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ON GENERALIZED DELANNOY PATHS*

JEAN-MICHEL AUTEBERT† AND SYLVIANE R. SCHWER‡

Abstract. A Delannoy path is a minimal path with diagonal steps in $\mathbb{Z}^2$ between two arbitrary points. We extend this notion to the n dimensions space $\mathbb{Z}^n$ and identify such paths with words on a special kind of alphabet: an S-alphabet. We show that the set of all the words corresponding to Delannoy paths going from one point to another is exactly one class in the congruence generated by a Thue system that we exhibit. This Thue system induces a partial order on this set that is isomorphic to the set of ordered partitions of a fixed multiset where the blocks are sets with a natural order relation. Our main result is that this poset is a lattice.

Key words. Delannoy path, Thue system, lattice

AMS subject classifications. 06B05, 68Q42

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1. Introduction.

A Delannoy path [11] is given as a path that can be drawn on a rectangular grid, starting from the southwest corner, going to the northeast corner, using only three kinds of elementary steps: *north*, *east*, and *northeast*. Hence they are minimal paths with diagonal steps. We generalize the notion of a Delannoy path to the hyperspace $\mathbb{Z}^n$, considering a hyperparallelipedic grid as a set of elementary steps: a step in each direction and the combinations of several of them, the diagonal steps.

We prove that, in a very natural way, an S-alphabet can be associated with the possible elementary steps in a Delannoy path in $\mathbb{Z}^n$, and consequently S-words with Delannoy paths themselves. These notions were introduced by Schwer [8], in a completely different context, for treating simultaneity problems.

We then define a Thue system on the set of S-words that turns out to be noetherian and confluent. This Thue system induces both an ordering on S-words and a congruence. Our main goal is to prove that each equivalence class for this congruence is with this order relation a lattice (Theorem 5.5). (This lattice is a nondistributive lattice as soon as $n > 2$.)

An equivalence class can be viewed as the set of all ordered partitions of a fixed multiset where the blocks are sets (not multisets). There is a transparent bijection between an equivalence class and an element of this set, and the order relation over partitions derived is a very natural one. In [9] are given some links between S-words and others mathematical objects.

Moreover, we exhibit a characterization of the S-words of a class (and so of generalized Delannoy paths going from a point to another) with a family of matrices having its coefficients in $\{-1, 0, 1\}$ (Theorem 4.2), and we prove that the order on S-words can be exactly transposed as the componentwise order on matrices induced by $-1 < 0 < 1$ (Theorem 4.6).

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2. Recalls. Concerning lattices, the notations follow [10, 4]. Recall that a lattice is an ordered set such that each pair of elements has a least upper bound and a greatest lower bound. A subset of a lattice is a *sublattice* if for the same order relation it is a lattice. It is a *distributive* lattice if the two operations associating, respectively, with two elements, their least upper bound and a greatest lower bound, are distributive with respect to each other. A lattice ordered by $\leq$ is modular if for all triples of elements (a, b, c) with $a \leq c$ the least upper bound of a and of the greatest lower bound of b and c is equal to the greatest lower bound of c and of the least upper bound of a and b. It is known [10] that every distributive lattice is modular and that the different chains going from one element to another all have the same length in a modular lattice.

Concerning formal languages, we follow [1, 5].

Let X be an alphabet, let X^* be the set of words over X, and let ε be the empty word. If f is a word in X^*, then $|f|$ is the length of f. A word g is a *prefix* of f if some word u exists such that $f = gu$.

Let R be a finite relation over X^*. The Thue system generated by R is the relation over X^*, denoted $\longrightarrow$, that is the smallest relation containing R and compatible with the concatenation product: $(u, v) \in R \Longrightarrow \forall f, g \in X^*, fug \longrightarrow fvg$.

We use freely the usual notions and notations, as can be found, for example, in [1] or [6]. In particular, $\longleftarrow$ denotes the symmetric relation of $\longrightarrow$, $\longleftrightarrow$ the symmetric closure of $\longrightarrow$, and $\longrightarrow^*$ its reflexive and transitive closure. Let set $[f] = \{g \in X^* \mid f \longleftrightarrow^* g\}$ and $\langle f \rangle = \{g \in X^* \mid f \longrightarrow^* g\}$. These notations are extended to languages $[L] = \bigcup_{f \in L}[f]$ and $\langle L \rangle = \bigcup_{f \in L}\langle f \rangle$.

We just recall here the properties [1] of Thue systems that we shall make use of: A *noetherian* system is a system for which no infinite chain exists. A system is *confluent* if $f \longrightarrow^* u$ and $f \longrightarrow^* v$ implies the existence of g such that $u \longrightarrow^* g$ and $v \longrightarrow^* g$. An element f is an *irreducible* element for $\longrightarrow$ if no other element g exists such that $f \longrightarrow g$.

In this paper, we make use of the notions of S-alphabet and S-word introduced by Schwer [8, 9].

Let X be an alphabet. An *S-alphabet* issued from X is a nonempty subset of $\widehat{X} = \{P \in 2^X \mid P \neq \emptyset\}$. $\widehat{X}$ is itself an S-alphabet. The elements of an S-alphabet are called S-letters. Let Y be an S-alphabet subset of $\widehat{X}$; the alphabet $\{x \in X \mid \exists y \in Y \ : \ x \in y\}$ is the *underlying alphabet* of Y. An *S-word* is a word written over an alphabet of S-letters. So we may make use of all the usual notations and definitions of the languages theory for S-words. It is, however, useful to introduce notations that put in relation S-words with the underlying alphabet.

Let $X = \{a_1, a_2, \ldots, a_n\}$, we define the homomorphism $\psi : \widehat{X}^* \longrightarrow \mathbb{N}^n$ by $\psi(P) = (\chi_P(a_1), \ldots, \chi_P(a_n))$, where χ_P is the characteristic function of P. This extends the usual notion of Parikh mapping [5]. The ith component of $\psi(f)$ is denoted $\psi_i(f)$.

We also define the homomorphism $\nu : \widehat{X}^* \longrightarrow \mathbb{N}$ by $\nu(P) = Card(P)$, i.e., $\nu(f) = \Sigma_{1 \leq i \leq n}\psi_i(f)$. So ν is the number of occurrences of letters appearing in all the S-letters of the S-word.

Let $\psi(f) = (p_1, p_2, \ldots, p_n)$; for $m \leq n$, and for $l \leq p_m$, we name *position* of the lth occurrence of the letter a_m the integer $1 + \nu(g)$, where g is the S-word that is the longest prefix of f such that $\psi_m(g) < l$.

To simplify the exposition of the examples, we write the different letters in a S-letter one after the other, without commas to separate them, and we write them in increasing order on the indices.

Example 2.1. *On the alphabet $\widehat{X}$ issued from $X = \{a_1, a_2, a_3\}$, consider the* *word* $f = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}\{a_2\}\{a_2\}$. *It is such that* $\psi(f) = (4, 4, 3)$.

For the letter a_1, *the longest prefixes* g_l *of* f *such that* $\psi_1(g_l) < l$ *when* l *equals* 1, 2, 3, *and* 4 *are, respectively,* $g_1 = \varepsilon$, $g_2 = \{a_1a_2\}\{a_3\}$, $g_3 = \{a_1a_2\}\{a_3\}\{a_1\}$, *and* $g_4 = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$, *and we have* $\nu(g_1) = 0$, $\nu(g_2) = 3$, $\nu(g_3) = 4$, *and* $\nu(g_4) = 6$. *The respective positions of the four occurrences of* a_1 *are then* $1, 4, 5$, *and* 7.

For the letter a_2, *the longest prefixes* g_l *of* f *such that* $\psi_2(g_l) < l$ *when* l *equals* 1, 2, 3, *and* 4 *are, respectively,* $g_1 = \varepsilon$, $g_2 = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$, $g_3 = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}$, *and* $g_4 = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}\{a_2\}$, *and we have* $\nu(g_1) = 0$, $\nu(g_2) = 6$, $\nu(g_3) = 9$, *and* $\nu(g_4) = 10$. *The respective positions of the four occurrences of* a_2 *are then* $1, 7, 10$, *and* 11.

For the letter a_3, *the longest prefixes* g_l *of* f *such that* $\psi_3(g_l) < l$ *when* l *equals* 1, 2, *and* 3 *are, respectively,* $g_1 = \{a_1a_2\}$, $g_2 = \{a_1a_2\}\{a_3\}\{a_1\}$, *and* $g_3 = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$, *and we have* $\nu(g_1) = 2$, $\nu(g_2) = 4$, *and* $\nu(g_4) = 6$. *The respective positions of the three occurrences of* a_3 *are then* $3, 5$, *and* 7.

3. The Thue system. We extend Delannoy paths to the hyperplane $\mathbb{Z}^n$; i.e., we consider minimal paths with diagonal steps between two arbitrary points.

We associate with each dimension a letter of an alphabet $X = \{a_1, a_2, \ldots, a_n\}$ and construct the S-alphabet $\widehat{X} = \{P \in 2^X \mid P \neq \emptyset\}$.

The interpretation is the following: the letter $\{a_i\}$ is a step in the dimension i, and more generally the letter $P \in \widehat{X}$ is a simultaneous step in each of the dimensions indicated by the letters of X that belong to P, called *diagonal step* if $Card(P) \geq 2$.

Let us give an arbitrary order over the letters of X by $a_1 < a_2 < \cdots < a_n$. This induces over the S-letters a partial order $P < Q \iff [\forall x \in P, \forall y \in Q : \ x < y]$.

We then define the Thue system, relation denoted $\longrightarrow$ on $\widehat{X}^*$, by the following: $\forall P, Q, R \in \widehat{X}$ such that $P < Q$ and $R = P \cup Q$, set $PQ \longrightarrow R$ and $R \longrightarrow QP$. Note that P and Q are disjoint.

In the case where $n = 2$, with $X = \{a, b\}$, we get $\widehat{X} = \{\{a\}, \{b\}, \{a, b\}\}$, and renaming, respectively, a, b, and c these three letters, the obtained system is precisely the system studied in [2].

Note that doing the bijection of X in itself, which maps a_i on a_{n+1-i}, or reversing the order over the letters of X, which leads exactly to the same relation, one gets $\longleftarrow$, the symmetric relation of $\longrightarrow$. Each property of $\longrightarrow$ is also a property of $\longleftarrow$ (and the converse).

Lemma 3.1. *If* f *and* g *are two words in the same class, their image under* ψ *is the same.*

Proof. By induction, it is sufficient to ensure that each application of a rule preserves the image under ψ. □

Lemma 3.2. *The set of all irreducible words for this Thue system is* $Irr = \{a_n\}^* \ldots \{a_2\}^*\{a_1\}^*$. *Symmetrically, the set of all irreducible words for the inverse Thue system is* $\{a_1\}^*\{a_2\}^* \ldots \{a_n\}^*$.

Proof. Clearly, a word in Irr has no subword being a left factor of a couple in the relation defining the Thue system, and so Irr is a set of irreducible words. Conversely, let f be an S-word not in Irr; then there is either in f an S-letter R containing at least two letters or there are two S-letters $\{a_i\}$ and $\{a_j\}$ with $i < j$ and $\{a_i\}$ is situated before $\{a_j\}$. In the latter case, there exist two such S-letters being consecutive, and the rule $\{a_i\}\{a_j\} \longrightarrow \{a_ia_j\}$ may be applied to f, which is not an irreducible word. In the former case, R can be partitioned between two subsets P and Q so that all the indices of the elements of P are smaller than the indices of the elements of Q, and the rule $R \longrightarrow QP$ may be applied to f, which is not an irreducible word. ∎

COROLLARY 3.3. *For each word, there is at most one irreducible word.*

Proof. It is sufficient to check that, among all words having the same image under ψ, there is only one belonging to Irr. ∎

LEMMA 3.4. *The Thue system is noetherian. As a consequence, the relation* $\longrightarrow^*$ *is an order relation.*

Proof. Let f be an S-word, and let P be an occurrence of one of its S-letters. Let $Post(P, f)$ denote the set of S-letters situated after P in f. To each letter a_m in P is attached the integer $Card(\{i > m \mid a_i \in P\}) + 2.\sum_{Q \in Post(P,f)} Card(\{i > m \mid a_i \in Q\})$, and let $\sigma(f)$ be the sum of these integers for all the occurrences of letters in f. It is easy to check that $f \longrightarrow g \Longrightarrow \sigma(f) > \sigma(g)$. As a consequence, the Thue system is noetherian. The relation $\longrightarrow^*$, which is by definition reflexive and transitive, is antisymmetric as well. It is so an order relation. ∎

COROLLARY 3.5. *The Thue system is confluent.*

Proof. Let f and g be two congruent words. As the system is noetherian, they each have an irreducible, and as they are congruent these irreducibles are but one. The two words can be derived on the same word. ∎

COROLLARY 3.6. *The following equality holds:* $[f] = \{g \in \widehat{X}^* \mid \psi(g) = \psi(f)\}$.

Proof. The inclusion $[f] \subset \{g \in \widehat{X}^* \mid \psi(g) = \psi(f)\}$ has already been established. Conversely, if two words have the same image under ψ, they have the same irreducible, and so are congruent. ∎

The n-uple $(p_1, p_2, \ldots, p_n)$ is characteristic of the class of words f satisfying $\psi(f) = (p_1, p_2, \ldots, p_n)$. This class is denoted $\mathfrak{L}(p_1, p_2, \ldots, p_n)$. The quotient $\widehat{X}^*/_{\longleftrightarrow^*}$ is isomorphic to $\mathbb{N}^n$ with componentwise addition.

Altogether, the following holds:

$$\mathfrak{L}(p_1, p_2, \ldots, p_n) = \{g \in \widehat{X}^* \mid \{a_1\}^{p_1}\{a_2\}^{p_2} \ldots \{a_n\}^{p_n} \longrightarrow^* g \longrightarrow^* \{a_n\}^{p_n} \ldots \{a_1\}^{p_1}\}.$$

In other words, $\langle \widehat{X}^*, \longrightarrow^* \rangle$ is a set with a partial order whose set of minimal elements is $\{a_1\}^{p_1}\{a_2\}^{p_2} \ldots \{a_n\}^{p_n}$ and set of maximal elements is $\{a_n\}^{p_n} \ldots \{a_2\}^{p_2}\{a_1\}^{p_1}$.

As noticed before, the set $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ is isomorphic to the set of ordered partitions $(B_1, \ldots, B_k)$ of the multiset $\{1^{p_1} \ldots, n^{p_n}\}$ where the B_i are sets. The covering relation is given by

$$(B_1, \ldots, B_k) \longrightarrow (B_1, \ldots, B_{i-2}, B_{i-1} \cup B_i, B_{i+1}, \ldots, B_k)$$

if $\max B_{i-1} < \min B_i$ and

$$(B_1, \ldots, B_{i-2}, B_{i-1} \cup B_i, B_{i+1}, \ldots, B_k) \longrightarrow (B_1, \ldots, B_k)$$

if $\max B_i < \min B_{i-1}$.

We proved formerly in [2] that $\mathfrak{L}(p_1, p_2)$ with the order relation $\longrightarrow^*$ is a distributive lattice.

The main difference between the case when $n = 2$ and the general case treated here when $n > 2$ is the following: though the order $a < b$ over $X = \{a, b\}$ can easily be extended to a total order over the S-alphabet by setting $\{a\} < \{a, b\} < \{b\}$, the natural generalization of this last: $P < R < Q$ if $\forall x \in P, \forall y \in Q : \ x < y$ and if $R = P \cup Q$, is not a linear order. This deeply changes the nature of the structure of $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ with the order relation $\longrightarrow^*$.

For instance, the following example shows that $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ is not, in general, a distributive lattice.

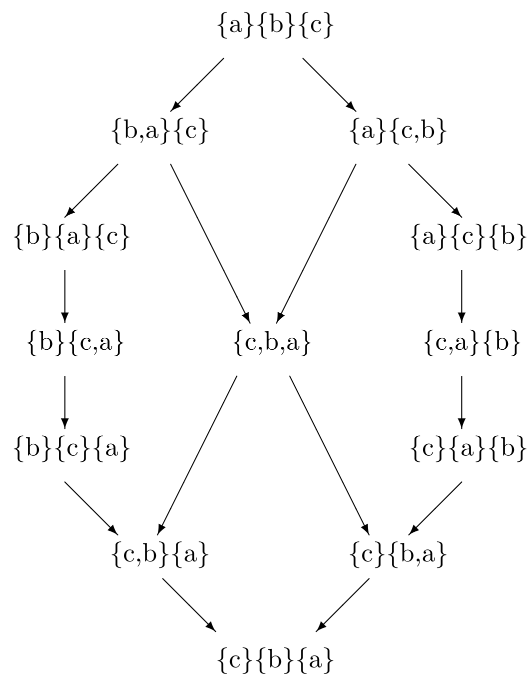

Fig. 3.1. $\mathfrak{L}(1,1,1)$.

Example 3.1. *The lattice of* $\mathfrak{L}(1,1,1)$*, represented in Figure* 3.1*, is not modular; hence it is not distributive.*

Nevertheless, it has been announced in [7] that, in the case where all p_i are equal to 1, $\mathfrak{L}(1,1,\dots,1)$ is a lattice. We prove here that it is also true in the general case.

4. The matrix associated to an S-word of $\mathfrak{L}(p_1,p_2,\dots,p_n)$. In what follows, all the S-words are words of $\mathfrak{L}(p_1,p_2,\dots,p_n)$, and we set $s=\Sigma p_i$.

It has already been indicated that the smallest word of $\mathfrak{L}(p_1,p_2,\dots,p_n)$ is the word $f_{\min}=\{a_1\}^{p_1}\{a_2\}^{p_2}\dots\{a_n\}^{p_n}$. For an integer i such that $1\le i\le \nu(f)$, we consider the occurrence of the letter in ith position in $f_{\min}$: it is, for some integers l and m, the lth occurrence of a letter a_m. Thus an integer i determines two integers l and m, defined by the relation $i=l+\Sigma_{1\le s<m}\ p_s$ with $l\le p_m$. We call *letter of rank* i in a word $f\in\mathfrak{L}(p_1,p_2,\dots,p_n)$ the occurrence of the lth letter a_m where l and m have been so determined. We set $m=r(i)$.

Example 4.1. *Let* $X=\{a_1,a_2,a_3\}$*. Considering as in the preceding example the word* $f=\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}\{a_2\}\{a_2\}$*, this word is such that* $\psi(f)=(4,4,3)$ *and* $\nu(f)=11$.

The letters of ranks $1,2,3,$ *and* 4 *are occurrences of the letter* a_1*, the letters of ranks* $5,6,7,$ *and* 8 *are occurrences of the letter* a_2*, and the letters of ranks* $9,10,$ *and* 11 *are occurrences of the letter* a_3.

The letter of rank 6 *is thus the second occurrence of the letter* a_2 *belonging to the S-letter* $\{a_1a_2a_3\}$ *that immediately follows the prefix* $\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$ *of* f*; its position is* 7.

Table 4.1 *gives explicitly the letters of all ranks and their positions.*

Definition 4.1. *Let* f *be a word of* $\mathfrak{L}(p_1,p_2,\dots,p_n)$*. The* matrix associated

TABLE 4.1

Rank	Letter	S-letter	Former prefix	Position
1	a_1	$\{a_1a_2\}$	ε	1
2	a_1	$\{a_1\}$	$\{a_1a_2\}\{a_3\}$	4
3	a_1	$\{a_1a_3\}$	$\{a_1a_2\}\{a_3\}\{a_1\}$	5
4	a_1	$\{a_1a_2a_3\}$	$\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$	7
5	a_2	$\{a_1a_2\}$	ε	1
6	a_2	$\{a_1a_2a_3\}$	$\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$	7
7	a_2	$\{a_2\}$	$\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}$	10
8	a_2	$\{a_2\}$	$\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}\{a_2\}$	11
9	a_3	$\{a_3\}$	$\{a_1a_2\}$	3
10	a_3	$\{a_1a_3\}$	$\{a_1a_2\}\{a_3\}\{a_1\}$	5
11	a_3	$\{a_1a_2a_3\}$	$\{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}$	7

*with f, denoted $M(f)$, is the matrix $\nu(f) \times \nu(f)$ whose element $M(f)[i,j]$ of the i*th *row and of the j*th *column is*

• -1 *if the position in f of the letter of rank i is smaller than the position in f of the letter of rank j;*

• 0 *if the position in f of the letter of rank i is equal to the position in f of the letter of rank j;*

• 1 *if the position in f of the letter of rank i is greater than the position in f of the letter of rank j.*

EXAMPLE 4.2. *Going further with the preceding example, the matrix associated to the word $f = \{a_1a_2\}\{a_3\}\{a_1\}\{a_1a_3\}\{a_1a_2a_3\}\{a_2\}\{a_2\}$ is*

	1	2	3	4	5	6	7	8	9	10	11
1	0	−1	−1	−1	0	−1	−1	−1	−1	−1	−1
2	1	0	−1	−1	1	−1	−1	−1	1	−1	−1
3	1	1	0	−1	1	−1	−1	−1	1	0	−1
4	1	1	1	0	1	0	−1	−1	1	1	0
5	0	−1	−1	−1	0	−1	−1	−1	−1	−1	−1
6	1	1	1	0	1	0	−1	−1	1	1	0
7	1	1	1	1	1	1	0	−1	1	1	1
8	1	1	1	1	1	1	1	0	1	1	1
9	1	−1	−1	−1	1	−1	−1	−1	0	−1	−1
10	1	1	0	−1	1	−1	−1	−1	1	0	−1
11	1	1	1	0	1	0	−1	−1	1	1	0

.

A word f is thus associated with a $\nu(f)\times\nu(f)$ matrix with coefficients in $\{-1, 0, 1\}$. Conversely, the matrix associated with a word f characterizes this word: it describes which occurrences of letters are situated in the same S-letter and the order of the occurrences of the letters with respect to each other.

The matrix associated with a word owns numerous properties. We list several of them:

• Constructively, a matrix $M(f)$ associated with a word f has only 0's in its diagonal and verifies ${}^tM(f) = -M(f)$.

Denote by $\mathcal{M}$ the set of $s \times s$ matrices M with entries in $\{-1, 0, 1\}$ verifying ${}^tM = -M$ (and hence $M[i,i] = 0\ \forall i$).

Moreover, the coefficients of the strict upper triangular part share two other properties:

• The first property, called the *commutativity property*, comes out from the commutativity of the occurrences of the same letter between themselves. This property leads us to divide the matrix in submatrices $p_i \times p_j$, just as we did on the example, indicating the orders in the positions of the occurrences of a same letter a_i with those

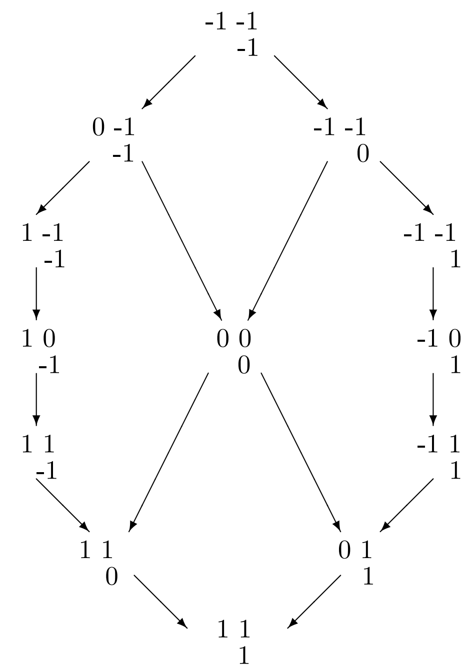

FIG. 4.1. *The lattice of transitivity.*

of another letter a_j. Denote $M_{i,j}$ the submatrix concerning the relationships between letters a_i and a_j.

This commutativity implies that, inside a submatrix $M_{i,j}$, supposing $i < j$,

(i) if i_1 and i_2 are the ranks of two letters a_i, and j_1 the rank of a letter a_j, then $i_1 < i_2$ and $(M[i_1, j_1] = 0$ or $M[i_1, j_1] = 1) \Longrightarrow M[i_2, j_1] = 1$;

(ii) if i_1 is the rank of a letter a_i, and j_1 and j_2 the ranks of two letters a_j, then $j_1 < j_2$ and $(M[i_1, j_1] = 0$ or $M[i_1, j_1] = -1) \Longrightarrow M[i_1, j_2] = -1$.

In the case where $i = j$, i.e., for the submatrix $M_{i,i}$ (square and centered on the diagonal), as we know that the diagonal is made of 0, the upper triangular part is then made of -1.

In what follows, $\mathcal{M}(p_1, \ldots, p_n)$ denotes the set of matrices in $\mathcal{M}$ verifying the commutativity property.

• The second property, called the *transitivity property*, comes from the transitivity of the order relation over the letters of the underlying alphabet: if $a_i < a_j$ and $a_j < a_k$, then $a_i < a_k$ and so the comparisons of the positions of the letters of ranks i and j on one hand, and j and k on the other hand, have an influence upon those of i and k. More precisely, $\forall i, j, k$ such that $i < j < k$, the triple $(M(f)[i,j], M(f)[i,k], M(f)[j,k])$, which we represent under the triangular shape under which it appears in the matrix $\begin{smallmatrix} M(f)[i,j] & M(f)[i,k] \\ & M(f)[j,k] \end{smallmatrix}$ belongs to the following set T_{13} of triples:

$$\left\{ \begin{smallmatrix} -1 & -1 \\ & -1 \end{smallmatrix}, \begin{smallmatrix} 0 & -1 \\ & -1 \end{smallmatrix}, \begin{smallmatrix} 1 & -1 \\ & -1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ & -1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ & -1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ & 0 \end{smallmatrix}, \begin{smallmatrix} -1 & -1 \\ & 0 \end{smallmatrix}, \begin{smallmatrix} -1 & -1 \\ & 1 \end{smallmatrix}, \begin{smallmatrix} -1 & 0 \\ & 1 \end{smallmatrix}, \begin{smallmatrix} -1 & 1 \\ & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} \right\},$$

which, ordered by the componentwise order on integers, is a lattice too (cf. Figure 4.1).

One should remark that it is the same lattice as $\mathfrak{L}(1, 1, 1)$.

In what follows, $\mathcal{M}^*(p_1, \ldots, p_n)$ denotes the set of matrices in $\mathcal{M}(p_1, \ldots, p_n)$ verifying the transitivity property.

We shall prove that these conditions do characterize the matrices associated with words f such that $\psi(f) = (p_1, \ldots, p_n)$ (and that, consequently, this association is a bijection between $[f]$ and $\mathcal{M}(p_1, \ldots, p_n)$), establishing the following theorem:

THEOREM 4.2. *Let M be a matrix of $\mathcal{M}$. It is the matrix associated with a word $f \in \mathfrak{L}(p_1, \ldots, p_n)$ if and only if it belongs to $\mathcal{M}^*(p_1, \ldots, p_n)$.*

Let M be a matrix of $\mathcal{M}(p_1, \ldots, p_n)$, and let $s = \Sigma_{j \leq n} p_n$. For all i such that $1 \leq i \leq s$, let pr_i be the number of integers $j > i$ such that $M[i, j] = 1$, and po_i the number of integers $k < i$ such that $M[k, i] = -1$, and we evaluate the integer $pl_i = pr_i + po_i$.

LEMMA 4.3. *For all $i \leq s$, the number of integers j verifying $pl_j < pl_i$ is exactly pl_i.*

Proof. Let i and j be two indices such that $i < j$. These two indices define an integer $x = M[i, j]$ and the following six vectors: V_i is the vector $M[h, i]$ for $1 \leq h < i$; V'_j is the vector $M[h, j]$ for $1 \leq h < i$; V''_j is the vector $M[h, j]$ for $i < h < j$; H'_i is the vector $M[i, h]$ for $i < h < j$; H''_i is the vector $M[i, h]$ for $j < h \leq s$; and H_j is the vector $M[j, h]$ for $j < h \leq s$, as indicated in Table 4.2.

TABLE 4.2

		i		j	
		V_i		V'_j	
i		0	H'_i	x	H''_i
				V''_j	
j				0	H_j

Let A be a vector; $|A|_1$ denotes the number of 1's in A and $|A|_{-1}$ denotes the number of -1's in A. In each case, we compare $|H''_i|_1$ and $|H_j|_1$ on one hand, $|V_i|_{-1}$ and $|V'_j|_{-1}$ on the other hand, and finally $|H'_i|_1$ and $|V''_j|_{-1}$, comparisons between vectors of same lengths.

Let i and j be two indices such that $r(i) < r(j)$ (and hence $i < j$).

— In the case where $x = M[i, j] = 1$, one gets $pr_i = |H'_i|_1 + 1 + |H''_i|_1$ and $po_i = |V_i|_{-1}$, and $pr_j = |H_j|_1$ and $po_i = |V'_j|_{-1} + |V''_j|_{-1}$.

The transitivity property implies, $\forall h > j$, $M[j, h] = 1 \Longrightarrow M[i, h] = 1$, and hence $|H''_i|_1 \geq |H_j|_1$, $\forall h < i$, $M[j, h] = -1 \Longrightarrow M[i, h] = -1$, and hence $|V_i|_{-1} \geq |V'_j|_{-1}$, and $\forall i < h < j$, $M[j, h] = -1 \Longrightarrow M[h, i] = 1$, and hence $|H'_i|_1 \geq |V''_j|_{-1}$. So $pl_i > pl_j$.

— In the case where $x = M[i, j] = -1$, one gets $pr_i = |H'_i|_1 + |H''_i|_1$ and $po_i = |V_i|_{-1}$, and $pr_j = |H_j|_1$ and $po_i = |V'_j|_{-1} + 1 + |V'_j|_{-1}$.

In the same way, the transitivity property implies $|H''_i|_1 \leq |H_j|_1$, $|V_i|_{-1} \leq |V'_j|_{-1}$, and $|H'_i|_1 \leq |V''_j|_{-1}$. So $pl_i < pl_j$.

— In the case where $x = M[i, j] = 0$, one gets $pr_i = |H'_i|_1 + |H''_i|_1$ and $po_i = |V_i|_{-1}$, and $pr_j = |H_j|_1$ and $po_i = |V'_j|_{-1} + |V'_j|_{-1}$.

In the same way, the transitivity property implies $|H''_i|_1 = |H_j|_1$, $|V_i|_{-1} = |V'_j|_{-1}$, and $|H'_i|_1 = |V''_j|_{-1}$. So $pl_i = pl_j$.

If i_1 and i_2 are two indices such that $r(i_1) = r(i_2)$ (corresponding to the same i) with $i_1 < i_2$, then, in the same way, following (i) one gets $pr_{i_1} \leq pr_{i_2}$, and following (ii) $po_{i_1} \leq po_{i_2}$, and hence $pl_{i_1} < pl_{i_2}$.

To verify the lemma, it is sufficient now for a fixed i to count down. □

To prove Theorem 4.2, it remains only to prove that the condition is sufficient. Let M be a matrix in $\mathcal{M}^*(p_1, \ldots, p_n)$; we are able to calculate for all i such that $1 \leq i \leq s$ the integer pl_i. A word $f \in \mathfrak{L}(p_1, \ldots, p_n)$ is then constructed by setting its letter of rank i to the position $1 + pl_i$. □

As the matrices associated with congruent words have the same size, they can be ordered by the comparison componentwise of the coefficients of these matrices.

DEFINITION 4.4. *Let f and g be two congruent words of X^*, and let $s = \nu(f) = \nu(g)$. f is* dominated *by g, which is denoted $f \preceq g$, if, for all integers i, j such that $0 < i < j \leq s$, $M(f)[i,j] \leq M(g)[i,j]$ holds.*

In the same way, M and N being two matrices of $\mathcal{M}$, the matrix M is *dominated* by N (or N *dominates* M), which is denoted $M \preceq N$, if, for all integers i, j such that $0 < i < j \leq s$, $M[i,j] \leq N[i,j]$ holds.

We introduce a distance between words in $\mathfrak{L}(p_1, \ldots, p_n)$.

DEFINITION 4.5. *Let d be the application from $\mathfrak{L}(p_1, \ldots, p_n)^2$ to $\mathbb{N}$, with $s = \Sigma p_i$, defined by*

$$d(f,g) = \sum_{0 \leq i < j \leq s} |M(f)[i,j] - M(g)[i,j]|.$$

This application is clearly a distance.

The next theorem is crucial.

THEOREM 4.6. $\langle f \rangle = \{g \in [f] \mid f \preceq g\}$.

Proof.

— Let us first prove the inclusion $\langle f \rangle \subseteq \{g \in [f] \mid f \preceq g\}$.

It is sufficient to prove that if $f \longrightarrow g$, then $f \preceq g$, since an easy induction on the number of rewriting rules applied to obtain a word $g \in \langle f \rangle$ from f then gives the result.

• If the applied rule is $PQ \longrightarrow R$ (with $R = P \cup Q$ and $[\forall x \in P, \forall y \in Q : \ x < y]$), then let i be the rank of a letter in P and j the rank of a letter in Q; then $i < j$ and $M(f)[i,j] = -1$, and $M(g)[i,j] = 0$. As these coefficients are the only ones that are changed, $\forall i < j, M(f)[i,j] \leq M(g)[i,j]$ holds.

• If the applied rule is $R \longrightarrow QP$ (with $R = P \cup Q$ and $[\forall x \in P, \forall y \in Q : \ x < y]$), then let i be the rank of a letter in P and j the rank of a letter in Q; then $i < j$ and $M(f)[i,j] = 0$, and $M(g)[i,j] = 1$. As these coefficients are the only ones that are changed, $\forall i < j, M(f)[i,j] \leq M(g)[i,j]$ holds.

— Let us now prove the converse inclusion.

The distance between words will allow us to make an induction on the distance between a word of the set $\{g \in [f] \mid f \preceq g\}$ and f itself.

Let $\mathcal{S}_n$ be the following property: $\{\forall f \in \widehat{X}^*, \forall g \in [f] \mid f \preceq g$ and $d(f,g) \leq n\} \Longrightarrow g \in \langle f \rangle$. We have to prove $\mathcal{S}_n$ for all integer n.

Let $g \in [f]$ be such that $f \preceq g$, and let $n = d(f,g)$.

— If n equals 0, since d is a distance, $g = f$ and $f \longrightarrow^* f$ holds. So $\mathcal{S}_0$ is true.

— Suppose that $n > 0$ and that $\mathcal{S}_{n-1}$ is true. Since $f \preceq g$, there must exist two indices i and j with $1 \leq i < j \leq s$ such that $M(f)[i,j] < M(g)[i,j]$.

Case 1. There are two indices i and j with $1 \leq i < j \leq s$ such that $M(f)[i,j] = 0$

and $M(g)[i,j] = 1$.

In this case, let R be the S-letter of f containing the two letters of ranks i and j; among the occurrences of letters in R, there are two verifying the same property as i and j and such that no letter in R has a rank which is an integer between their respective ranks; let P be the set of the letters in R of rank smaller or equal to the smallest of their two ranks, and let Q be the set of the others; $R \longrightarrow QP$ is then a rule of the Thue system. Then let f' be the word obtained from f by substituting to the occurrence of the S-letter R the two S-letters word QP.

Case 2. It is not the case.

Then $\exists i$ and j with $1 \le i < j \le s$ such that $M(f)[i,j] = -1$ and $M(g)[i,j] \ge -1$; we first show that there exist two such indices with, moreover, the condition that the letters of rank i and j are in two consecutive S-letters of f: if not, let k be the rank of a letter inside an intermediate S-letter; if $i < k < j$, then $M(f)[i,k] = M(f)[k,j] = -1$ and either $M(g)[i,k] > -1$, or $M(g)[k,j] > -1$, and so we have the same situation for letters in S-letters that are strictly nearer; if $i < j < k$, then $M(f)[k,j] = 1$, and according to the transitivity property $M(f)[i,k] = -1$, and since $M(g)[k,j] > M(f)[k,j]$, $M(g)[k,j] = 1$, and according to the transitivity property $M(g)[i,k] = 1$, and also in this case we have the same situation for letters in S-letters that are strictly nearer; if $k < i < j$ symmetrically we get the same result.

Supposing now that the letters of rank i and j verifying $1 \le i < j \le s$, $M(f)[i,j] = -1$, and $M(g)[i,j] > -1$ are in two consecutive S-letters in f, say P and Q, and that $j - i$ is the smallest possible, let us show now that i is the largest among the ranks of letters in P: if there is in P a letter of rank $i' > i$, then $M(g)[i,i'] = 0$ because otherwise (if $M(g)[i,i'] = 1$) we would be in Case 1 and if $i' > j$, $M(f)[i',j] = 1$, hence $M(g)[i',j] = 1$, and according to the transitivity property $M(g)[i,i'] = 1$, and again we would be in Case 1, and if $i' \le j$, $M(g)[i',j] \ge -1$ would contradict $j - i$ the smallest possible, and $M(g)[i',j] = -1$ implies according to the transitivity property $M(g)[i,i'] = 1$, and again we would be in Case 1.

Symmetrically, one can prove that j is the smallest among the ranks of letters in Q, and so if $R = P \cup Q$, $PQ \longrightarrow R$ is a rule of the Thue system. Then let f' be a word obtained from the word f replacing the occurrence of the two S-letters word PQ by the S-letter R.

In the two cases, clearly $f \longrightarrow f'$ (and hence $g \in [f']$), and f' is *dominated* by g and $d(f', g) < n$; hence, according to the induction hypothesis, $f' \longrightarrow^* g$. So $f \longrightarrow^* g$ holds, and $\mathcal{S}_n$ is true. □

Noticing that the triples of T_{13} are precisely the upper triangular parts of the matrices attached to the S-words of $\mathfrak{L}(1,1,1)$, we have just proved that the order between S-words of $\mathfrak{L}(1,1,1)$ and the order between the triples of T_{13} are in a complete correspondence, justifying our former remark that it is the same lattice.

5. $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ is a lattice. Let f and g be two congruent S-words: $f \longleftrightarrow^* g$ with $\nu(f) = \nu(g) = s$. Since the relation $\longrightarrow^*$ is confluent, $\langle f \rangle \cap \langle g \rangle \neq \emptyset$ holds. Let h be an S-word in $\langle f \rangle \cap \langle g \rangle$. The matrix associated with h verifies the following: $\forall i < j, M(f)[i,j] \le M(h)[i,j]$ and $\forall i < j, M(g)[i,j] \le M(h)[i,j]$. Let U be the matrix of $\mathcal{M}$ having in its upper triangular part the following coefficients: $\forall i < j,\ U[i,j] = \mathrm{Max}\{M(f)[i,j], M(g)[i,j]\}$. This matrix has *ipso facto* the commutativity property of matrices in $\mathcal{M}(p_1, \ldots, p_n)$, but it may not have the transitivity property, and so it may not be a matrix in $\mathcal{M}^*(p_1, \ldots, p_n)$.

EXAMPLE 5.1. *Let* $f = \{a_1a_4\}\{a_2a_3a_4\}\{a_3\}$ *and* $g = \{a_1a_3a_4\}\{a_3\}\{a_2a_4\}$. *Their associated matrices are*

$M(f) =$

	1	2	3	4	5	6
1	0	−1	−1	−1	0	−1
2	1	0	0	−1	1	0
3	1	0	0	−1	1	0
4	1	1	1	0	1	1
5	0	−1	−1	−1	0	−1
6	1	0	0	−1	1	0

and $M(g) =$

	1	2	3	4	5	6
1	0	−1	0	−1	0	−1
2	1	0	1	1	1	0
3	0	−1	0	−1	0	−1
4	1	−1	1	0	1	−1
5	0	−1	0	−1	0	−1
6	1	0	1	1	1	0

and the matrix U *is*

	1	2	3	4	5	6
1	0	−1	0	−1	0	−1
2	1	0	1	1	1	0
3	0	−1	0	−1	1	0
4	1	−1	1	0	1	1
5	0	−1	−1	−1	0	−1
6	1	0	0	−1	1	0

.

One can remark that, for example, the triple

$$\begin{matrix} U[1,3] & U[1,5] \\ & U[3,5] \end{matrix} = \begin{matrix} 0 & 0 \\ & 1 \end{matrix}$$

does not belong to the set T_{13}*.*

However, since U comes from matrices having this transitivity property through the Max operation, among the 14 triples contradicting this property, half of them cannot be in U, namely, the triples
$\begin{smallmatrix} 0 & 0 \\ & -1 \end{smallmatrix}$, $\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 0 & 1 \\ & -1 \end{smallmatrix}$, $\begin{smallmatrix} -1 & 0 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} -1 & 0 \\ & -1 \end{smallmatrix}$, $\begin{smallmatrix} -1 & 1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} -1 & 1 \\ & -1 \end{smallmatrix}$.

Let us verify for example that $\begin{smallmatrix} 0 & 0 \\ & -1 \end{smallmatrix}$ cannot be in U: this triple comes from two triples of T_{13} $\begin{smallmatrix} x & y \\ & -1 \end{smallmatrix}$ and $\begin{smallmatrix} x' & y' \\ & -1 \end{smallmatrix}$ with $x, x', y, y' \leq 0$. Hence $y = y' = -1$, and so we get a contradiction with $0 = \text{Max}\{y, y'\} = -1$. ◻

The other triples receive an analogous treatment.

So the only triples not in T_{13} that can be found in U are the following 7:
$\begin{smallmatrix} 0 & 0 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 0 & -1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 0 & -1 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 0 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix}$.
They are the inverses of the others.

Let T_{20} be the set of triples obtained adding these seven triples to T_{13}.

If T is a subset of the set T_{27} of all the possible triples, let $\mathcal{M}^T(p_1, \ldots, p_n)$ be the set of matrices M in $\mathcal{M}(p_1, \ldots, p_n)$ such that all the triples $(M[i,j], M[i,k], M[j,k])$ belong to T. In particular, $\mathcal{M}^{T_{27}}(p_1, \ldots, p_n) = \mathcal{M}(p_1, \ldots, p_n)$ and $\mathcal{M}^{T_{13}}(p_1, \ldots, p_n) = \mathcal{M}^*(p_1, \ldots, p_n)$.

It is remarkable that, for each of the seven new triples there exists, in the set T_{13} of allowed triples, a unique minimum triple that is bigger than it, respectively:

$\begin{smallmatrix} 0 & 1 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 0 & 0 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 0 & 1 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 1 \\ & 0 \end{smallmatrix}$, $\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}$.

Let $\odot$ be the operation over $\{-1, 0, 1\}$ defined by the table

$\odot$	−1	0	1
−1	−1	−1	−1
0	−1	0	1
1	−1	1	1

.

We define an operation $\bigodot$ over the matrices in $\mathcal{M}$ by the following: $M \bigodot N$ is the matrix in $\mathcal{M}$ whose coefficients of the upper triangular part are $M \bigodot N[i,k] = \mathrm{Max}_{i \leq j \leq k} M[i,j] \odot N[j,k]$.

EXAMPLE 5.2. *Going further with the preceding example, we obtain for* $U \bigodot U$

	1	2	3	4	5	6
1	0	−1	0	−1	1	0
2	1	0	1	1	1	1
3	0	−1	0	−1	1	0
4	1	−1	1	0	1	1
5	−1	−1	−1	−1	0	−1
6	0	−1	0	−1	1	0

LEMMA 5.1. *Let M be a matrix of $\mathcal{M}(p_1, \dots, p_n)$. $M \bigodot M$ is a matrix dominating M belonging to $\mathcal{M}(p_1, \dots, p_n)$.*

Proof.

— *$M \bigodot M$ dominates M.*

Since $\forall i$ and j such that $i < k$, $M \bigodot M[i,k] = \mathrm{Max}_{i \leq j \leq k} M[i,j] \odot M[j,k] = \mathrm{Max}\{M[i,i] \odot M[i,k], \mathrm{Max}_{i<j\leq k} M[i,j] \odot M[j,k]\}$ holds, and since $M[i,i] = 0$, $M[i,i] \odot M[i,k] = M[i,k]$.

— $M \bigodot M$ has the commutativity property.

First, clearly in a submatrix $M_{i,i}$ the coefficients above the diagonal have value -1; moreover, in a submatrix $M_{i,k}$ with $i < k$, if i_1 and i_2 are the ranks of two letters a_i, and k_1 is the rank of a letter a_k, since $M \bigodot M[i_1, k_1] = 0$ or $M \bigodot M[i_1, k_1] = 1$ $\Longrightarrow \exists j \mid i_1 \leq j \leq k_1$ and $M[i_1, j] = 0$ or $M[i_1, j] = 1$ and $M[j, k_1] = 0$ or $M[j, k_1] = 1$; but M having itself the commutativity property, if $i_1 < i_2$, ($M[i_1, j] = 0$ or $M[i_1, j] = 1$) $\Longrightarrow M[i_2, j] = 1$, and hence $M[i_2, j] \odot M[j, k_1] = 1$, and $M \bigodot M[i_2, k_1] = 1$; in the same way, if i_1 is the rank of a letter a_i, and k_1 and k_2 are the ranks of two letters a_k, $k_1 < k_2$ and $M \bigodot M[i_1, k_1] = 0$ or $M \bigodot M[i_1, k_1] = -1 \Longrightarrow M \bigodot M[i_1, k_2] = -1$. ◻

Setting $U^{(1)} = U$ and $U^{(i+1)} = U^{(i)} \bigodot U^{(i)}$, starting from U and iterating the operation as long as the obtained matrix does not have the transitivity property, we get a strictly increasing (for the order $\preceq$) sequence of matrices in $\mathcal{M}(p_1, \dots, p_n)$: $U^{(1)} \prec U^{(2)} \prec \dots$. The process stops after repeating a finite number of times the operation, and one gets a matrix, denoted U^*, belonging to $\mathcal{M}^*(p_1, \dots, p_n)$.

According to Theorem 4.2, there exists a word of $\mathfrak{L}(p_1, p_2, \dots, p_n)$ having this matrix as its associated matrix. Let $f \bigtriangledown g$ be this word. It is a word in the class of f and g.

EXAMPLE 5.3. *Going further with the preceding example, $U \bigodot U$ owns the transitivity property. Hence we get $U^* = U \bigodot U$ which is the matrix associated to the word $f \bigtriangledown g = \{a_4\}\{a_1 a_3 a_4\}\{a_3\}\{a_2\}$.*

LEMMA 5.2. *Let M be a matrix of $\mathcal{M}^{T_{20}}(p_1, \dots, p_n)$. $M \bigodot M$ belongs to $\mathcal{M}^{T_{20}}(p_1, \dots, p_n)$.*

Proof. According to the preceding lemma, $M \bigodot M \in \mathcal{M}^{T_{27}}(p_1, \dots, p_n)$. Let us review the seven possible cases of triples $\begin{smallmatrix} M \odot M[i,j] & M \odot M[i,k] \\ & M \odot M[j,k] \end{smallmatrix}$ that do not belong to T_{20}.

— Case where $M \bigodot M[i,j] = -1$, $M \bigodot M[i,k] > -1$ and $M \bigodot M[j,k] < 1$.

In this case, $M[i,j] = -1$, and since $M \bigodot M[i,k] > -1$, there exists $j' \neq j$ such that $M[i,j'] > -1$ and $M[j',k] > -1$. Suppose that $j' < j$. Since $M \bigodot M[i,k] > -1$, $M[j',j] = -1$ holds. But $M[j,k] < 1$, and so the triple $\begin{smallmatrix} M[j',j] & M[j',k] \\ & M[j,k] \end{smallmatrix}$ is not in T_{20}, a contradiction. If $j' > j$, since $M[i,j] = -1$ and $M[i,j'] > -1$, $M[j,j'] = 1$ holds because this triple is in T_{20}, or $M[j,j'] = 1$ and $M[j',k] > -1$ implies $M \bigodot M[j,k] = 1$, a contradiction.

— Case where $M \bigodot M[i,j] = 0$, $M \bigodot M[i,k] > -1$, and $M \bigodot M[j,k] = -1$.

In this case, $M[j,k] = -1$, and since $M \bigodot M[i,k] > -1$, there exists $j' \neq j$ such that $M[i,j'] > -1$ and $M[j',k] > -1$. Suppose that $j < j'$. Since $M \bigodot M[j,k] = -1$, $M[j,j'] = -1$ holds. But $M[i,j] < 1$, and so the triple $\begin{smallmatrix} M[i,j] & M[i,j'] \\ & M[j,j'] \end{smallmatrix}$ is not in T_{20}, a contradiction. If $j > j'$, since $M[j,k] = -1$ and $M[j',k] > -1$, $M[j',j] = 1$ holds because this triple is in T_{20}, or $M[j',j] = 1$ and $M[i,j'] > -1$ implies $M \bigodot M[i,j] = 1$, a contradiction.

— Case where $M \bigodot M[i,j] = 0$, $M \bigodot M[i,k] = 1$ and $M \bigodot M[j,k] = 0$.

In this case, $M[i,j] < 1$ and $M[j,k] < 1$, and since $M \bigodot M[i,k] = 1$, there exists $j' \neq j$ such that $M[i,j'] = 1$ and $M[j',k] \geq 0$ or the converse. Suppose that $j' < j$. Since $M \bigodot M[i,j] = 0$, if $M[i,j'] = 1$, $M[j',j] = -1$ holds. But $M[j,k] < 1$, and so the triple $\begin{smallmatrix} M[j',j] & M[j',k] \\ & M[j,k] \end{smallmatrix}$ is not in T_{20}, a contradiction, and if $M[i,j'] = 0$, and hence $M[j',k] = 1$, which with $M[j,k] < 1$ implies $M[j',j] = 1$. Then $M[i,j'] = 0$ and $M[j',j] = 1$ and hence $M \bigodot M[i,j] = 1$, a contradiction with the hypothesis. If $j' > j$, then $M[j',k] > -1$ and $M \bigodot M[j,k] = 0$ implies that $M[j,j'] < 1$, which with $M[i,j] < 1$ implies either $M[i,j'] = -1$, a contradiction with the hypothesis, or $M[i,j] = M[j,j'] = M[i,j'] = 0$; but $M[i,j'] = 0 \Longrightarrow M[j',k] = 1$, which with $M[j,j'] = 0$ implies $M \bigodot M[j,k] = 1$, a contradiction with the hypothesis. ◻

Lemma 5.3. *If M is a matrix of $\mathcal{M}^*(p_1,\ldots,p_n)$ and N is a matrix of $\mathcal{M}^{T_{20}}(p_1,\ldots,p_n)$ that does not have the transitivity property, then $M \succeq N \Longrightarrow M \succeq N \bigodot N$.*

Proof. Suppose that M does not dominate $N \bigodot N$. Then there exist i and k such that $i < k$ and $N[i,k] \leq M[i,k] < N \bigodot N[i,k]$. Hence, there exists an integer j with $i < j < k$ such that $N \bigodot N[i,k] = N[i,j] \odot N[j,k] > N[i,k]$. So, the triple $\begin{smallmatrix} N[i,j] & N[i,k] \\ & N[j,k] \end{smallmatrix}$ does not belong to T_{13}. Let us review the seven possible cases:

— If $N[j,k] = 1$ and hence $N[i,j] > -1$, then $M[j,k] = 1$ and $M[i,j] > -1$ because M dominates N, and $M[i,k] < N \bigodot N[i,k] = 1$. In all cases, the triple $\begin{smallmatrix} M[i,j] & M[i,k] \\ & M[j,k] \end{smallmatrix}$ does not belong to T_{13}, a contradiction with $M \in \mathcal{M}^*(p_1,\ldots,p_n)$.

— If $N[i,j] = 1$ and $N[j,k] = 0$, then $M[i,j] = 1$ and $M[j,k] > -1$ because M dominates N, and $M[i,k] < N \bigodot N[i,k] = 1$. In all cases, the triple $\begin{smallmatrix} M[i,j] & M[i,k] \\ & M[j,k] \end{smallmatrix}$ does not belong to T_{13}, a contradiction with $M \in \mathcal{M}^*(p_1,\ldots,p_n)$.

— Last, if $N[i,j] = N[j,k] = 0$ and hence $N[i,k] > -1$, then $N \bigodot N[i,k] = 0$ and $M[i,k] < N \bigodot N[i,k] \Longrightarrow M[i,k] = -1$, and M dominates N implies $M[i,j] > -1$ and $M[j,k] > -1$. In all cases, the triple $\begin{smallmatrix} M[i,j] & M[i,k] \\ & M[j,k] \end{smallmatrix}$ does not belong to T_{13}, a contradiction with $M \in \mathcal{M}^*(p_1,\ldots,p_n)$. ◻

Proposition 5.4. *$\forall h \in \langle f \rangle \cap \langle g \rangle$, $f \bigtriangledown g \preceq h$ holds.*

Proof. Per absurdo, let $h \in \langle f \rangle \cap \langle g \rangle$ be such that $h \neq f \bigtriangledown g$, and let $M(h)$ be its associated matrix. So $M(h)$ dominates U. Hence $M(h) \succeq U^{(1)}$. If $U^{(1)}$ shares the transitivity property, $U^{(1)} = U^*$ holds, and hence $M(h) \succeq U^*$. Otherwise, the preceding lemma shows that $M(h) \succeq U^{(2)}$, and iterating until $U^{(i)} = U^*$, in all cases, $M(h) \succeq U^*$ holds. U^* being the matrix associated with $f \bigtriangledown g$, $f \bigtriangledown g \preceq h$ is true. ◻

We can now state the following theorem.

Theorem 5.5. *The relation $\longrightarrow^*$ gives to $\mathfrak{L}(p_1,p_2,\ldots,p_n)$ a structure of lattice.*

Proof. Proposition 5.4 means that the word $f \bigtriangledown g$ is a least upper bound of f and g over $[f]$, and $\mathfrak{L}(p_1,p_2,\ldots,p_n)$ has a structure of semilattice.

Symmetrically, $\longrightarrow^*$ confers to $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ a structure of lattice. □

As soon as $n > 2$, the lattice $\mathfrak{L}(p_1, p_2, \ldots, p_n)$ has got $\mathfrak{L}(1, 1, 1)$ as a sublattice. So it is not a modular lattice, hence not a distributive lattice.

Remarks. Since taking the inverse order on the letters of the underlying alphabet leads to the inverse relation of $\longrightarrow^*$, the least upper bound of the mirror images of two congruent S-words is the mirror image of the greatest lower bound of these two words.

Concerning the calculus of matrix U^*, recall that the operation $\bigodot$ replaces a triple in $T_{20} \backslash T_{13}$ by the triple in T_{13} that is the smallest bigger than itself and that this is always done by only increasing the value of the right upper element of the triangle given by the triple. As a consequence, the entries in the matrix that are just above the diagonal are unchanged by the operation, and clearly with each iteration at least one new parallel to the diagonal is definitively set. If s is the dimension of the matrix and if i is the integer for which $U^* = U^{(i)}$, $i \leq s - 2$ holds.

In [3], we present a complete C program, taking advantage of these remarks, computing the least upper bound and the greatest lower bound of two S-words with the method developed in this paper.

6. Conclusion. We have presented the formalism of S-words that we think is beneficial for treating Delannoy paths. The S-alphabets allow us to describe exactly the set of considered elementary steps. If someone would change the rule allowing only a part of the set of diagonal steps (for instance, only diagonal steps over the faces of a cube), one has only to consider the corresponding S-alphabet, a subalphabet of the S-alphabet we considered, and to proceed to the intersection with the set of words over this subalphabet.

We have associated with S-words, and hence to Delannoy paths, matrices that characterize them. Whatever the rule is, this allows us to order these Delannoy paths by means of the "domination" order, which is nothing more than the componentwise natural order, restricted to the upper triangular part, over these matrices.

The rules could be changed even more drastically to give the possibility of having diagonal steps composed of several elementary steps in a dimension. To describe such paths one has only to make use of multi-S-alphabets, i.e., multisets of letters. In this case, the commutativity property of the associated matrices would be weakened to the following:

In a submatrix $M_{i,j}$, supposing $i < j$,

(i) if i_1 and i_2 are the ranks of two letters a_i, and j_1 the rank of a letter a_j, then $i_1 < i_2 \Longrightarrow M[i_1, j_1] \leq M[i_2, j_1]$;

(ii) if i_1 is the rank of a letter a_i, and j_1 and j_2 the ranks of two letters a_j, then $j_1 < j_2 \Longrightarrow M[i_1, j_1] \geq M[i_1, j_2]$.

An essential part of our work was to exhibit a Thue system that allows us to define the set of Delannoy paths going from one point to another as a class for the congruence generated by the system and to prove that the rewriting process defines an order that coincides with the one of the associated matrices. We think that, if necessary, it would be possible for other rules to exhibit such a Thue system.

Appendix. Table of the sets of triples T_{13}, T_{20}, and T_{27}. We represent a triple $(M[i, j], M[i, k], M[j, k])$ under the triangular shape it appears in the matrices:

$$\begin{matrix} M[i,j] & M[i,k] \\ & M[j,k] \end{matrix}\ .$$

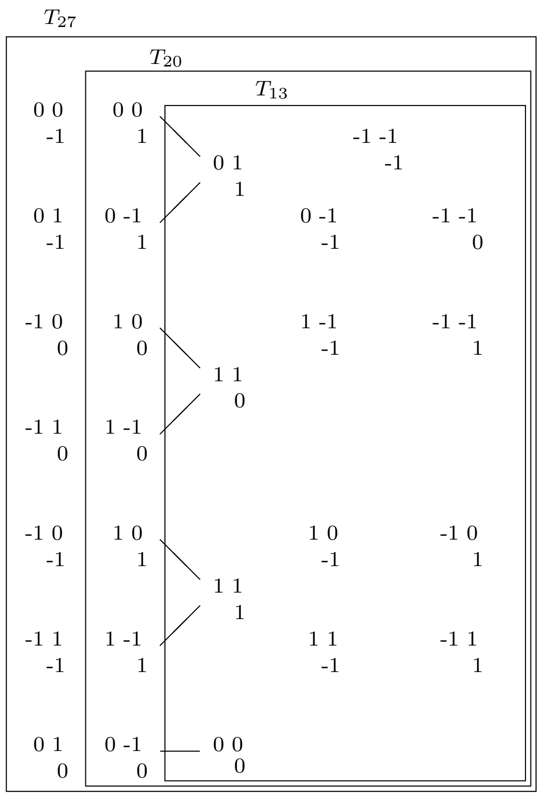

FIG. A.1. *The triples of* T_{13}*,* T_{20}*, and* T_{27}*.*

The triples of $T_{20}\backslash T_{13}$ are connected to the triples of T_{13} that cover them. These latter are obtained by replacing the right upper element by the value given by the operation $\odot$ applied to the other two elements of the triple.

REFERENCES

[1] J.-M. AUTEBERT, *Langages Algébriques*, Masson, Paris, 1987.
[2] J.-M. AUTEBERT, M. LATAPY, AND S. R. SCHWER, *Le treillis des chemins de Delannoy*, Discrete Math., 258 (2002) pp. 225–234.
[3] J.-M. AUTEBERT AND S. R. SCHWER, *Chemins de Delannoy généralisés*, LIPN internal report 2001-04, Villetaneuse, France, 2001.
[4] B. A. DAVEY AND H. A. PRIESTLY, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, UK, 1990.
[5] S. GINSBURG, *The Mathematical Theory of Context Free Languages*, McGraw-Hill, New York, 1966.
[6] M. JANTZEN, *Confluent String Rewriting*, EATCS Monogr. Theoret. Comput. Sci. 14, Springer-Verlag, Berlin, 1988.
[7] D. KROB, M. LATAPY, J.-C. NOVELLI, H. D. PHAN, AND S. R. SCHWER, *Pseudo-permutations* I: *First combinatorial and lattice properties*, in Proceedings of the 13th International Conference on Formal Power Series & Algebraic Combinatorics, Arizona State University, Tempe, AZ, 2001.

[8] S. R. Schwer, *Dépendances temporelles : les mots pour le dire*, LIPN internal report, Villetaneuse, France, 1997.

[9] S. R. Schwer, *S-arrangements avec répétitions*, C. R. Acad. Sci. Paris Ser. I, 334 (2002), pp. 261–266.

[10] G. Szász, *Théorie des treillis*, Dunod, Paris, 1971.

[11] E. W. Weisstein, *CRC Concise Encyclopaedia of Mathematics*, CRC Press, Boca Raton, FL, 2000.