

AN EXPRESSION FOR GENERALIZED FIBONACCI NUMBERS

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An interesting expression for the Fibonacci numbers is presented here which relies on the modulo three value of the subscript.

$$(1a) \quad F_{3a} = 1 - \sum_{i=0}^{a-1} \binom{2i+a-1}{3i} (-1)^{a-i} 8^i \quad (a > 0)$$

$$(1b) \quad F_{3a+1} = 1 - 2 \sum_{i=0}^{a-1} \binom{2i+a}{3i+1} (-1)^{a-i} 8^i \quad (a \geq 0)$$

$$(1c) \quad F_{3a+2} = 1 - 4 \sum_{i=0}^{a-1} \binom{2i+a+1}{3i+2} (-1)^{a-i} 8^i \quad (a \geq 0)$$

This is a special case of a more general expression for the generalized Fibonacci numbers [1].

$$(2a) \quad V_{n,m} = 1 \quad (m = -n + 1, \dots, 0)$$

$$(2b) \quad V_{n,m} = \sum_{k=1}^n V_{n,m-k}$$

It is seen that $F_m = V_{2,m-2}$.

It is interesting to note that these numbers arise in the analysis of polyphase merge-sorting with $n+1$ tapes. The $V_{n,m}$ represent the total number of strings on all the tapes and also the length of strings (assuming initial length of 1) at each step of the polyphase merge process. A description of the polyphase merge-sort can be found in [2].

The general expression can be written as:

$$(3) \quad V_{n,a(n+1)+b} = 1 + 2^{b-1} (n-1) \sum_{i=0}^a \binom{in+a+b-1}{a-i} (-1)^{a+i} (2^{n+1})^i$$

(b = 1, \dots, n+1), (a \geq 0).

Let

$$f_n(x) = \sum_{m=0}^{\infty} V_{n,m} x^m$$

It follows immediately that

$$(1 - x - x^2 - \dots - x^n)f_n(x) = 1 + (n-1)x + (n-2)x^2 + \dots + x^{n-1} .$$

Therefore

$$\begin{aligned} f_n(x) &= \frac{1 + (n-1)x + (n-2)x^2 + \dots + x^{n-1}}{1 - x - x^2 - \dots - x^n} \\ (4) \quad &= \frac{1 + (n-2)x - x^2 - \dots - x^n}{1 - 2x + x^{n+1}} \\ &= \frac{1}{1-x} + \frac{(n-1)x}{1-2x+x^{n+1}} \end{aligned}$$

If

$$\frac{1}{1-2x+x^{n+1}} = \sum_{m=0}^{\infty} W_{n,m} x^m$$

the sequence $W_{n,m}$ is defined by:

$$(5a) \quad W_{n,m} = 0 \quad (m < 0)$$

$$(5b) \quad W_{n,m} = 1 \quad (m = 0)$$

$$(5c) \quad W_{n,m} = 2W_{n,m-1} - W_{n,m-n-1} \quad (m > 0)$$

$$\text{From Eq. (4)} \quad V_{n,m} = 1 + (n-1)W_{n,m-1} \quad (m > 0)$$

Theorem:

$$(6) \quad W_{n,a(n+1)+b} = \sum_{j=0}^{\infty} \binom{a+b+nj}{(n+1)j+b} 2^{b+(n+1)j} (-1)^{a-j}$$

(This formula immediately yields the identity (3).)

Proof:

$$\frac{1}{1-2x+x^{n+1}} = \sum_{m=0}^{\infty} (2x - x^{n+1})^m$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{k} 2^k (-1)^{m-k} x^{(n+1)m-nk}$$

Rearranging this sum in terms of powers of x , let $(n+1)a + b = (n+1)m - nk$. It follows that $k = b \pmod{(n+1)}$, so $k = (n+1)j + b$ for some $j \geq 0$. Changing the sum on m and k into a sum on a , b and j , and noting that $m = a + b + nj$, results in:

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} x^{(n+1)a+b} \sum_{j=0}^{\infty} \binom{a+b+nj}{(n+1)j+b} 2^{(n+1)j+b} (-1)^{a-j}$$

This completes the proof. A similar method was used by Polya [3] to solve another recurrence relation.

Another interesting expression which arises from this analysis is the general expression for the numbers defined by:

$$(7a) \quad U_{n,m} = 0 \quad (m = -n+1, \dots, -1)$$

$$(7b) \quad U_{n,m} = 1 \quad (m = 0)$$

$$(7c) \quad U_{n,m} = \sum_{i=1}^m U_{n,m-i} \quad (m > 0)$$

It is seen that $F_m = U_{2,m-1}$.

These numbers also arise in the analysis of polyphase merge-sorting; they represent the number of strings produced at each step of the process.

The general expression is:

$$U_{n,a(n+1)+b} = 2^{b-1} \sum_{i=0}^a \left\{ \binom{in+a+b}{a-i} + \binom{in+a+b-1}{a-i-1} \right\} (-1)^{a-m} (2^{n+1})^n$$

($b = 0, \dots, n$), ($a+b > 0$)

The proof is similar to the above and is omitted.

ACKNOWLEDGEMENT

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REFERENCES

1. E. P. Miles, Jr., "Generalized Fibonacci Numbers and Associated Matrices," Amer. Math. Month. 67 (October 1960), pp 745—752.
2. R. L. Gilstad, "Polyphase Merge Sorting—an Advanced Technique," Proc. Eastern Joint Computer Conference, Dec. 1960.
3. G. Polya, Induction and Analogy in Mathematics, Princeton, Chapter 5, p. 77.

FIBONACCI YET AGAIN

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Consider a triangle such that the square of one side equals the product of the other two sides.

Then we have sides: X , \sqrt{XY} , and Y ; say $X > Y$.

Eliminating an common factor we may set $X = a^2$, $Y = b^2$, so that the "reduced" sides become a^2 , ab , b^2 .

Then, for a triangle, we must have $ab + b^2 > a^2$ which requires $(\sqrt{5} - 1)/2 < b/a < (\sqrt{5} + 1)/2$.

Hence a sufficient condition for a triangle that meets the requirements is

$$F_{2n-1}/F_{2n} < b/a < F_{2n}/F_{2n-1} \quad \text{with} \quad X = ka^2, \quad Y = kb^2 .$$

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