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ON GENERALIZATIONS OF THE POMPEIU FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we determine the general solution of the functional equations

$$f(x + y + xy) = p(x) + q(y) + g(x)h(y), \qquad (\forall\, x, y \in \Re_\star)$$

and

$$f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \qquad (\forall\, x, y \in \Re)$$

which are generalizations of a functional equation studied by Pompeiu. We present a method which is simple and direct to determine the general solutions of the above equations without any regularity assumptions.

KEY WORDS AND PHRASES: Pompeiu functional equation, multiplicative function, logarithmic function, exponential function.

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1. INTRODUCTION

Let $\Re$ be the set of all real numbers and $\Re_o$ denote the set of nonzero reals. Further, let $\Re_\star = \Re \setminus \{-1\}$, that is the set of real numbers except negative one. A function $M : \mathcal{D} \to \Re$ is said to be *multiplicative* if and only if $M(xy) = M(x)M(y)$ for all $x, y \in \mathcal{D}$, where $\mathcal{D} = \Re$ or $\Re_o$. A function $E : \Re \to \Re$ is called *exponential* if and only if $E(x + y) = E(x)E(y)$ for all $x, y \in \Re$. A function $L : \Re_o \to \Re$ is said to be *logarithmic* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in \Re_o$. A comprehensive treatment of these functions can be found in the book of Aczel and Dhombres [1].

If $\mathbf{G} = \Re_\star$, then $(\mathbf{G}, \circ)$ is an abelian group where the group operation is defined as

$$x \circ y = x + y + xy.$$

A characterization of the homomorphisms of the group $(\mathbf{G}, \circ)$ can be obtained by solving the functional equation

$$f(x + y + xy) = f(x) + f(y) + f(x)f(y). \qquad \text{(PE)}$$

This functional equation is known as the *Pompeiu functional equation* [3,4].

Suppose that $f : \Re \to \Re$ satisfies (PE). Then the only solution f of the Pompeiu equation (PE) is given by

$$f(x) = M(x+1) - 1, \tag{1.1}$$

where M is multiplicative.

To see this, add 1 to both sides of (PE) and write $F(x) = 1 + f(x)$. Then (PE) reduces to $F(x+y+xy) = F(x)F(y)$. Now replacing x by $x-1$ and y by $y-1$, we obtain $M(xy) = M(x)M(y)$, where $M(x) = F(x-1)$. Thus, M is multiplicative and $f(x) = F(x) - 1 = M(x+1) - 1$, which is (1.1).

In a special case, f is an automorphism of the field $\Re$. Suppose M is also additive. Then M is a ring homomorphism of $\Re$. If M is a nontrivial homomorphism, then $f(x) = M(x) = x$, that is, f is an automorphism of the field $\Re$.

In this paper, we determine the general solution of the functional equations

$$f(x+y+xy) = p(x) + q(y) + g(x)h(y), \qquad (\forall\, x, y \in \Re_\star) \tag{FE1}$$

and

$$f(ax+by+cxy) = f(x) + f(y) + f(x)f(y), \qquad (\forall\, x, y \in \Re) \tag{FE2}$$

which are generalizations of the Pompeiu functional equation (PE). We present a method which is simple and direct to determine the general solutions of (FE1) and (PE2) without any regularity assumptions. For other related functional equations, the interested reader should refer to [2] and [5].

2. SOME PRELIMINARY RESULTS

The following two lemmas will be instrumental for establishing the main result of this paper.

LEMMA 1. Let $g, h : \Re_o \to \Re$ satisfy the functional equation

$$g(xy) = g(y) + g(x)\,h(y) \tag{2.1}$$

for all $x, y \in \Re_o$. Then for all $x, y \in \Re_o$, $g(x)$ and $h(y)$ are given by

$$g(x) = 0, \qquad h(y) = arbitrary; \tag{2.2}$$

$$g(x) = L(x), \qquad h(y) = 1; \tag{2.3}$$

$$g(x) = \alpha\,[M(x) - 1], \qquad h(y) = M(y), \tag{2.4}$$

where $M : \Re_o \to \Re$ is a multiplicative map not identically one, $L : \Re_o \to \Re$ is a logarithmic function not identically zero and α is an arbitrary nonzero constant.

PROOF. If $g \equiv 0$, then h is arbitrary and they satisfy the equation (2.1). Hence we have the solution (2.2). We assume hereafter that $g \not\equiv 0$.

Interchanging x with y in (2.1) and comparing the resulting equation to (2.1), we get

$$g(y)\,[h(x) - 1] = g(x)\,[h(y) - 1]. \tag{2.5}$$

Suppose $h(x) = 1$ for all $x \in \Re_o$. Then (2.1) yields $g(xy) = g(y) + g(x)$ and hence the function $g : \Re_o \to \Re$ is logarithmic. This yields the solution (2.3).

Finally, suppose $h(y) \neq 1$ for some y. Then from (2.5), we have

$$g(x) = \alpha\,[h(x) - 1], \tag{2.6}$$

where α is a nonzero constant, since $g \not\equiv 0$. Using (2.6) in (2.1), and simplifying, we obtain

$$h(xy) = h(x)\,h(y). \tag{2.7}$$

Hence, $h : \Re_o \to \Re$ is a multiplicative function. This gives the asserted solution (2.4) and the proof of the lemma is now complete.

LEMMA 2. The general solutions $f, g, h : \Re_o \to \Re$ of the functional equation

$$f(xy) = f(x) + f(y) + \alpha\, g(x) + \beta\, h(y) + g(x)h(y) \qquad (\forall\ x, y \in \Re_o) \tag{2.8}$$

where α and β are apriori chosen constants, have values $f(x), g(x)$ and $h(y)$ given, for all $x, y \in \Re_o$, by

$$\left.\begin{aligned} &f(x) = L(x) + \alpha\beta \\ &g(x) \text{ is arbitrary} \\ &h(y) = -\alpha; \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} &f(x) = L(x) + \alpha\beta \\ &g(x) = -\beta \\ &h(y) \text{ is arbitrary}; \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} &f(x) = L_o(x) + \frac{1}{2}\,c\,L_1^2(x) + \alpha\beta \\ &g(x) = c\,L_1(x) - \beta \\ &h(y) = L_1(y) - \alpha; \end{aligned}\right\} \tag{2.11}$$

$$\left.\begin{aligned} &f(x) = L(x) + \gamma\delta\,[M(x) - 1] + \alpha\beta \\ &g(x) = \gamma\,[M(x) - 1] - \beta \\ &h(y) = \delta\,[M(y) - 1] - \alpha, \end{aligned}\right\} \tag{2.12}$$

where $M : \Re_o \to \Re$ is a multiplicative map not identically one, $L_o, L_1, L : \Re_o \to \Re$ are logarithmic functions with L_1 not identically zero, and c, δ, γ are arbitrary nonzero constants.

PROOF. Interchanging x with y in (2.8) and comparing the resulting equation to (2.8), we obtain

$$[\,\alpha + h(y)\,]\,[\,\beta + g(x)\,] = [\,\alpha + h(x)\,]\,[\,\beta + g(y)\,]. \tag{2.13}$$

Now we consider several cases.

Case 1. Suppose $h(y) = -\alpha$ for all $y \in \Re_o$. Then (2.8) yields

$$f(xy) = f(x) + f(y) - \alpha\beta. \tag{2.14}$$

Hence

$$f(x) = L(x) + \alpha\beta, \tag{2.15}$$

where $L : \Re_o \to \Re$ is a logarithmic function. Hence we have the asserted solution (2.9).

Case 2. Suppose $g(x) = -\beta$ for all $x \in \Re_o$. Then (2.8) yields

$$f(xy) = f(x) + f(y) - \alpha\beta.$$

Hence, as before,

$$f(x) = L(x) + \alpha\beta,$$

where $L : \Re_o \to \Re$ is a logarithmic function. Thus we have the asserted solution (2.10).

Case 3. Now we assume $h(x) \neq -\alpha$ for some $x \in \Re_o$ and $g(x) \neq -\beta$ for some $x \in \Re_o$. From (2.13), we get

$$\beta + g(y) = c\,[\alpha + h(y)], \tag{2.16}$$

where c is a nonzero constant.

Using (2.8), we compute

$$\begin{aligned} f(x \cdot yz) = f(x) + f(y) + f(z) + \alpha\, g(y) + \beta\, h(z) \\ + g(y)h(z) + \alpha\, g(x) + \beta\, h(yz) + g(x)h(yz). \end{aligned} \tag{2.17}$$

Again, using (2.8), we have

$$\begin{aligned} f(xy \cdot z) = f(x) + f(y) + f(z) + \alpha\, g(x) + \beta\, h(y) \\ + g(x)h(y) + \alpha\, g(xy) + \beta\, h(z) + g(xy)h(z). \end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we obtain

$$[\alpha + h(z)]\,[g(y) - g(xy)] = [\beta + g(x)]\,[h(y) - h(yz)], \qquad \forall\, x, y \in \Re_o. \tag{2.19}$$

Since $g(x) \neq -\beta$ for some $x \in \Re_o$, there exists a $x_o \in \Re_o$ such that $g(x_o) + \beta \neq 0$. Letting $x = x_o$ in (2.19), we have

$$h(yz) = h(y) + [\alpha + h(z)]\, k(y), \tag{2.20}$$

where

$$k(y) = \frac{g(yx_o) - g(y)}{g(x_o) + \beta}. \tag{2.21}$$

The general solution of (2.20) can be obtained from Lemma 1 (add α to both sides). Hence, taking into consideration that $h(y) + \alpha \not\equiv 0$, we have

$$h(y) = L_1(y) - \alpha. \tag{2.22}$$

or

$$h(y) = \delta\,[M(y) - 1] - \alpha, \tag{2.23}$$

where L_1 is logarithmic not identically zero, M is multiplicative not identically one, and δ is an arbitrary constant.

Now we consider two subcases.

Subcase 3.1. From (2.22) and (2.16), we have

$$g(y) = c\, L_1(y) - \beta. \tag{2.24}$$

Using (2.22) and (2.24) in (2.8), we get

$$f(xy) = f(x) + f(y) + c\, L_1(x) L_1(y) - \alpha\beta. \tag{2.25}$$

Defining

$$L_o(x) := f(x) - \frac{1}{2}\, c\, L_1^2(x) - \alpha\beta, \tag{2.26}$$

we see that (2.25) reduces to

$$L_o(xy) = L_o(x) + L_o(y)$$

for all $x, y \in \Re_o$, that is, L_o is logarithmic and from (2.26), we have

$$f(x) = L_o(x) + \frac{1}{2}\, c\, L_1^2(x) + \alpha\beta. \tag{2.27}$$

Hence (2.27), (2.24) and (2.22) yield the asserted solution (2.11).

Subcase 3.2. Finally, from (2.23) and (2.16), we obtain

$$g(y) = \delta\, c\, [M(y) - 1] - \beta. \tag{2.28}$$

With (2.23) and (2.28) in (2.8), we have

$$f(xy) = f(x) + f(y) - \alpha\beta + c\,\delta^2\,[M(x) - 1]\,[M(y) - 1]. \tag{2.29}$$

Defining

$$L(x) := f(x) - c\,\delta^2\,[M(x) - 1] - \alpha\beta, \tag{2.30}$$

we see that (2.29) reduces to

$$L(xy) = L(x) + L(y)$$

for all $x, y \in \Re_o$, that is, L is a logarithmic function. Using (2.30), we have

$$f(x) = L(x) + \gamma\delta\,[M(x) - 1] + \alpha\beta, \tag{2.31}$$

where $\gamma = c\,\delta$. Hence (2.31), (2.28) and (2.23) yield the asserted solution (2.12). This completes the proof of the lemma.

3. SOLUTION OF THE FUNCTIONAL EQUATION (FE1)

Now we are ready to present the general solution of (FE1) using Lemma 2.

THEOMEM 1. The functions $f, p, q, g, h : \Re_\star \to \Re$ satisfy the functional equation

$$f(x + y + xy) = p(x) + q(y) + g(x)\,h(y) \tag{FE1}$$

for all $x, y \in \Re_\star$ if and only if, for all $x, y \in \Re_\star$,

$$\left.\begin{aligned} f(x) &= L(x+1) + \alpha\beta + a + b \\ p(x) &= L(x+1) + b \\ q(y) &= L(y+1) + \alpha\beta + a + \beta h(y) \\ g(x) &= -\beta \\ h(y) &\text{ is arbitrary;} \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} f(x) &= L(x+1) + \alpha\beta + a + b \\ p(x) &= L(x+1) + +\alpha\beta + b + \alpha g(x) \\ q(y) &= L(y+1) + a \\ g(x) &\text{ is arbitrary} \\ h(y) &= -\alpha; \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} f(x) &= L(x+1) + \gamma\delta\,[M(x+1)-1] + \alpha\beta + a + b \\ p(x) &= L(x+1) + (\delta+\alpha)\gamma\,[M(x+1)-1] + b \\ q(y) &= L(y+1) + (\gamma+\beta)\delta\,[M(y+1)-1] + a \\ g(x) &= \gamma\,[M(x+1)-1] - \beta \\ h(y) &= \delta\,[M(y+1)-1] - \alpha; \end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned} f(x) &= L_o(x+1) + \frac{1}{2}\,c\,L_1^2(x+1) + \alpha\beta + a + b \\ p(x) &= L_o(x+1) + \frac{1}{2}\,c\,L_1^2(x+1) + \alpha c\,L_1(x+1) + b \\ q(y) &= L_o(y+1) + \frac{1}{2}\,c\,L_1^2(y+1) + \beta\,L_1(y+1) + a \\ g(x) &= c\,L_1(x+1) - \beta \\ h(y) &= L_1(y+1) - \alpha, \end{aligned}\right\} \tag{3.4}$$

where $M : \Re_o \to \Re$ is a multiplicative function not identically one, $L_o, L_1, L : \Re_o \to \Re$ are logarithmic maps with L_1 not identically zero, and $\alpha, \beta, \gamma, \delta, a, b, c$ are arbitrary real constants.

PROOF. First, we substitute $y = 0$ in (FE1) and then we put $x = 0$ in (FE1) to obtain

$$p(x) = f(x) - a + \alpha\, g(x) \tag{3.5}$$

and

$$q(y) = f(y) - b + \beta\, h(y), \tag{3.6}$$

where $a := q(0)$, $b := p(0)$, $\alpha := -h(0)$, $\beta := -g(0)$. Using (3.5) and (3.6) in (FE1), we have

$$f(x + y + xy) = f(x) + f(y) - a - b + \alpha\, g(x) + \beta\, h(y) + g(x)\, h(y) \tag{3.7}$$

for $x, y \in \Re_*$. Replacing x by $u - 1$ and y by $v - 1$ in (3.7) and then defining

$$F(u) := f(u-1) - a - b, \qquad G(u) := g(u-1), \qquad H(u) := h(u-1) \tag{3.8}$$

for all $u \in \Re_o$, we obtain

$$F(uv) = F(u) + F(v) + \alpha\, G(u) + \beta\, H(v) + G(u)\, H(v) \tag{3.9}$$

for all $u, v \in \Re_o$. The general solution of (3.9) can now be obtained from Lemma 2. The first two solutions of Lemma 2 (see (2.9) and (2.10)) together with (3.5) and (3.6) yield the solutions (3.1) and (3.2). The next two solutions of Lemma 2 (that is, solution (2.11) and (2.12)) yield together with (3.5) and (3.6) the asserted solutions (3.3) and (3.4). This completes the proof of the theorem.

4. SOLUTION OF THE FUNCTIONAL EQUATION (FE2)

Let a, b and c be real parameters. We consider the functional equation

$$f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \qquad \forall\, x, y \in \Re. \tag{FE2}$$

The only constant solutions of (FE2) are $f \equiv 0$ and $f \equiv -1$. So we look for nonconstant solutions of the functional equation (FE2).

Substitution of $x = 0 = y$ in (FE2) yields $f(0)\,[f(0) + 1] = 0$. Hence, either $f(0) = 0$ or $f(0) = -1$. Now we consider two cases.

Case 1. Suppose $f(0) = -1$. Then $x = 0$ in (FE2) gives $f(by) = f(0)$, so that when $b \neq 0$, f is a constant which is not the case. Similarly by putting $y = 0$ in (FE2), we get f is a constant when $a \neq 0$.

Suppose $a = 0 = b$. If c is also zero, then (FE2) is $[1+f(x)]\,[1+f(y)] = 0$ since $f(0) = -1$. That is f is a constant. So, assume $c \neq 0$. Then replacing x by $\frac{x}{c}$ and y by $\frac{y}{c}$ in (FE2), we obtain

$$M(xy) = M(x)\,M(y), \tag{4.1}$$

where $M : \Re \to \Re$ is a multiplicative map with $M(x) = 1 + f\left(\frac{x}{c}\right)$. Hence

$$f(x) = M(cx) - 1 \tag{4.2}$$

is a solution of (FE2) with $f(0) = -1$, $a = 0 = b$, $c \neq 0$.

Case 2. Suppose $f(0) = 0$. Let $a = 0$. Then $y = 0$ in (FE2) gives $f \equiv 0$ which is not the case. So, $a \neq 0$. Similarly $b \neq 0$. Setting $x = 0$ and $y = 0$ separately in (FE2), we get

$$f(by) = f(y) \qquad \text{and} \qquad f(ax) = f(x) \tag{4.3}$$

so that (FE2) becomes

$$f(ax + by + cxy) = f(ax) + f(by) + f(ax)\,f(by). \tag{4.4}$$

Suppose $c = 0$. Then replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$ in (4.4) we have

$$E(x + y) = E(x)\,E(y)$$

where $E : \Re \to \Re$ given by

$$E(x) = 1 + f(x) \tag{4.5}$$

is an exponential map. Further, from (4.3) and (4.5), we get

$$E(ax) = E(x) = E(bx)$$

and since $E(x)\,E(-x) = 1$, so we get

$$E((a - b)x) = 1 = E((a - 1)x). \tag{4.6}$$

If $a \neq b$, then E is a constant map and so f is also a constant function. If $a \neq 1$, then E and so f is a constant. Hence $a = 1 = b$. Thus by (4.5)

$$f(x) = E(x) - 1$$

is a solution of (FE2) with $a = b = 1$, $c = 0$.

Finally, let $a \neq 0$, $b \neq 0$ and $c \neq 0$. Set $\alpha = \frac{c}{ab}$. Replacing x by $\frac{x}{a\alpha}$ and y by $\frac{y}{b\alpha}$ in (4.4), we obtain

$$F(x + y + xy) = F(x)\,F(y), \tag{4.7}$$

where

$$F(x) = 1 + f\left(\frac{x}{\alpha}\right). \tag{4.8}$$

Changing x to $x - 1$ and y to $y - 1$ in (4.7) we have

$$M(xy) = M(x)\,M(y),$$

where $M : \Re \to \Re$ is multiplicative and

$$M(x) = F(x-1). \tag{4.9}$$

Thus by (4.8) and (4.9), we have

$$f(x) = F(\alpha x) - 1 = M(1+\alpha x) - 1. \tag{4.10}$$

If we use (4.10) in (4.3), and recall that $\alpha = \frac{c}{ba}$, we get

$$M\left(1 + \frac{c}{a}x\right) = M\left(1 + \frac{c}{b}x\right) = M\left(1 + \frac{c}{ab}x\right). \tag{4.11}$$

Recall that, since M is multiplicative, $M(x)M\left(\frac{1}{x}\right) = 1$ (otherwise if $M(1) = 0$, then $M \equiv 0$ so that $f \equiv -1$). Changing separately x to $\frac{ax}{c}$ and x to $\frac{bx}{c}$ in (4.11), we obtain

$$M(1+x) = M\left(1 + \frac{x}{b}\right) = M\left(1 + \frac{x}{a}\right). \tag{4.12}$$

Similarly, replacing x by $\frac{abx}{c}$ in (4.11), we have

$$M(1+x) = M(1+ax) = M(1+bx). \tag{4.13}$$

Replacing x by $x-1$ in (4.13), we obtain $M(x) = M(1 + a(x-1))$ which yields

$$M\left(\frac{1-a+ax}{x}\right) = 1 \qquad \text{if } x \neq 0.$$

Suppose $a \neq 1$. Changing x to $(1-a)x$, we have $M\left(a + \frac{1}{x}\right) = 1$ and thus (again replacing x by $\frac{1}{x-a}$) we have $M(x) = 1$ when $x \neq a$. Similarly, if $b \neq 1$, we get $M(x) = 1$ when $x \neq 0$, b.

Hence, $M(x) = 1$ for all x which leads to f is a constant. Therefore $a = 1 = b$. Then from (4.10), we obtain

$$f(x) = M(1+cx) - 1 \tag{4.14}$$

where $M : \Re \to \Re$ is multiplicative. Thus we have proved the following theorem.

THEOREM 2. The function $f : \Re \to \Re$ is a solution of (FE2) if and only if $f(x)$, for every $x \in \Re$, is given by

$$f(x) = \begin{cases} M(cx) - 1 & \text{if } a = 0 = b,\ c \neq 0 \\ E(x) - 1 & \text{if } a = 1 = b,\ c = 0 \\ M(cx+1) - 1 & \text{if } a = 1 = b,\ c \neq 0 \\ k & \text{otherwise,} \end{cases}$$

where $M : \Re \to \Re$ is multiplicative, $E : \Re \to \Re$ is exponential, and k is a constant satisfying $k(k+1) = 0$.

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