The main idea is to consider a characteristic direction as a real point at infinity of the variety in complex projective space and to use the homogeneity of the space.

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§ 2. Partial differential operators with constant coefficients, of general type.

Consider a matrix of polynomial coefficients with m rows and n columns

$$P(X)=\begin{pmatrix} P_{11}(X) & \cdots & P_{1n}(X) \\ \cdots & \cdots & \cdots \\ P_{m1}(X) & \cdots & P_{mn}(X) \end{pmatrix} \tag{1}$$

where $P_{jk}(X)$ is a polynomial of l variables $X=(X_1, \cdots, X_l)$. As a trick for making the following discussions uniform, we put $P_{jk}(X)=0$ $(j>m\,;\,k=1, \cdots, n)$ and allow the matrix to have infinitely many rows with coefficients all zero except for a finite number. Without this convention, we would be obliged to distinguish between the two cases $m \geqq n$ and $m<n$. Replacing the variables $X=(X_1, \cdots, X_l)$ by the differential operators $\frac{1}{i}\frac{\partial}{\partial x}=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_l}\right)$ we get a matrix of partial differential operators with constant coefficients of the most general type. Here i denotes the imaginary unit.

Consider the following system of homogeneous partial differential equations

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)U=0\,, \tag{2}$$

where $U(x)=\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix}$ is an unknown vector function of l variables $x=(x_1, \cdots, x_l)$. The operation of differentiation should be understood in the sense of distribution of L. Schwartz [**10**], and unknown functions are distributions. In the investigation of solutions of the equation (2), an essential role is played by the ideal $\mathfrak{a}$ generated by all the (n, n)-minors of the polynomial matrix $P(X)$. It is easy to see that the equation

$$Q\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u_k=0 \tag{3}$$

holds for any k and for any $Q(X) \in \mathfrak{a}$. In fact, by eliminating all the unknown functions other than u_k, it is seen that (3) holds when Q is one of the (n, n)-minors; and, since these minors generate the ideal, (3) is true for any $Q(X) \in \mathfrak{a}$.

§ 3. Algebraic-geometric preliminaries.

The notations defined in this paragraph will be of constant use in the sequel.

Let $\boldsymbol{C}$ be the complex number field and $\mathfrak{a}$ an ideal of the polynomial ring $\boldsymbol{C}[X_1, \cdots, X_l]$. Let $\mathfrak{a}^*$ be the homogeneous ideal canonically constructed from $\mathfrak{a}$ (see [**2**]). This means that if $\varphi: \boldsymbol{C}[X_0, X_1, \cdots, X_l] \rightarrow \boldsymbol{C}[X_1, \cdots, X_l]$ is the homomorphism carrying each polynomial $F(X_0, X_1, \cdots, X_l) \in \boldsymbol{C}[X_0, X_1, \cdots, X_l]$ into

$F(1, X_1, \cdots, X_l) \in \boldsymbol{C}[X_1, \cdots, X_l]$, and if H is the set of all the homogeneous elements of $\boldsymbol{C}[X_0, X_1, \cdots, X_l]$, then $\mathfrak{a}^* = \varphi^{-1}(\mathfrak{a}) \cap H$. Let $\boldsymbol{C}^l$ be the complex affine space of l dimensions and $\boldsymbol{P}^l$ the complex projective space of l dimensions that contains $\boldsymbol{C}^l$ canonically, i.e. a point in $\boldsymbol{C}^l$ with coordinates $(\zeta_1, \cdots, \zeta_l)$ is considered as a point in $\boldsymbol{P}^l$ with homogeneous coordinates $(1, \zeta_1, \cdots, \zeta_l)$.

Let V be the affine variety defined by $\mathfrak{a}$, V^* the projective variety defined by $\mathfrak{a}^*$. A point $p \in \boldsymbol{P}^l$ is called a point at infinity of V if p is in V^* but not in V. In later applications, we shall only be concerned with real points at infinity, that is, points whose homogeneous coordinates are all real except for a common multiplier. But until the last step, no special treatment is necessary (see the condition (7) in § 4).

Let p_0 be a point at infinity of V[1], V_0^* an irreducible component of V^* which contains p_0 and V_0 the affine part of V_0^*, i.e. $V_0 = V_0^* \cap \boldsymbol{C}^l$.

We shall use the following propositions which are elementary facts from algebraic geometry.

PROPOSITION 1. *V_0 is not empty.*

PROOF. This follows from the fact that the decomposition of $\mathfrak{a}^*$ which is carried over by φ^{-1} from a primary decomposition of $\mathfrak{a}$ is also a primary decomposition.

PROPOSITION 2. *In projective space, any pair of points on an irreducible algebraic variety can be connected by an irreducible algebraic curve on the variety.*

This may be obtained by taking generic hypersurface sections successively and using Bertini's theorem (see [**11**, § 47], and also [8]).

PROPOSITION 3. (Reduction theorem of algebraic curves). *For any irreducible algebraic curve Γ, there exists a normalization $\tilde{\Gamma}$, i.e. $\tilde{\Gamma}$ is a non-singular algebraic curve and there exists a mapping $f: \tilde{\Gamma} \to \Gamma$ which is birational and regular.*

As for proof, see for instance [**2**, III, p. 157, Th. V].

§ 4. Characteristic directions.

The aim of this paragraph is to establish Theorem 1 which plays an essential role in § 5.

Since V_0 is not empty, we can take $p_1 \in V_0$ and a curve Γ which passes through p_0 and p_1. Obviously Γ intersects the hyperplane at infinity at only a finite number of points. Let q_0 be a point in $f^{-1}(p_0)$ ($\subseteqq \tilde{\Gamma}$; see Proposition

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1) If $\mathfrak{a}$ is the ideal stated in § 2, and if the corresponding V has no point at infinity, then the operator $P\left(\frac{1}{i} \frac{\partial}{\partial x}\right)$ is elliptic (see § 6). We assume in this paragraph the existence of p_0.

3 in § 3) and parametrize a neighbourhood of q_0 (relative to $\tilde{\Gamma}$) by a regular algebraic function defined in a neighbourhood of the origin in the complex plane. Such a parametrization is possible because $\tilde{\Gamma}$ is non-singular. Carrying the parametrization over into a neighbourhood of p_0 by f, we get a parametrization of a neighbourhood of p_0 in a branch of Γ. Let $p_0=(0, \zeta_1^0, \cdots, \zeta_l^0)$, where we may assume $\zeta_1^0 \neq 0$ without loss of generality. Then we can take a neighbourhood N in which the homogeneous coordinates of each point can be put in the form $p=\left(\frac{\zeta_0}{\zeta_1}, 1, \frac{\zeta_2}{\zeta_1}, \cdots, \frac{\zeta_l}{\zeta_1}\right)$. And so $\left(\frac{\zeta_0}{\zeta_1}, \frac{\zeta_2}{\zeta_1}, \cdots, \frac{\zeta_l}{\zeta_1}\right)$ can be considered as local affine coordinates in N. By the above parametrization, there are regular algebraic functions $f_0(t), f_2(t), \cdots, f_l(t)$ defined in a neighbourhood of the origin of the complex t-plane such that every point in N, which has the local affine coordinates $\frac{\zeta_0}{\zeta_1}=f_0(t)$, $\frac{\zeta_2}{\zeta_1}=f_2(t), \cdots, \frac{\zeta_l}{\zeta_1}=f_l(t)$, lies on Γ. Since Γ intersects the hyperplane at infinity at only a finite number of points, we can assume that

$$f_0(0)=0 \quad \text{and} \quad f_0(t) \neq 0 \quad \text{for} \quad 0<|t| \leqq \varepsilon, \tag{4}$$

if we take a sufficiently small positive constant ε. The second relation in (4) shows that all the points corresponding to t $(0<|t| \leqq \varepsilon)$ lie on the affine variety V. And so, returning to the canonical affine coordinates (see § 3), the points can be represented by vectors

$$\zeta(t)=\left(\frac{1}{f_0(t)}, \frac{f_2(t)}{f_0(t)}, \cdots, \frac{f_l(t)}{f_0(t)}\right), \qquad 0<|t| \leqq \varepsilon. \tag{5}$$

Now, we change the parameter t into a new one s by inverting the algebraic function: $\frac{1}{s\zeta_1^0}=f_0(t)$. The inverse function might be, in general, multivalued and so we take as the domain Δ of the variable s the exterior of a disc with centre at the origin, slit along the negative imaginary axis. Taking a certain branch, $t=g(s)$ is a regular algebraic function of $s \in \Delta$. Equation (5) assumes the form

$$\zeta(s)=(s\zeta_1^0, sg_2(s), \cdots, sg_l(s)) \qquad s \in \Delta, \tag{6}$$

where $g_2, \cdots, g_l$ are regular algebraic functions of $s \in \Delta$ with $g_k(\infty)=\zeta_k^0$, $(k=2, \cdots, l)$. We can rewrite it in the from $\zeta(s)=s\zeta^0+\eta(s)$, with $\zeta^0=(\zeta_1^0, \cdots, \zeta_l^0)$ and $\lim_{\substack{s\to\infty \\ s\in\Delta}} \frac{\eta(s)}{s}=0$. Each component of $\eta(s)=(\eta_1(s), \cdots, \eta_l(s))$ being a regular algebraic function of $s \in \Delta$, we have the Puiseux expansions around $s=\infty$ of the form

$$\frac{\eta_k(s)}{s}=s^{-\lambda_k}\sum_{\nu=0}^{\infty} c_{k,\nu} s^{\overline{\mu_k}^{\,\nu}}, \qquad \mu_k>0, \qquad (k=1, \cdots, l).$$

Here, all λ_k being positive, we can take a real constant ρ such that Max$(1-\lambda_1,$

$\cdots, 1-\lambda_l; 0) < \rho < 1$, and so $\lim\limits_{\substack{s\to\infty \\ s\in\Delta}} \frac{1}{s^\rho}\eta(s) = 0$. Now we cannot transform, in general, the parameter s so that ζ^0 and $\eta(s)$ become orthogonal in the sense that $\sum_{k=1}^{l} \zeta_k^0 \eta(s) = 0$. But, if the condition

(7) $$(\zeta_1^0)^2 + \cdots + (\zeta_l^0)^2 \neq 0$$

holds, then there exists a complex orthogonal transformation T by which ζ^0 goes over into $\tilde{\zeta}^0 = (\alpha, 0, \cdots, 0)$ with $\alpha^2 = (\zeta_1^0)^2 + \cdots + (\zeta_l^0)^2$. In this coordinate system the parametrization (6) takes the form $\tilde{\zeta}(s) = (s\alpha, s\tilde{g}_2(s), \cdots, s\tilde{g}_l(s)) = s(\alpha, 0, \cdots, 0) + (0, s\tilde{g}_2(s), \cdots, s\tilde{g}_l(s))$ with $\tilde{g}_k(\infty) = 0$ $(k = 2, \cdots, l)$. Returning to the original coordinates by T^{-1}, we get a parametrization

(8) $$\zeta(s) = s\zeta^0 + \eta(s) \quad \text{with} \quad \lim_{s\to\infty} \frac{1}{s^\rho}\eta(s) = 0 \quad \text{and} \quad \langle \zeta^0, \eta(s) \rangle = \sum_{k=1}^{l} \zeta_k^l \eta_k(s) = 0\,.$$

The condition (7) holds whenever ζ^0 is a non-vanishing real vector.

DEFINITION 1. A non-vanishing real vector $\xi = (\xi_1, \cdots, \xi_l)$ is called a *characteristic direction* of an ideal $\mathfrak{a}$, if the point in P^l with homogeneous coordinates $(0, \xi_1, \cdots, \xi_l)$ is a point at infinity of the variety V.

REMARK. When the ideal $\mathfrak{a}$ is that in § 2, a characteristic direction is a common real zero of the principal parts, i. e. the homogeneous parts of highest degree of all the polynomials in $\mathfrak{a}$ which is generated by all the (n, n)-minors of the matrix $P(X)$. And so, this definition coincides with the usual one when the matrix (1) is square (see [**9**]).

Thus we have proved the first part of the following

THEOREM 1. *For a non-vanishing real vector $\xi = (\xi_1, \cdots, \xi_l)$ to be a characteristic direction of an ideal $\mathfrak{a}$, it is necessary that there exist vectors of the form*

$$\zeta(s) = s\xi + \eta(s)$$

depending on a complex parameter $s \in \Delta$, with the conditions

(i) *$\zeta(s) \in V$ for all $s \in \Delta$, and each component of $\zeta(s)$ is a regular algebraic function of $s \in \Delta$;*

(ii) *ξ is orthogonal to $\eta(s)$ for all $s \in \Delta$, i. e. $\langle \xi, \eta(s) \rangle = \sum_{k=1}^{l} \xi_k \eta_k(s) = 0$;*

(iii) *there exists a real constant ρ such that $0 < \rho < 1$ and $\lim\limits_{\substack{s\to\infty \\ s\in\Delta}} \frac{1}{s^\rho}\eta(s) = 0$.*

And, it is sufficient that there exist a sequence of complex numbers $\{s_\nu\}$ and a sequence of vectors $\{\eta^{(\nu)}\}$ $(\nu = 1, 2, \cdots)$ such that

(iv) *$\zeta^{(\nu)} = s_\nu \xi + \eta^{(\nu)} \in V$ for all ν;*

(v) *$\lim\limits_{\nu\to\infty} |s_\nu| = \infty$ and $\lim\limits_{\nu\to\infty} \frac{1}{s_\nu}\eta^{(\nu)} = 0$.*

PROOF. We have only to prove the second part. The homogeneous coordinates of $\zeta^{(\nu)}$ are $(1, s_\nu\xi_1+\eta_1^{(\nu)}, \cdots, s_\nu\xi_l+\eta_l^{(\nu)})=\left(\frac{1}{s_\nu}, \xi_1+\frac{\eta_1^{(\nu)}}{s_\nu}, \cdots, \xi_l+\frac{\eta_l^{(\nu)}}{s_\nu}\right)$, and this obviously tends to $(0, \xi_1, \cdots, \xi_l)$. Since V^* is closed in $\boldsymbol{P}^l$, $(0, \xi_1, \cdots, \xi_l)\in V^*$. This means that ξ is a characteristic direction.

For a vectors $\zeta=(\zeta_1, \cdots, \zeta_l)\in \boldsymbol{C}^l$, the norm $\|\zeta\|$ is defined as $\|\zeta\|=(|\zeta_1|^2+\cdots+|\zeta_l|^2)^{\frac{1}{2}}$; for a polynomial $F(X)\in \boldsymbol{C}[X_1, \cdots, X_l]$, the affine variety defined by $F(\zeta)=0$ is denoted by V_F; and for a subset A of $\boldsymbol{C}^l$ the distance between ζ and A is defined by $d(\zeta, A)=\inf_{\zeta'\in A}\|\zeta-\zeta'\|$. Then, Lech's theorem (see [**6**]) reads as follows.

THEOREM 2. *For any given ideal $\mathfrak{a}$, there exist a polynomial $L(X)\in\mathfrak{a}$ and a positive constant c such that for any real vector ξ we have*

$$d(\xi, V_L)\geqq c\, d(\xi, V)\,. \tag{9}$$

By (9), every real point of V_L belongs also to V. Futher, the following corollary holds.

COROLLARY. *Every real point at infinity of V_L is also a real point at infinity of V.*

PROOF. Let $(0, \xi_1, \cdots, \xi_l)$ be a real point at infinity of V_L and put $\xi=(\xi_1, \cdots, \xi_l)$, then by Theorem 1, there exists $\tilde{\zeta}(s)=s\xi+\tilde{\eta}(s)$ with the conditions (i), (ii), (iii) with $V_L, \tilde{\zeta}, \tilde{\eta}$ replacing V, ζ, η. Let s tend to ∞ along the real axis $s=\sigma$ in Δ. By (9), $d(\sigma\xi, V_L)\geqq c\, d(\sigma\xi, V)$ and so there exists $\eta(\sigma)$ such that $\sigma\xi+\eta(\sigma)\in V$ and $\|\tilde{\eta}(\sigma)\|\geqq c\|\eta(\sigma)\|$, This implies, by the second part of Theorem 1, that ξ is a point at infinity of V.

§ 5. Construction of null solutions.

In this paragraph, following the method of Hörmander [**3**][2], we shall construct null solutions of the equation (2) for any characteristic direction ξ with the aid of Theorem 1 in the preceeding paragraph.

DEFINITION 2. Let ξ be a non-vanishing real vector. By a *null solution* of the differential operator $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ in the direction ξ, we mean a vector function $U(X)=\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x)\end{pmatrix}$ which satisfies the following conditions:

(i) U is a C^∞-solution of (2), i.e. each component of U is a complex-valued indefinitely continuously differentiable function of l variables $x=(x_1, \cdots, x_l)$ defined in the whole real affine space $\boldsymbol{R}^l$ of l dimensions and U satisfies the

2) In [**3**, p. 217], "sin" should be replaced by "cos".

homogeneous equation (2);

(ii) U is non-trivial, i.e. U does not vanish identically on $\boldsymbol{R}^l$;

(iii) $U(x)=0$ on the half space of $\boldsymbol{R}^l$ defined by the inequality $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_l\xi_l > 0$.

Let ξ be a characteristic direction of the ideal $\mathfrak{a}$ in § 2 and let $\zeta(s) = s\xi + \eta(s)$, $s \in \varDelta$, be as is in Theorem 1. Put $P_{jk}(s) = P_{jk}(\zeta(s))$ and $P(s) = (P_{jk}(s))$. Since $\zeta(s) \in V$, the rank of $P(s)$ is always $\leqq n-1$ for $s \in \varDelta$. Since the total number of non-trivial minors of any order of $P(s)$ is finite and each of them is algebraic, the totality of zero points in $\varDelta$ of the non-trivial minors are finite in number. Therefore, there exists a positive number M such that every point $s = \sigma + i\tau$ with $\tau > M$ belongs to $\varDelta$ and is not a zero point. Then, in the region D defined by $\tau > M$, every minor of $P(s)$ always differs from zero or vanishes identically. Let the rank of $P(s)$ be r which is, by the above argument, independent of $s \in D$. We can assume without loss of generality that

$$\begin{vmatrix} P_{11}(s) \cdots P_{1r}(s) \\ \cdots\cdots\cdots\cdots \\ P_{r1}(s) \cdots P_{rr}(s) \end{vmatrix} \neq 0, \qquad \text{for any} \quad s \in D. \tag{10}$$

Here always $r \leqq n-1$. Consider the simultaneous linear equations

$$\sum_{k=1}^{r} P_{jk}(s) C_k'(s) = P_{j\,r+1}(s) \qquad (j = 1, \cdots, r). \tag{11}$$

We can solve (11) by Cramèr's formula, since (10) holds. Put $C_{r+1}'(s) = -1$, $C_{r+2}'(s) = \cdots = C_n'(s) = 0$. Clearly $C_1'(s), \cdots, C_n'(s)$ are all algebraic, dividing them by $K \cdot s^N$ with sufficiently large K and N, we can assume that the resulting functions $C_1(s), \cdots, C_n(s)$ satisfy the following conditions:

$$\begin{cases} C_1(s), \cdots, C_n(s) \text{ are regular and algebraic functions of } s \in D; \\ \sum_{k=1}^{n} P_{jk}(s) C_k(s) = 0, \qquad j = 1, 2, \cdots, \qquad s \in D; \\ |C_k(s)| \leqq 1, \qquad k = 1, \cdots, n; \\ C_{r+1}(s) \neq 0, \qquad s \in D. \end{cases} \tag{12}$$

There is no restriction on j because of the assumption on the rank.

Now, we consider the following contour integral in D:

$$u_k(x) = \int_{i\tau-\infty}^{i\tau+\infty} C_k(s) e^{i\langle x, \zeta(s)\rangle} e^{-\left(\frac{s}{i}\right)^{\rho'}} ds \qquad (k = 1, \cdots, n). \tag{13}$$

Here τ is fixed $(\tau > M)$ and ρ' is a real constant such that $0 < \rho < \rho' < 1$ where ρ is the constant in Theorem 1. Here we define $\left(\frac{s}{i}\right)^{\rho'}$ so that it is real and positive when s is on the positive imaginary axis.

Now, we prove that the vector function $U=\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, where u_k is the function defined by (13), satisfies the three conditions in Definition 2.

Condition (i): Since $\Re\left\{i\langle x, s\xi+\eta(s)\rangle-\left(\frac{s}{i}\right)^{\rho'}\right\}=-\langle x, \tau\xi\rangle-\langle x, \eta'(s)\rangle-|s|^{\rho}\cos\theta \leqq c'|s|^{\rho}-|s|^{\rho'}\cos\frac{\pi\rho'}{2}\leqq -c|s|^{\rho'}$ for large $|s|$, by Theorem 1 (where c, c' are some positive constants, $\eta'(s)=\Im\eta(s)$, $\theta=\rho'\arg\frac{s}{i}$), the integral in (13) is uniformly convergent after an arbitrary number of differentiations with respect to $x=(x_1,\cdots,x_l)$ under the integral sign. And so, U is C^∞ and differentiation can be done under the integral sign. Further, obviously

$$\sum_{k=0}^{n} P_{jk}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u_k=\int_{i\tau-\infty}^{i\tau+\infty}\sum_{k=1}^{n}P_{jk}(\zeta(s))C_k(s)e^{i\langle x,\zeta(s)\rangle}e^{-\left(\frac{s}{i}\right)^{\rho'}}ds=0\,,$$

$j=1,2,\cdots$, since (12) holds.

Condition (ii): Take the $(r+1)$-st component of U and put $x=\lambda\xi$ where λ is a real parameter. Then, by the condition (ii) of Theorem 1,

$$\begin{aligned} u_{r+1}(\lambda\xi) &= \int_{i\tau-\infty}^{i\tau+\infty} C_{r+1}(s)e^{i\lambda s\|\xi\|^2}e^{-\left(\frac{s}{i}\right)^{\rho'}}ds \\ &= e^{-\lambda\tau\|\xi\|^2}\int_{-\infty}^{+\infty} C_{r+1}(\sigma+i\tau)e^{i\lambda\|\xi\|^2\sigma}e^{-\left(\frac{\sigma+i\tau}{i}\right)^{\rho'}}d\sigma \\ &= e^{-\lambda\tau\|\xi\|^2}\mathfrak{F}\left(C_{r+1}(\sigma+i\tau)e^{-\left(\frac{\sigma+i\tau}{i}\right)^{\rho'}}\right)(\lambda\|\xi\|^2)\,. \end{aligned}$$

Here $\mathfrak{F}$ denotes the Fourier transformation on the real line $\boldsymbol{R}$. Then, the uniqueness theorem of Fourier transformation asserts that this last expression does not vanish identically for $\lambda\in\boldsymbol{R}$, since the transformed function does not vanish by (12).

Condition (iii): It is not difficult to see, by Cauchy's theorem, that the value of the integral (13) does not depend on τ. And it is obvious by (12) that

$$|u_k(x)|\leqq e^{-\tau\langle x,\xi\rangle}\int_{-\infty}^{+\infty}e^{-c|\sigma|^{\rho'}}d\sigma\,.$$

Letting $\tau\to+\infty$ we have $u_k(x)=0$ for any k if $\langle x,\xi\rangle>0$.

Thus we have proved the sufficiency in the following

Theorem 3. *For the existence of null-solution of a partial differential operator* $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ *in the direction* ξ, *it is necessary and sufficient that* ξ *be a characteristic direction of the ideal* $\mathfrak{a}$.

Proof. We have only to prove the necessity. And that follows from Holmgren's theorem (see [5]), since the equation (3) holds for any $Q\in\mathfrak{a}$.

§ 6. Regularity of solutions.

For the regularity of solutions of partial differential equations, many definitions are given (see Definition 4 below) and these definitions are identical (with the only exception of hypoellipticity) if the equations have constant coefficients. This fact is already known when the matrix (1) is square, and we prove it here in the general case.

DEFINITION 3. We call a differential operator $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ *elliptic* if the corresponding ideal $\mathfrak{a}$ has no characteristic direction.

DEFINITION 4. We call a partial differential operator $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$

(i) *hypoelliptic* if each component of any solution of the homogeneous equation (2) is indefinitely continuously differentiable;

(ii) *analytic-hypoelliptic* if each component of any solution of the homogeneous equation is an analytic function of the real variables $x=(x_1, \cdots, x_l)$;

(iii) *pseudoanalytic* if the solutions of (2) which vanish in a non-empty open set (however small) are identically zero.

(iv) We say that $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ has the *unique continuation property* if the solutions of (2) which have a zero of infinite order at some point $\in \boldsymbol{R}^l$ are identically zero.

THEOREM 4. *The following conditions on a partial differential operator with constant coefficients* $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ *are equivalent:*

(a) *it is elliptic;*

(b) *it is analytic-hypoelliptic;*

(c) *it has the unique continuation property;*

(d) *it is pseudoanalytic.*

PROOF. (a) implies (b). Since, by equation (3), $L\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u_k=0$ holds for any k where L is Lech's polynomial (see § 4). By the corollary to Theorem 2 $L\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ is a single elliptic differential operator. That implies that each u_k is an analytic function of $x=(x_1, \cdots, x_l)$, since "(a) implies (b)" holds for a single equation (see [**9**]).

(b) implies (c), and (c) implies (d). These two implications are trivial.

(d) implies (a). This is clear from the existence of null solutions.

A simple example of elliptic system of partial differential equations is the so-called Cauchy-Riemann equations for the real and imaginary parts of a holomorphic function of several complex variables.

Some other related results will be reported in [**7**].

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