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On generalized derivatives and formal powers for pseudoanalytic functions

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Abstract

We consider pseudoanalytic functions depending on two or three real variables. They are characterized by the corresponding Bers-Vekua equations. In the case of two dimensions we use the complex notation whereas for the case of three variables the concept of complex quaternions serves for our investigations. In a particular plane case we give an explicit representation of formal powers with which a complete system of solutions of the corresponding Bers-Vekua equation can be given. By an example we show how the concept of formal powers may also be applied to the case of three variables.

1 Introduction

The definition of the pseudoanalytic functions in the plane by L. Bers [2] is based on a derivation with respect to a pair of complex functions thereby assigning the role of the complex units 1 and i to two arbitrary functions. I.N. Vekue [9] considered a generalized Cauchy-Riemann system which lead him to generalized analytic functions.

Such pseudoanalytic functions in the plane have many applications. For example V.V. Kravchenko [4], [5] succeeded in treating the Schrödinger equation in two dimensions using the theory of pseudoanalytic functions. He constructed a complete set of solutions of this equation using formal powers.

In the following we give some definitions and results from Ber's theory. Let Ω be a simply connected domain in $\mathbb{R}^2$. A pair (F, G) of two complex functions with real Hölder-continuous first derivatives is called a generating pair if it satisfies the condition $\Im(\bar{F}G) > 0$ in Ω. The functions

$$a_{(F,G)} = \frac{F_{\bar{z}}\bar{G} - \bar{F}G_{\bar{z}}}{F\bar{G} - \bar{F}G}, \qquad b_{(F,G)} = \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\bar{G} - \bar{F}G}$$
$$A_{(F,G)} = -\frac{\bar{F}G_z - F_z\bar{G}}{F\bar{G} - \bar{F}G}, \qquad B_{(F,G)} = \frac{FG_z - F_zG}{F\bar{G} - \bar{F}G}$$

are called the characteristic coefficients of the pair (F, G).

Every complex valued function w defined in a subdomain of Ω admits the unique representation $w = \Phi F + \Psi G$ with real valued functions Φ, Ψ determined uniquely by w. The (F, G)-derivative $\dot{w}(z_0), z_0 \in \Omega$, is defined by

$$\dot{w}(z_0) \equiv \frac{d_{(F,G)} w(z_0)}{dz} := \lim_{z \to z_0} \frac{w(z) - \Phi(x_0, y_0) F(z) - \Psi(x_0, y_0) G(z)}{z - z_0} . \tag{1}$$

A function w for which the limit (1) exists for all $z_0 \in \Omega$ is called a (F, G)-pseudoanalytic function or short a pseudoanalytic function. The (F, G)-derivative exists if and only if $\Phi_{\bar{z}} F + \Psi_{\bar{z}} G = 0$ which is equivalent to the so called Bers-Vekua equation

$$w_{\bar{z}} = a_{(F,G)} w + b_{(F,G)} \bar{w} .$$

The derivative can be written in the form

$$\dot{w} = \Phi_z F + \Psi_z G = w_z - A_{(F,G)} w - B_{(F,G)} \bar{w} .$$

Consider two generating pairs (F, G) and (F_1, G_1) in Ω. A pair (F_1, G_1) is called successor of the pair (F, G) and (F, G) is called predecessor of (F_1, G_1) if

$$a_{(F_1,G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)} .$$

It follows that if w is a (F, G)-pseudoanalytic function and (F_1, G_1) is a successor of (F, G) then $\dot{w}$ is a (F_1, G_1)-pseudoanalytic function.

A sequence of generating pairs $\{(F_l, G_l)\}_{l \in \mathbb{Z}}$ is called a generating sequence if (F_{l+1}, G_{l+1}) is a successor of (F_l, G_l). If $(F_0, G_0) = (F, G)$ we say that (F, G) is embedded in the sequence $\{(F_l, G_l)\}$.

Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of a (F, G)-pseudoanalytic function w by the recursion formula

$$w^{[0]} = w , \quad w^{[l+1]} = \frac{d_{(F_l,G_l)} w^{[l]}}{dz} , \ l = 0, 1, 2, \ldots$$

Now formal powers $Z_l^{(n)}(a, z_0; z)$ with center $z_0 \in \Omega$, coefficient a and exponent n can be introduced by the following relations

$$Z_l^{(0)}(a, z_0; z) = \lambda F_l + \mu G_l , \ \lambda, \mu \in \mathbb{R} \quad \text{with} \quad \lambda F_l(z_0) + \mu G_l(z_0) = a \tag{2}$$

$$Z_l^{(n+1)}(a, z_0; z) = (n+1) \int_{z_0}^{z} Z_{l+1}^{(n)}(a, z_0; \zeta) \, d_{(F_l,G_l)} \zeta , \ n = 0, 1, 2, \ldots$$

Here the (F, G)-integral is given by

$$\int_\Gamma w(\zeta) \, d_{(F,G)} \zeta = \frac{1}{2} \left(F(z_1) \, \Re \int_\Gamma G^* w \, d\zeta + G(z_1) \, \Re \int_\Gamma F^* w \, d\zeta \right)$$

where Γ is a rectifiable curve leading from z_0 to z_1 and the functions F^* and G^* are defined as

$$F^* = -\frac{2\bar{F}}{F\bar{G} - \bar{F}G} , \quad G^* = \frac{2\bar{G}}{F\bar{G} - \bar{F}G} .$$

These formal powers have the following properties:

(P1) $Z_l^{(n)}(a, z_0; z)$ is a (F_l, G_l)-pseudoanalytic function of z,

(P2) for $a = a_1 + ia_2, a_1, a_2 \in \mathbb{R}$, we have

$$Z_l^{(n)}(a_1 + ia_2, z_0; z) = a_1 Z_l^{(n)}(1, z_0; z) + a_2 Z_l^{(n)}(i, z_0; z)\,,$$

(P3)
$$\frac{d_{(F_l,G_l)} Z_l^{(n)}}{dz} = n\, Z_{l+1}^{(n-1)}\,,$$

(P4)
$$Z_l^{(n)}(a, z_0; z) = a(z - z_0)^n + o((z - z_0)^n) \quad \text{for} \quad z \to z_0\,.$$

With such formal powers we can build the series

$$w(z) = \sum_{n=0}^{\infty} Z_0^{(n)}(a_n, z_0; z)$$

which is uniformly convergent in some neighbourhood of z_0. The limit is a (F_0, G_0)-pseudoanalytic function again. The series can be differentiated term by term giving

$$w^{[r]} = \sum_{n=r}^{\infty} n(n-1)\dots(n-r+1)\, Z_r^{(n-r)}(a_n, z_0; z)$$

from which the Taylor formula for the coefficients a_n is obtained as

$$a_n = \frac{1}{n!} w^{[n]}(z_0)\,.$$

Although the principle of constructing the formal powers is obvious, there exist only few explicit representations of such a complete system of solutions of a Bers-Vekua equation.

2 Formal powers for a class of pseudoanalytic functions

Here we consider the differential equation

$$w_{\bar{z}} = \frac{m}{\eta}\,\bar{w}\,, \quad m \in \mathbb{R},\ \eta = z + \bar{z}\,.$$

A generating pair of the pseudoanalytic functions described by this equation is given by $F = \eta^m\,, G = i\eta^{-m}$ leading to the characteristic coefficients

$$a_{(F,G)} = 0\,,\ b_{(F,G)} = \frac{m}{\eta}\,,\ A_{(F,G)} = 0\,,\ B_{(F,G)} = \frac{m}{\eta}\,.$$

The pair (F, G) can be embedded in the generating sequence $\{(F_l, G_l)\}_{l \in \mathbb{Z}}$ with

$$F_0 = F_{2k} = \eta^m, F_1 = F_{2k+1} = i\eta^m, G_0 = G_{2k} = \frac{i}{\eta^m}, G_1 = G_{2k+1} = -\frac{1}{\eta^m}, k \in \mathbb{Z}$$

with the corresponding characteristic coefficients

$$a_{(F_l,G_l)} = 0\,,\ b_{(F_l,G_l)} = \frac{(-1)^l m}{\eta}\,,\ A_{(F_l,G_l)} = 0\,,\ B_{(F_l,G_l)} = \frac{(-1)^l m}{\eta}\,.$$

For the formal powers $Z_l^{(n)} = Z_l^{(n)}(a, z_0; z)$ we can formulate the properties (P1)–(P4) in the following way:
(P1) leads to the differential equation

$$\frac{\partial}{\partial \bar{z}}\, Z_l^{(n)} = (-1)^l \frac{m}{\eta}\, \overline{Z_l^{(n)}} \tag{3}$$

and property (P3) gives

$$\frac{d_{(F_l,G_l)} Z_l^{(n)}}{dz} = \frac{\partial}{\partial z}\, Z_l^{(n)} - (-1)^l \frac{m}{\eta} \overline{Z_l^{(n)}} = n\, Z_{l+1}^{(n-1)}\,. \tag{4}$$

Combining (3) and (4) we have

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) Z_l^{(n)} = n\, Z_{l+1}^{(n-1)}\,. \tag{5}$$

With the operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$ eq. (5) can be written as

$$\frac{\partial}{\partial y}\, Z_l^{(n)} = i\, n\, Z_{l+1}^{(n-1)}\,. \tag{6}$$

For the sake of simplicity we choose $z_0 = \alpha, \alpha \in \mathbb{R}^+$. In view of property (P2) it is sufficient to determine the formal powers $Z_l^{(n)}(1, \alpha; z)$ and $Z_l^{(n)}(i, \alpha; z)\,.$

First let us compute the formal powers $Z_l^{(0)}(a, \alpha; z), a = 1, i,$ for which we have from (2) the conditions

$$a = 1:\ \lambda(2\alpha)^m + \mu\frac{i}{(2\alpha)^m} = 1\,, \quad a = i:\ \hat{\lambda}(2\alpha)^m + \hat{\mu}\frac{i}{(2\alpha)^m} = i\ , \quad \text{for } l \text{ even}\,,$$

$$a = 1:\ \lambda i(2\alpha)^m - \mu\frac{1}{(2\alpha)^m} = 1\,, \quad a = i:\ \hat{\lambda} i(2\alpha)^m - \hat{\mu}\frac{1}{(2\alpha)^m} = i\ , \quad \text{for } l \text{ odd}\,,$$

which gives the desired result:

$$Z_l^{(0)}(1, \alpha; z) = \left(\frac{x}{\alpha}\right)^{(-1)^l m}\,,\ Z_l^{(0)}(i, \alpha; z) = i\,\left(\frac{x}{gives\alpha}\right)^{(-1)^{l+1} m}\,.$$

Now from equation (6) in connection with (3) and considering the property (P4) we get the formal powers

$$Z_0^{(k)}(1, \alpha; z) = k! \sum_{l=0}^{k} \frac{1}{l!} (i\,y)^l\, \varphi_{k-l}(x)$$

with

$$\varphi_0(x) = \left(\frac{x}{\alpha}\right)^{(-1)^k m}, \quad \varphi_l(x) = x^{(-1)^{k+l} m} \int_\alpha^x \xi^{(-1)^{k+l+1} m} \varphi_{l-1}(\xi)\, d\xi\,,\ l = 1, \ldots, k$$

and

$$Z_0^{(k)}(i, \alpha; z) = i\, k! \sum_{l=0}^{k} \frac{1}{l!} (i\, y)^l\, \psi_{k-l}(x)$$

with

$$\psi_0(x) = \left(\frac{x}{\alpha}\right)^{(-1)^{k+1} m}, \quad \psi_l(x) = x^{(-1)^{k+l+1} m} \int_\alpha^x \xi^{(-1)^{k+l} m} \psi_{l-1}(\xi)\, d\xi\,,\ l = 1, \ldots, k\,.$$

3 Pseudoanalytic functions in the space

To consider three dimensional problems and pseudoanalytic functions defined there we use the algebra of complex quaternions which are called biquaternions also. This algebra is denoted by $\mathbb{H}(\mathbb{C})$ and defined by

$$\mathbb{H}(\mathbb{C}) = \left\{a : a = \sum_{k=0}^{3} a_k\, i_k\,,\ \ a_k \in \mathbb{C}\right\}$$

where the i_k are the standard basic quaternions with $i_3 = i_1 i_2$, and the complex imaginary unit i commutes with the $i_k, k = 1, 2, 3$. The quaternionic conjugation is given by $\bar{a} := a_0 - a_1 i_1 - a_2 i_2 - a_3 i_3$. Denote by $\mathbb{H}_k, k = 1, 2$, the set of reduced complex quaternions which have the form $a = a_0 + a_k i_k$, $a_0, a_k \in \mathbb{C}$. H.R. Malonek [7], [8] extended the concept of generating functions introduced by L. Bers [2] for the representation of pseudoanalytic functions in the space in the following way:

Let $F, G \in C^1(\Omega_0)$ be two arbitrary functions defined in $\Omega_0 \subset \mathbb{R}^3$ with values in $\mathbb{H}_1$ and $M, N \in C^1(\Omega_0)$ two arbitrary functions defined in Ω_0 with values in $\mathbb{H}_2$. Neither F, G nor M, N belong to the set of zero divisors and we require

$$G(x)\, \bar{F}(x) - \bar{G}(x)\, F(x) \neq 0 \quad \text{and} \quad N(x)\, \bar{M}(x) - \bar{N}(x)\, M(x) \neq 0 \quad \text{for} \quad x \in \Omega_0\,.$$

The pair (F, G) is called the $\mathbb{H}_1$-generating pair $\mathcal{H}_1$ and (M, N) the $\mathbb{H}_2$-generating pair $\mathcal{H}_2$.

Following Malonek a $\mathbb{H}(\mathbb{C})$-valued function $w : \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ written in the form

$$w(x) = F(x)\, [M(x)\Phi(x) + N(x)\Psi(x)] + G(x)\, [M(x)\mu(x) + N(x)\nu(x)]$$

is said to be (left) pseudoanalytic in the space with respect to the generating pairs $\mathcal{H}_1$ and $\mathcal{H}_2$ if the auxiliary function

$$W(x) = w(x) - \Big[F(x)\, [M(x)\Phi_* + N(x)\Psi_*] + G(x)\, [M(x)\mu_* + N(x)\nu_*]\Big]$$

with $\Phi_* = \Phi(x_*), \Psi_* = \Psi(x_*), \mu_* = \mu(x_*)$ and $\nu_* = \nu(x_*)$, (which is zero at x_*) satisfies the condition $D\,W(x_*) = 0$ for $x_* \in \Omega \subset \Omega_0$ arbitrary but fixed. Here D denotes the Dirac operator defined by

$$D := \sum_{k=1}^{3} i_k\, \partial_k \quad \text{with} \quad \partial_k = \frac{\partial}{\partial x_k}\,.$$

This leads to the generalized Bers-Vekua equation

$$Dw = aw + bJ_1(w) + cJ_2(w) + dJ_1J_2(w)\,. \tag{7}$$

The linear mappings $J_k : \mathbb{H}(\mathbb{C}) \to \mathbb{H}(\mathbb{C})$ are defined by their action on the basic quaternions: $J_l(i_l) = -i_l, l = 1, 2,\ J_k(i_l) = i_l, k \neq l, k, l = 1, 2$. The characteristic coefficients a, b, c and d in (7) are given by

$$\begin{aligned}
a &= D[FM]\bar{N}AB\bar{G} - D[FN]\bar{M}AB\bar{G} - D[GM]\bar{N}AB\bar{F} + D[GN]\bar{M}AB\bar{F}\\
b &= -D[FM]\bar{N}ABG + D[FN]\bar{M}ABG + D[GM]\bar{N}ABF - D[GN]\bar{M}ABF\\
c &= -D[FM]NAB\bar{G} + D[FN]MAB\bar{G} + D[GM]NAB\bar{F} - D[GN]MAB\bar{F}\\
d &= D[FM]NABG - D[FN]MABG - D[GM]NABF + D[GN]MABF
\end{aligned}$$

Such generalized pseudoanalytic functions appear for example in connection with the Dirac–equation with a vectorial electromagnetic potential (see e.g. [3], [6]). The Dirac equation with vector potential is related to the normal Dirac equation without potential like pseudoanalytic functions are related to holomorphic functions.

In particular in [1] the following generalized Bers-Vekua equation

$$D^* w = \frac{m}{x_1}\,\bar{w}\,,\ m \in \mathbb{R}\,, \tag{8}$$

with $D^* = -i_1 D$ was considered. For $m \in \mathbb{N}$ a connection between the solutions of (8) and the monogenic functions – i.e. those functions which obey the differential equation $D^*\varphi = 0$ – was presented. For the case $m = 1$ equation (8) can be given in the form

$$D^* w = \frac{1}{2x_1}(-w + J_1(w) + J_2(w) + J_1J_2(w))$$

and the functions

$$F = 1\,,\ G = i_1 \quad \text{and} \quad M = x_1^{-1}(x_1 - x_3 i_2)\,,\ N = x_1^{-1} i_2$$

represent two generating pairs of the generalized pseudoanalytic functions described by it. Now we can define a generalized derivative of w in the sense of Bers in the form

$$\dot{w} := \overline{D^*} w + \frac{1}{2x_1}(w + J_1(w) - J_2(w) + J_1J_2(w))$$

which is a solution of the Bers-Vekua equation

$$D^*\dot{w} = \frac{1}{2x_1}(-\dot{w} + J_1(\dot{w}) - J_2(\dot{w}) + J_1J_2(\dot{w}))\,.$$

That means that $\dot{w}$ is a pseudoanalytic function again. This could be considered as a hint to the existence of some generating sequences which could imply the possibility to define corresponding formal powers in the case of three dimensions also.

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