KYUNGPOOK Math. J. 52(2012), 307-317
http://dx.doi.org/10.5666/KMJ.2012.52.3.307

On Generalizations of the Hadamard Inequality for (α, m)-Convex Functions

Erhan Set*
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
e-mail : erhanset@yahoo.com

Maryam Sardari
Institute for Advanced Studies in Basic Sciences, P. O. Box 45195-1159, Zanjan, Iran
e-mail : m_sardari@iasbs.ac.ir

Muhamet Emin Ozdemir
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
e-mail : emos@atauni.edu.tr

Jamal Rooin
Institute for Advanced Studies in Basic Sciences, P. O. Box 45195-1159, Zanjan, Iran
e-mail : rooin@iasbs.ac.ir

Abstract. In this paper we establish several Hadamard-type integral inequalities for (α, m) −convex functions.

1. Introduction

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature.

Theorem 1. *If f is convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

* Corresponding Author.
Received March 10, 2010; accepted September 23, 2011.
2010 Mathematics Subject Classification: 26A51, 26D15.
Key words and phrases: Hadamard inequality, convex functions, (α, m)-convex function.

Proof. See [1]. □

If the function f is concave, (1.1) can be written as following:

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

For recent results related to the Hadamard inequality are given in [9], [10] and [17].

In the literature, the concepts of $m-$convexity and $(\alpha, m)-$convexity are well known. The concept of $m-$convexity was first introduced by G. Toader in [18] (see also [5], [6]) and it is defined as follows:

The function $f:[0,b]\to\mathbb{R}$ is said to be $m-$convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have:

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y). \tag{1.2}$$

The class of $(\alpha, m)-$convex functions was also first introduced in [8] and it is defined as follows:

The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $(\alpha, m)-$convex , where $(\alpha,m)\in[0,1]^2$, if we have

$$f(tx+m(1-t)y) \leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y) \tag{1.3}$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

It can be easily seen that for $(\alpha,m)\in\{(0,0),\ (1,1)\ (1,m)\}$ one obtains the following classes of functions: increasing, convex and $m-$convex functions respectively. The interested reader can find more about partial ordering of convexity in [15, P. 8,280]. For many papers connected with $m-$convex and $(\alpha, m)-$convex functions see ([2], [3], [6], [11], [12], [13], [14], [19]) and the references therein. There are similar inequalities for $s-$convex and $h-$convex functions in [7] and [16], respectively.

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequality for $m-$convex functions.

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.4}$$

Some generalizations of this result can be found in [2], [3].

In [4] S. S. Dragomir established two new Hadamard-type inequalities for $m-$convex functions. They are given in the following Theorems.

Theorem 3. *Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1 [a, b] \cap L_1 \left[\frac{a}{m}, \frac{b}{m}\right]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx. \tag{1.5}$$

Theorem 4. *Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $f \in L_1 [am, b]$ where $0 \leq a < b$, then the following inequality holds:*

$$\frac{1}{m+1} \left[\frac{1}{mb - a} \int_a^{mb} f(x)dx + \frac{1}{b - ma} \int_{ma}^b f(x)dx \right] \leq \frac{f(a) + f(b)}{2}. \tag{1.6}$$

The goal of this paper is to obtain new inequalities like those given in Theorems 1, 2, 3, 4, but now for the class of (α, m) $-$convex functions.

2. Inequalities for (α, m)-convex functions

The following theorem is a generalization of the Hadamard inequality.

Theorem 5. *Let $f : [0, \infty) \to \mathbb{R}$ be an (α, m) $-$convex function with $0 \leq a < b$ and $(\alpha, m) \in [0, 1] \times (0, 1]$. If $f \in L_1 \left[m^2 a, (2-m) b\right] \cap L_1 \left[ma, \frac{(2-m)b}{m}\right]$, then the following inequalities hold:*

$$\begin{aligned} & f\left(\tfrac{2-m}{2} b + \tfrac{m}{2} (ma)\right) \\ & \leq \tfrac{1}{2^\alpha} \tfrac{1}{b(2-m) - m^2 a} \left\{ \int_{m^2 a}^{(2-m)b} \left[f(x) + (2^\alpha - 1) m \right.\right. \\ & \left.\left. \times f\left(\tfrac{(2-m)b}{m} \left(1 - \tfrac{x - m^2 a}{2b - mb - m^2 a}\right) + m \tfrac{x - m^2 a}{2b - mb - m^2 a} a \right) \right] dx \right\} \\ & \leq \tfrac{1}{2^\alpha (\alpha+1)} \left[f((2-m) b) \right. \\ & \left. + (\alpha + (2^\alpha - 1)) mf(am) + (2^\alpha - 1) \alpha m^2 f\left(\tfrac{(2-m)b}{m^2}\right) \right]. \end{aligned} \tag{2.1}$$

Proof. Let $U_1 = t(2-m) b + (1-t) m^2 a$ and $U_2 = (1-t)(2-m) b + tm^2 a$, where $t \in [0, 1]$ is arbitrary. Then we get

$$f\left(\frac{U_1 + U_2}{2}\right) = f\left(\frac{2-m}{2} b + \frac{m}{2} (ma)\right).$$

By the (α, m) −convexity of f we can write the following inequality:

$$\begin{aligned} f\left(\tfrac{2-m}{2}b+\tfrac{m}{2}(ma)\right) &= f\left(\tfrac{U_1+U_2}{2}\right) \\ &\le \tfrac{1}{2^\alpha} f(U_1)+\left(1-\tfrac{1}{2^\alpha}\right) mf\left(\tfrac{U_2}{m}\right) \\ &= \tfrac{1}{2^\alpha}\left[f(U_1)+(2^\alpha-1)\, mf\left(\tfrac{U_2}{m}\right)\right], \end{aligned}$$

or

$$\begin{aligned} f\left(\tfrac{2-m}{2}b+\tfrac{m}{2}(ma)\right) \le \tfrac{1}{2^\alpha}\Big[& f\left(t(2-m)b+(1-t)m^2a\right) \\ & +(2^\alpha-1)\, mf\left(\tfrac{(1-t)(2-m)b}{m}+tma\right)\Big]. \end{aligned}$$

Integrating over $t\in[0,1]$, we get

$$\begin{aligned} & f\left(\tfrac{2-m}{2}b+\tfrac{m}{2}(ma)\right) \\ & \le \tfrac{1}{2^\alpha}\int_0^1 \Big[f\left(t(2-m)b+(1-t)m^2a\right) \\ & +(2^\alpha-1)\, mf\left(\tfrac{(1-t)(2-m)b}{m}+tma\right)\Big]\, dt \qquad (2.2) \\ & = \tfrac{1}{2^\alpha}\tfrac{1}{b(2-m)-m^2a}\Big\{\int_{m^2a}^{(2-m)b}\Big[f(x)+(2^\alpha-1)\, m \\ & \times f\left(\tfrac{(2-m)b}{m}\left(1-\tfrac{x-m^2a}{2b-mb-m^2a}\right)+m\tfrac{x-m^2a}{2b-mb-m^2a}a\right)\Big]\, dx\Big\}, \end{aligned}$$

where we used the change of the variable $x=t(2-m)b+(1-t)m^2a$ or $t=\frac{x-m^2a}{2b-mb-m^2a}$ and so

$$\int_0^1 f\left(t(2-m)b+(1-t)m^2a\right) dt = \frac{1}{(2-m)b-m^2a}\int_{m^2a}^{(2-m)b} f(x)\, dx$$

and

$$\begin{aligned} & \int_0^1 f\left(\tfrac{(1-t)(2-m)b}{m}+tma\right) dt \\ & = \tfrac{1}{(2-m)b-m^2a} \\ & \times \int_{m^2a}^{(2-m)b} f\left(\tfrac{(2-m)b}{m}\left(1-\tfrac{x-m^2a}{2b-mb-m^2a}\right)+m\tfrac{x-m^2a}{2b-mb-m^2a}a\right) dx. \end{aligned}$$

This completes the proof of the first inequality in (2.1).

Next, by the (α, m) −convexity of f, we also have

$$\begin{aligned} f\left(t(2-m)b+(1-t)m^2a\right) &= f(t(2-m)b+m(1-t)ma) \\ &\le t^\alpha f((2-m)b)+m(1-t^\alpha) f(ma) \end{aligned}$$

and

$$f\left(\frac{(1-t)(2-m)b}{m}+tma\right) = f\left(t\,(ma)+m\,(1-t)\left(\frac{(2-m)b}{m^2}\right)\right)$$
$$\leq t^\alpha f\,(ma)+m\,(1-t^\alpha)\,f\left(\frac{(2-m)b}{m^2}\right).$$

So

$$\frac{1}{2^\alpha}\left[f\left(t\,(2-m)\,b+(1-t)\,m^2a\right)+(2^\alpha-1)\,mf\left(\frac{(1-t)(2-m)b}{m}+tma\right)\right]$$

(2.3) $$\leq \frac{1}{2^\alpha}\left\{t^\alpha f\,((2-m)\,b)+m\,(1-t^\alpha)\,f\,(ma)\right.$$
$$\left.+(2^\alpha-1)\,m\left[t^\alpha f\,(ma)+m\,(1-t^\alpha)\,f\left(\frac{(2-m)b}{m^2}\right)\right]\right\}.$$

Integrating (2.3) over t on $[0,1]$, we get

$$\frac{1}{2^\alpha}\frac{1}{(2-m)b-m^2a}\left\{\int_{m^2a}^{(2-m)b}\left[f\,(x)+(2^\alpha-1)\,m\right.\right.$$
$$\left.\left.\times f\left(\frac{(2-m)b}{m}\left(1-\frac{x-m^2a}{2b-mb-m^2a}\right)+m\frac{x-m^2a}{2b-mb-m^2a}a\right)\right]dx\right\}$$
$$\leq \frac{1}{2^\alpha(\alpha+1)}\left\{f\,((2-m)\,b)\right.$$
$$\left.+\left(\alpha+(2^\alpha-1)\right)mf\,(am)+(2^\alpha-1)\,\alpha m^2 f\left(\frac{(2-m)b}{m^2}\right)\right\}.$$

This completes the proof of the second inequality in (2.1). □

Remark 1. Choosing $(\alpha, m) = (1,1)$ in (2.1), from the first and the second inequalities of (2.1), respectively, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{1}{b-a}\left[\int_a^b\left[f(x)+f(a+b-x)\right]dx\right]$$
$$= \frac{1}{2}\frac{1}{b-a}\left[\int_a^b f(x)dx+\int_a^b f(x)dx\right]$$
$$= \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{4}\left[f(b)+2f(a)+f(b)\right]$$
$$= \frac{f(a)+f(b)}{2}.$$

Note that, we used

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

and so

$$\int_a^b [f(x)+f(a+b-x)]\,dx = 2\int_a^b f(x)dx.$$

Clearly, we can drop the assumption $f \in L_1\left[m^2a,(2-m)\,b\right] \cap L_1\left[ma,\frac{(2-m)b}{m}\right] = L_1\,[a,b]$, and in this case (2.1) exactly becomes the Hermite-Hadamard inequalities for $(\alpha,m)=(1,1)$.

Theorem 6. *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If $0\le a<b<\infty$ and $f\in L_1\,[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f\,(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1},\frac{f\,(b)+\alpha m f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \tag{2.4}$$

Proof. Since f is $(\alpha,m)-$convex, we have

$$f\,(tx+m\,(1-t)\,y) \le t^\alpha f\,(x)+m\,(1-t^\alpha)\,f\,(y)$$

for all $x,y\ge 0$, which gives:

$$f\,(ta+(1-t)\,b) \le t^\alpha f\,(a)+m\,(1-t^\alpha)\,f\left(\frac{b}{m}\right)$$

and

$$f\,(tb+(1-t)\,a) \le t^\alpha f\,(b)+m\,(1-t^\alpha)\,f\left(\frac{a}{m}\right)$$

for all $t\in[0,1]$. Integrating on $[0,1]$, we obtain

$$\int_0^1 f\,(ta+(1-t)\,b)\,dt \le \frac{f\,(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}$$

and

$$\int_0^1 f\,(tb+(1-t)\,a)\,dt \le \frac{f\,(b)+\alpha m f\left(\frac{a}{m}\right)}{\alpha+1}.$$

However,

$$\int_0^1 f\,(ta+(1-t)\,b)\,dt = \int_0^1 f\,(tb+(1-t)\,a)\,dt = \frac{1}{b-a}\int_a^b f(x)dx$$

and the inequality (2.4) is obtained. □

Remark 2. The inequality (2.4) yields inequality (1.4) for $\alpha=1$.

Theorem 7. *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in$*

$(0,1]^2$. *If* $0 \leq a < b < \infty$ *and* $f \in L_1[a,b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, *then the following inequalities hold:*

$$
\begin{aligned}
f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{2^\alpha(b-a)} \int_a^b \left[f(x) + m\left(2^\alpha - 1\right) f\left(\tfrac{x}{m}\right)\right] dx \\
&\leq \tfrac{1}{2^{\alpha+1}(\alpha+1)} \left[(f(a) + f(b))\right. \\
&+ m\left(\alpha + 2^\alpha - 1\right)\left(f\left(\tfrac{a}{m}\right) + f\left(\tfrac{b}{m}\right)\right) \\
&+ \left.\alpha m^2\left(2^\alpha - 1\right)\left(f\left(\tfrac{a}{m^2}\right) + f\left(\tfrac{b}{m^2}\right)\right)\right].
\end{aligned} \tag{2.5}
$$

Proof. By the (α, m) −convexity of f, we have

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &= f\left(\frac{x}{2} + m\frac{y}{2m}\right) \\
&\leq \frac{1}{2^\alpha} f(x) + m\left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{y}{m}\right) \\
&= \frac{1}{2^\alpha}\left[f(x) - mf\left(\frac{y}{m}\right)\right] + mf\left(\frac{y}{m}\right)
\end{aligned}
$$

for all $x, y \in [0, \infty)$.

Now, if we choose $x = ta + (1-t)b$ and $y = (1-t)a + tb$, we deduce

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha}\left[f(ta + (1-t)b) - mf\left(\frac{(1-t)a + tb}{m}\right)\right] + mf\left(\frac{(1-t)a + tb}{m}\right) \\
&= \frac{1}{2^\alpha}\left[f(ta + (1-t)b) + m\left(2^\alpha - 1\right) f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)\right]
\end{aligned}
$$

for all $t \in [0, 1]$.

Integrating over $t \in [0, 1]$, we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha}\left[\int_0^1 f(ta + (1-t)b)\, dt\right. \\
&\left.+ m\left(2^\alpha - 1\right)\int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt\right].
\end{aligned} \tag{2.6}
$$

Taking into account that

$$
\int_0^1 f(ta + (1-t)b)\, dt = \frac{1}{b-a}\int_a^b f(x)dx
$$

and

$$
\int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt = \frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx,
$$

we deduce from (2.6) the first inequality in (2.5).

Next, by the (α, m) −convexity of f, we also have

$$\begin{aligned} &\frac{1}{2^{\alpha}}\left[f\left(ta+(1-t)b\right)+m\left(2^{\alpha}-1\right)f\left(t\frac{b}{m}+(1-t)\frac{a}{m}\right)\right] \\ \leq\ &\frac{1}{2^{\alpha}}\left[t^{\alpha}f\left(a\right)+m\left(1-t^{\alpha}\right)f\left(\frac{b}{m}\right)\right. \\ &\left.+m\left(2^{\alpha}-1\right)\left(t^{\alpha}f\left(\frac{b}{m}\right)+m\left(1-t^{\alpha}\right)f\left(\frac{a}{m^{2}}\right)\right)\right]. \end{aligned} \tag{2.7}$$

Integrating over t on $[0,1]$, we get

$$\begin{aligned} &\frac{1}{2^{\alpha}(b-a)}\int_{a}^{b}\left(f(x)+m\left(2^{\alpha}-1\right)f(\frac{x}{m})\right)dx \\ \leq\ &\frac{1}{2^{\alpha}}\left[f\left(a\right)\int_{0}^{1}t^{\alpha}dt+mf\left(\frac{b}{m}\right)\int_{0}^{1}\left(1-t^{\alpha}\right)dt+\right. \\ &+m\left(2^{\alpha}-1\right)f\left(\frac{b}{m}\right)\int_{0}^{1}t^{\alpha}dt \\ &\left.+m^{2}\left(2^{\alpha}-1\right)f\left(\frac{a}{m^{2}}\right)\int_{0}^{1}\left(1-t^{\alpha}\right)dt\right] \\ =\ &\frac{1}{2^{\alpha}(\alpha+1)}\left[f\left(a\right)+m\left(\alpha+2^{\alpha}-1\right)f\left(\frac{b}{m}\right)+\alpha m^{2}\left(2^{\alpha}-1\right)f\left(\frac{a}{m^{2}}\right)\right]. \end{aligned} \tag{2.8}$$

Similarly, changing the roles of a and b, we get

$$\begin{aligned} &\frac{1}{2^{\alpha}(b-a)}\int_{a}^{b}\left(f(x)+m\left(2^{\alpha}-1\right)f(\frac{x}{m})\right)dx \\ \leq\ &\frac{1}{2^{\alpha}(\alpha+1)}\left[f\left(b\right)+m\left(\alpha+2^{\alpha}-1\right)f\left(\frac{a}{m}\right)+\alpha m^{2}\left(2^{\alpha}-1\right)f\left(\frac{b}{m^{2}}\right)\right]. \end{aligned} \tag{2.9}$$

Now adding (2.8) and (2.9) with each other, we obtain the second inequality in (2.5). □

Remark 3. Choosing $\alpha = 1$ in the first part of (2.5), we get (1.5).

Remark 4. The inequality (2.5) yields the Hadamard inequality (1.1) for $\alpha = 1$ and $m = 1$.

Theorem 8. *Let $f : [0,\infty) \to \mathbb{R}$ be an (α, m)−convex function with $(\alpha, m) \in (0,1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq\frac{1}{2}\left[\frac{f\left(a\right)+f\left(b\right)+\alpha m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{\alpha+1}\right]. \tag{2.10}$$

Proof. By the $(\alpha, m)-$convexity of f, we can write

$$f(ta+(1-t)b) \leq t^{\alpha} f(a) + m(1-t^{\alpha}) f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a) \leq t^{\alpha} f(b) + m(1-t^{\alpha}) f\left(\frac{a}{m}\right)$$

for all $t \in [0,1]$.

Adding the above inequalities, we get

$$\begin{aligned} & f(ta+(1-t)b) + f(tb+(1-t)a) \\ \leq & t^{\alpha} f(a) + m(1-t^{\alpha}) f\left(\frac{b}{m}\right) + t^{\alpha} f(b) + m(1-t^{\alpha}) f\left(\frac{a}{m}\right). \end{aligned}$$

Integrating over $t \in [0,1]$, we obtain

$$\begin{aligned} (2.11) \qquad & \int_0^1 f(ta+(1-t)b)\,dt + \int_0^1 f(tb+(1-t)a)\,dt \\ \leq & \int_0^1 t^{\alpha}(f(a)+f(b))\,dt + \int_0^1 m(1-t^{\alpha})\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)dt \\ = & \frac{f(a)+f(b)}{\alpha+1} + \frac{m\alpha}{\alpha+1}\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right) \\ = & \frac{f(a)+f(b)+m\alpha\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{\alpha+1}. \end{aligned}$$

As it is easy to see that

$$\int_0^1 f(ta+(1-t)b)\,dt = \int_0^1 f(tb+(1-t)a)\,dt = \frac{1}{b-a}\int_a^b f(x)dx,$$

from (2.11) we deduce the desired result, namely, the inequality (2.10). □

Remark 5. The inequality (2.10) yields the right side of the Hadamard inequality (1.1) for $\alpha = 1$ and $m = 1$.

Theorem 9. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha, m)-$convex function with $(\alpha, m) \in (0,1]^2$. If $f \in L_1[am, b]$ where $0 \leq a < b$, then the following inequality holds:*

$$\begin{aligned} (2.12) \qquad & \frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \\ \leq & \frac{1}{(\alpha+1)}\left[(f(a)+f(b))(1+m\alpha)\right]. \end{aligned}$$

Proof. By (α, m) −convexity of f, for all $t \in [0,1]$, we can write:

$$\begin{aligned} f\left(ta + m\left(1-t\right)b\right) &\leq t^{\alpha} f\left(a\right) + m\left(1-t^{\alpha}\right) f\left(b\right), \\ f\left(tb + m\left(1-t\right)a\right) &\leq t^{\alpha} f\left(b\right) + m\left(1-t^{\alpha}\right) f\left(a\right), \\ f\left(\left(1-t\right)a + mtb\right) &\leq \left(1-t\right)^{\alpha} f\left(a\right) + m\left(1-\left(1-t\right)^{\alpha}\right) f\left(b\right), \\ f\left(\left(1-t\right)b + mta\right) &\leq \left(1-t\right)^{\alpha} f\left(b\right) + m\left(1-\left(1-t\right)^{\alpha}\right) f\left(a\right). \end{aligned}$$

Adding the above inequalities with each other, we get:

$$\begin{aligned} & f\left(ta + m\left(1-t\right)b\right) + f\left(tb + m\left(1-t\right)a\right) \\ & + f\left(\left(1-t\right)a + mtb\right) + f\left(\left(1-t\right)b + mta\right) \\ & \leq \left[t^{\alpha} + m\left(1-t^{\alpha}\right) + \left(1-t\right)^{\alpha} + m\left(1-\left(1-t\right)^{\alpha}\right)\right]\left(f(a) + f(b)\right). \end{aligned}$$

Now integrating over $t \in [0,1]$ and taking into account that:

$$\int_0^1 f\left(ta + m\left(1-t\right)b\right)dt = \int_0^1 f\left(\left(1-t\right)a + mtb\right)dt = \frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f\left(tb + m\left(1-t\right)a\right)dt = \int_0^1 f\left(\left(1-t\right)b + mta\right)dt = \frac{1}{b-ma}\int_{ma}^{b} f(x)dx,$$

we obtain the inequality (2.12). □

Remark 7. Choosing $\alpha = 1$ in (2.12), we obtain (1.6).

References

[1] A. G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colombiana Math., **28**(1994), 7-12.

[2] M. K. Bakula, M. E. Özdemir, J. Pečarić, *Hadamard tpye inequalities for $m-$convex and (α, m)-convex functions*, J. Inequal. Pure and Appl. Math., **9(4)**(2008), Art. 96.

[3] M. K. Bakula, J. Pečarić, M. Ribičić, *Companion inequalities to Jensen's inequality for $m-$convex and (α, m)-convex functions*, J. Inequal. Pure and Appl. Math., **7(5)**(2006), Art. 194.

[4] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for $m-$convex functions*, Tamkang J. Math., **3(1)**(2002).

[5] S. S. Dragomir, C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000. [online: http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html].

[6] S. S. Dragomir, G. Toader, *Some inequalities for $m-$convex functions*, Studia Univ. Babes-Bolyai, Mathematica, **38(1)**(1993), 21-28.

[7] U. S. Kırmacı, M. K. Bakula, M. E. Özdemir, J. Pečarić, *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. and Comp., **193**(2007), 26-35.

[8] V. G. Miheşan, A generalization of the convexity, Seminar on Functial Equations, Approx. and Convex, Cluj-Napoca (Romania) 1993.

[9] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.

[10] C. P. Niculescu, L. E. Persson, Convex Functions and Their Applications, A Contemporary Approach, Canadian Math. Series Books in Mathematics, Springer, New York, 2006.

[11] M. E. Özdemir, M. Avci, E. Set, *On some inequalities of Hermite–Hadamard type via $m-$convexity*, Appl. Math. Lett., **23(9)**(2010) 1065-1070.

[12] M. E. Özdemir, H. Kavurmaci, E. Set, *Ostrowski's type inequalities for $(\alpha, m)-$convex functions*, Kyungpook Math. J., **50**(2010), 371-378.

[13] M. E. Özdemir, M. Avcı and H. Kavurmacı, *Hermite–Hadamard-type inequalities via $(\alpha, m)-$convexity*, Computers and Mathematics with Applications, **61**(2011), 2614-2620.

[14] M. E. Özdemir, E. Set and M. Z. Sarıkaya, *Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions*, Hacettepe J. of. Math. and Statistics, **40**(2011), 219-229.

[15] J. E. Pečarić, F. Proschan, Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, INC, 1992.

[16] M. Z. Sarikaya, A. Saglam and H. Yıldırım, *On some Hadamard–type inequalities for h-convex functions*, J. Math. Ineq., **2(3)**(2008), 335-341.

[17] A. Sağlam, H. Yıldırım and M. Z. Sarıkaya, *Some new inequalities of Hermite-Hadamard type*, Kyungpook Math. J., **50**(2010) 399-410.

[18] G. Toader, *Some generalizations of the convexity*, Proc. Colloq. Approx. Opt. Cluj-Napoca, (1984), 329-338.

[19] G. Toader, *The hierarchy of convexity and some classic inequalities*, J. Math. Ineq., **3(3)**(2009), 305-313.