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On Generalizations of Extending Modules

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Abstract. A module M is said to be SIP-extending if the intersection of every pair of direct summands is essential in a direct summand of M. SIP-extending modules are a proper generalization of both SIP-modules and extending modules. Every direct summand of an SIP-module is an SIP-module just as a direct summand of an extending module is extending. While it is known that a direct sum of SIP-extending modules is not necessarily SIP-extending, the question about direct summands of an SIP-extending module to be SIP-extending remains open. In this study, we show that a direct summand of an SIP-extending module inherits this property under some conditions. Some related results are included about C_{11} and SIP-modules.

1. Introduction

Throughout this paper all rings are associative with unity and R always denotes such a ring. Modules are unital and for an abelian group M, we use M_R (resp. $_RM$) to denote a right (resp. left) R-module. Let M be a R-module and N a submodule of M. We use $N \leq_e M$ and $N \leq_d M$ to denote that N is essential in M and N is a direct summand of M, respectively. Moreover we use $End(M_R)$ and $r(m)$ to denote the ring of endomorphisms of M and the right annihilator in R of an element m in M, i.e., $r(m) = \{r \in R : m.r = 0\}$. Recall that a ring is called *Abelian* if every idempotent is central. For any unexplained terminology please see [1] and [5].

A module M_R has the *Summand Intersection Property*, SIP, if the intersection of every pair of direct summands of M_R is a direct summand of M_R. The study of modules having SIP was motivated by the following result of Kaplansky [7]: every free module over any principal ideal domain has SIP. The Summand Intersection Property has been studied by many authors (see e.g. [2], [3], [6], [8] and [17].)

Recall that a module M is called an *extending module* (or a *CS-module*) if every submodule is essential in a direct summand of M. In [5] and [11], extending modules were studied in detail.

The concept of C_{11}-modules was introduced in [15] as a generalization of extending modules. A module M is called C_{11}-module (or satisfies C_{11})[15] if every

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submodule of M has a complement which is a direct summand. It is known that a direct summand of a C_{11}-module is not a C_{11}-module, in general (see [16, Exercise 4]). A module is called a C_{11}^{+}-module if its every direct summand is a C_{11}-module [15]. In this paper we further the study of SIP-extending modules and we show that if M is a C_{11}-module which is also SIP-extending then every direct summand of M is a C_{11}-module, i.e., M is C_{11}^{+}-module (see Proposition 7). In the main result we show that if M is an SIP-extending module such that $End(M_R)$ is Abelian then every direct summand of M is SIP-extending.

2. SIP-extending modules

In [9], a module M is called an *SIP-extending module* provided that the intersection of every pair of direct summands of M is essential in a direct summand of M. We say a ring R is a right *SIP-extending ring* if the module R_R is an SIP-extending module, i.e., for every pair of idempotents e, c in R there exists $g^2 = g \in R$ such that $eR \cap cR$ is essential in gR. Examples of SIP-extending modules include every extending (hence every injective) module, every uniform module, every semisimple module and every module having the SIP (e.g. any Baer module [13]). The concept of an SIP-extending module is a proper generalization of both SIP-modules and extending modules, as shown by the following examples.

Example 1. Let F be any field and V be a F-vector space with $dimV_F \geq 2$. Let

$$R = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} : a \in F, v \in V \right\},$$

be the trivial extension of F by V. Then R is a right SIP-extending ring however since $dimV_F \geq 2$, R is not right extending ring.

Example 2([4, Exercise 1.5])**.** Let F be a field and

$$T = \left\{ \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, x, y \in F \right\}.$$

Let

$$e = e^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$c = c^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $eT \cap cT$ is nilpotent. Hence $eT \cap cT$ is not a direct summand of T. It follows that T does not have SIP. However it is a right SIP-extending ring.

It is well known that every direct summand of SIP-modules is an SIP-module and every direct summand of extending modules is an extending module. This result led us to the following question.

Question: Let M be an SIP-extending module and N be a direct summand of M. Is N an SIP-extending module?

In [9] we have provided a positive answer to the direct summand question under the condition that the summand is the unique closure of each of its essential submodules.

Proposition 3([9, Lemma 6])**.** *Let M be an SIP-extending module, and let N be a direct summand of M. Suppose that N is the unique closure in M of any of its essential submodules. Then N is also an SIP-extending module.*

Definition 4([14])**.** Let M be a module. If every submodule has a unique closure in M then M is called UC-module.

Proposition 5. *Let M be a UC-module. Then M has SIP if and only if M is SIP-extending.*

Proof. It is clear that if M has SIP then M is SIP-extending. Conversely, let S_1 and S_2 be direct summands of M. Then by hypothesis $S_1 \cap S_2 \leq_e P$ for some $P \leq_d M$. By the main theorem in [14], intersection of two closed submodules is closed hence $S_1 \cap S_2 = P$. Thus M has SIP. □

The following lemma is proved in [9, Theorem 8].

Lemma 6. *Let M be a C_{11}-module and E be a submodule of M. If for every direct summand D of M, $E \cap D$ is essential in a direct summand of E then E is a C_{11}-module.*

Lemma 7. *Let M be a C_{11}-module. If M is SIP-extending then every direct summand of M is C_{11} (i.e., M has C_{11}^{+}).*

Proof. By Lemma 6 and the definition of SIP-extending. □

Recall that R is said to Abelian if every idempotent of R is central. Note that every finite dimension module has an Abelian endomorphism ring by [11]. We have the following result for SIP-extending Abelian rings.

Proposition 8. *Let R be an Abelian ring then*

i) *R is SIP-extending (SIP) if and only if $R[x]$ is SIP-extending (SIP).*

ii)*R is SIP-extending (SIP) if and only if $R[[x]]$ is SIP-extending (SIP).*

Proof. Since R is an Abelian ring, the result follows by [10, Lemma 8]. □

Next, we provide an answer to the direct summand question for an SIP-extending module under another special condition. The result shows that a fully invariant direct summand of an SIP-extending module inherits the property. It also completes the sufficiency part of [9, Theorem 11] in which only the necessity was established.

Theorem 9. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of fully invariant submodules M_i of M where I is an index set. Then M is an SIP-extending module if and only if M_i, $\forall i \in I$ is an SIP-extending module.*

Proof. Let M be an SIP-extending module and M_i be a fully invariant direct summand of M. If L and K are direct summand of M_i then there exist $P \leq_d M$ ($M = P \oplus Q$, for some $Q \leq M$) such that $L \cap K \leq_e P$. Since M_i is a fully invariant direct summand of M and $M = P \oplus Q$ then $M_i = (M_i \cap P) \oplus (M_i \cap Q)$. Therefore $L \cap K \leq_e M_i \cap P \leq_d M_i$. So M_i is an SIP-extending module. Converse follows from [9, Theorem 11]. We include a brief outline for the convenience of the reader. Let S be any direct summand of M. So $S = \oplus(S \cap M_i)$. Now let S, T be direct summands of M. Hence, $S \cap T = \oplus[(S \cap M_i) \cap (T \cap M_i)]$. Therefore, there exists a direct summand K_i of M_i which contains $(S \cap M_i) \cap (T \cap M_i)$ as an essential submodule. □

Corollary 10. *M_R is an SIP-extending module such that $End(M_R)$ is Abelian. Then every direct summand of M is SIP-extending.*

Proof. Let M be an SIP-extending module and M_1 be a direct summand of M. Since $End(M_R)$ is Abelian M_1 is a fully invariant submodule of M. By Theorem 9, M_1 is an SIP-extending module. □

Definition 11. Let M be a R-module. M is said to be multiplication module if for each $X \leq M$ there exists $A_R \leq R_R$ such that $X = MA$

Corollary 12. *Let M be an SIP-extending module, then any direct summand of M is SIP-extending if M satisfies any of the following conditions.*
(i) *$M_R = R_R$ and R is Abelian.*
(ii) *M is a multiplication module and R is commutative.*

Proof. (i) Immediate by Corollary 10.
(ii) Assume that M is multiplication module and R is commutative. Note that every submodule of a multiplication module is fully invariant. Now Theorem 9 yields the result. □

Recall that a module M satisfies the C_3 condition whenever K, L are direct summand of M with $K \cap L = 0$ then $K \oplus L \leq_d M$ (see [11]). Note that the $\mathbb{Z}$-module $(\mathbb{Z} \oplus \mathbb{Z})$ is an SIP-extending which does not satisfy the C_3 condition (see, for example [3]). Now we provide an example which shows that a module satisfying the C_3 property does not have to be SIP-extending either.

Example 13. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of upper triangular matrices over F, $N = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $L = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ left ideals of R and $M =$

R/L. Let $U = N \oplus M$. Then ${}_RU$ satisfies the C_3 condition and is a UC-module but does not have the SIP as a left R-module. By Proposition 5 , U is not SIP-extending as a left R-module.

We conclude this paper with some results for C_{11}-modules. Recall that a module M is said to satisfy the full (finite) exchange property if for any module G and any two direct sum decompositions $G = M' \oplus N = \oplus_{i \in I} A_i$ where $M' \cong M$ and I is any (finite) index set, there are submodules B_i of A_i, $i \in I$, such that $G = M' \oplus (\oplus_{i \in I} B_i)$.

It was shown in [12] that every quasi-continuous module (i.e., an extending module with C_3 condition) satisfies the full exchange property whenever it satisfies the finite exchange property. We can weaken the extending condition (C_1) to C_{11} under an additional chain condition.

Theorem 14. *Let M_R be a C_{11}-module which satisfies C_3 condition. If M has ACC on $r(m)$, $m \in M$ then the finite exchange property of M implies the full exchange property.*

Proof. By [15, Lemma 4.6 (a)] and [11, Theorem 2.17] M has a decomposition into indecomposable submodules. This yields the full exchange property. □

Corollary 15. *Let R be a right Noetherian ring and M_R be a C_{11}-module which has C_3 then the finite exchange property implies the full exchange property for M.*

Proof. It follows from Theorem 14. □

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