STUDIA MATHEMATICA 103 (1) (1992)

On generalized inverses in C^*-algebras

by

ROBIN HARTE (Belfast) and MOSTAFA MBEKHTA (Lille)

Abstract. We investigate when a C^*-algebra element generates a closed ideal, and discuss Moore–Penrose and commuting generalized inverses.

0. Suppose A is a ring, with identity 1 and invertible group A^{-1}. Then we call an element $a \in A$ *regular* if $a \in aAa$, giving it a *generalized inverse* $b \in A$ for which

$$a = aba. \tag{0.1}$$

Generalized inverses breed *idempotents*: if (0.1) holds then $ba = p = p^2$ and $ab = q = q^2$ satisfy $aA = qA$ and $Aa = Ap$ with, for each $x \in A$,

$$ax = 0 \Leftrightarrow px = 0 \quad \text{and} \quad xa = 0 \Leftrightarrow xq = 0.$$

The generalized inverse $b \in A$ of (0.1) can be *normalized*: if $c = bab$ then

$$a = aca \quad \text{and} \quad c = cac.$$

The passage from b to c does not alter the projections p and q; conversely, $c = bab$ is determined by p and q:

1. THEOREM. *If $a = aba$ then*

$$b'a = ba \text{ and } ab' = ab \Rightarrow b'ab' = b'ab = bab, \tag{1.1}$$

and if also $e = e^2$ and $f = f^2$ with $Aa = Ae$ and $aA = fA$ then

$$c = ebf \Rightarrow a = aca \text{ with } ca = e \text{ and } ac = f.$$

Proof. Once stated, this is very routine. ∎

Associated with $a \in A$ are the *multiplication operators* $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ from A to A; when in particular A is a Hausdorff *topological ring* (addition and multiplication continuous) then L_a and R_a are both

1991 *Mathematics Subject Classification*: 46L05.

continuous and each of the "kernel ideals"

$$a^{-1}(0) = L_a^{-1}(0) = \{x \in A : ax = 0\},$$
$$a_{-1}(0) = R_a^{-1}(0) = \{x \in A : xa = 0\}$$

closed. The same is not always the case for the "range ideals":

2. THEOREM. *If $a \in A$ is regular then so are L_a and R_a, and the range ideals $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed.*

Proof. If $a = aba$ then $L_a = L_a L_b L_a$ and $R_a = R_a R_b R_a$; hence also $aA = qA = (1-q)^{-1}(0)$ and $Aa = Ap = (1-p)_{-1}(0)$. ∎

Much of this note is concerned with the converse of Theorem 2; we begin with an observation about "neighbouring" idempotents ([1], Proposition 19.1; [5], Theorem 26; [6], Proposition I.4.2):

3. THEOREM. *If $p = p^2$ and $q = q^2$ then*

$$p(p-q)^2 = (p-q)^2 p \quad \text{and} \quad q(p-q)^2 = (p-q)^2 q.$$

If in particular

$$1-(p-q)^2 = r^{-1} \in A^{-1}$$

then $e = qpr$ is idempotent, with

$$eA = qA \quad \text{and} \quad e^{-1}(0) = p^{-1}(0),$$
$$Ae = Ap \quad \text{and} \quad e_{-1}(0) = q_{-1}(0).$$

Proof. This is easily checked, once the formula is given. ∎

Kovarik [7] calls this the "poor man's path" between p and q. The product pq of idempotents p and q need not be idempotent ([3], Theorem 2.5.4):

4. THEOREM. *If $p = p^2$ and $q = q^2$ in A satisfy*

$$(1-q)(1-p) = 0$$

then pq is idempotent, with

$$qA = p^{-1}(0) + pqA \quad \text{and} \quad p^{-1}(0) \cap pqA = \{0\},$$
$$Ap = q_{-1}(0) + Apq \quad \text{and} \quad q_{-1}(0) \cap Apq = \{0\}.$$

Proof. Again routine. ∎

If the ring A has an *involution* $* : A \to A$ then we can introduce the concept of a "Moore–Penrose inverse", in the sense of a (normalized) generalized inverse $b(= bab)$ for $a = aba \in A$ for which

$$ba = (ba)^* \quad \text{and} \quad ab = (ab)^*,$$

i.e., the induced projections are *self-adjoint*—strictly, this is implied when we call an idempotent a "projection". For example the adjoint is a Moore–Penrose inverse for a *partial isometry*, $a \in A$ for which $a = aa^*a$.

When they exist, Moore–Penrose inverses are unique, and double commute with an element and its adjoint:

5. THEOREM. *If $b \in A$ and $b' \in A$ are Moore–Penrose inverses for $a \in A$ then $b' = b$. If b is the Moore–Penrose inverse of $a \in A$ then*

$$ca = ac \text{ and } ca^* = a^*c \Rightarrow cb = bc. \tag{5.1}$$

Proof. With $p = ba$ and $p' = b'a$, by (0.1),

$$p'p = b'aba = b'a = p' \quad \text{and} \quad pp' = bab'a = ba = p.$$

Taking adjoints gives $p' = (p')^* = p^*(p')^* = pp' = p$, and similarly $q' = q$; now use (1.1). Towards (5.1) we claim that if b is the Moore–Penrose inverse for a in A and $p = ba$ the induced idempotent then

$$ca = ac \Rightarrow pc = pcp, \tag{5.2}$$
$$ca^* = a^*c \Rightarrow pcp = cp. \tag{5.3}$$

For (5.2) argue

$$pc = bac = bca = bcaba = bacba = pcp, \tag{5.4}$$

and similarly for (5.3); alternatively take adjoints in (5.2). Thus if c is in the double commutant of a and a^* then c commutes with $p = ba$, and similarly, or by "reversal of products", with $q = ab$. But now also $bc = babc = bcab = bacb = cbab = cb$, giving (5.1). ∎

We shall write a^+ for the Moore–Penrose inverse of $a \in A$. For example if $a \in A$ has Moore–Penrose inverse $b = a^+$ then also

$$\exists (a^*)^+ = (a^+)^* \tag{5.5}$$

and hence the Moore–Penrose inverse of a self-adjoint element will also be self-adjoint. In general (notice for example $a^* = a^*aa^+$)

$$(a^+)^{-1}(0) = (a^*)^{-1}(0) \quad \text{and} \quad a^+A = a^*A.$$

The Moore–Penrose inverse exists for all regular elements when A is a C^*-algebra [1], [3], [4]:

6. THEOREM. *If a is regular in a C^*-algebra A then it has a Moore–Penrose inverse $a^+ \in A$.*

Proof. We use the construction of Theorem 3 and apply Theorem 1: if $p = p^2 \in A$ then

$$1-(p-p^*)^2 = 1 + (p-p^*)^*(p-p^*)$$

is invertible [1], [3], [4] by the B^*-condition, so that Theorem 3 applies with

$$e = [p] = p^*p(1-(p-p^*)^2)^{-1},$$

and also, after "reversal of products", with

$$f = [q^*] = qq^*(1-(q-q^*)^2)^{-1},$$

where $a = aba$ with $ba = p$ and $ab = q$. We may now apply Theorem 1: evidently $a^+ = [p]b[q^*]$ is a Moore–Penrose inverse for a. ■

It is rather clear, if A has an involution, that a regular $\Leftrightarrow$ a^* regular; in a C^*-algebra we have more:

7. THEOREM. *If $a \in A$ for a C^*-algebra A then*

(7.1) $$a \text{ regular} \Leftrightarrow a^*a \text{ regular} \Leftrightarrow aa^* \text{ regular}.$$

In particular,

(7.2) $$(a^*a)^+ = a^+(a^+)^* \quad \text{and} \quad (aa^*)^+ = (a^+)^*a^+.$$

Proof. If more generally A has an involution and $a = aba \in A$ with $ab = (ab)^*$ then

$$a^*abb^*a^*a = a^*ab(ab)^*a = a^*ababa = a^*a,$$

so that $a^*a \in A$ is also regular; thus if in particular a has a Moore–Penrose inverse then both aa^* and a^*a must be regular. By Theorem 6 this applies to all regular $a \in A$ when A is a C^*-algebra. Conversely, if for example a^*a is regular then there is $c \in A$ for which $a^*a = a^*aca^*a$, which gives

$$(a - aca^*a)^*(a - aca^*a) = 0 \in A.$$

Now if the B^*-condition holds in A it follows that $a = aca^*a$; this with a similar argument for aa^* finishes the proof of (7.1). For the first part of (7.2) observe that if $a = aba$ with self-adjoint ba and ab then

$$bb^*a^*a = b(ab)^*a = baba = (ba)^*ba \text{ self-adjoint. } ■$$

Our main result is a converse for Theorem 2:

8. THEOREM. *If $a \in A$ for a C^*-algebra A then*

(8.1) $$aA \text{ closed} \Rightarrow a \text{ regular},$$

and hence also

$$aA = \operatorname{cl} aA \Leftrightarrow Aa = \operatorname{cl} Aa.$$

Proof. Begin with the special case of a *positive* element, which always has a "square root" [1], [3], [4]:

$$0 \le a \in A \Rightarrow \exists a^{1/2} = (a^{1/2})^* \in \operatorname{cl} aA.$$

If aA is closed it follows that there is $c \in A$ for which $a^{1/2} = ac$; but now

$$acc^*a = ac(ac)^* = (a^{1/2})^2 = a.$$

For general $a \in A$ the product a^*a is positive, and satisfies

$$\|ax\| = \|(a^*a)^{1/2}x\| \quad \text{for each } x \in A,$$

so that, since also $\operatorname{dist}(x, a^{-1}(0)) = \operatorname{dist}(x, (a^*a)^{-1/2}(0))$,

$$aA \text{ closed} \Leftrightarrow (a^*a)^{1/2}A \text{ closed}.$$

Now (8.1) follows from two applications of Theorem 7: if aA is closed then so is cA for the positive element $c = (a^*a)^{1/2}$, which by the first part of this argument is also regular; but now $a^*a = c^*c$ is regular, and hence also a. ■

With the help of the square root we can also see that the Moore–Penrose inverse of a positive element will always be positive:

$$0 \le a \Rightarrow a^+ = a^+aa^+ = (a^{1/2}a^+)^*a^{1/2}a^+.$$

We shall call an element $a \in A$ *decomposably regular* if $a \in aA^{-1}a$, i.e., it has an *invertible* generalized inverse, and *simply polar* ([2], Definition 3.1; [3], Definition 7.3.5) if $a \in a\operatorname{comm}(a)a$, i.e., it has a *commuting* generalized inverse. This is a very strong condition to impose:

9. THEOREM. *A normalized commuting generalized inverse is unique. If $a \in A$ has a commuting generalized inverse then it is decomposably regular, and*

(9.1) $$A = aA + a^{-1}(0) \quad \text{with } aA \cap a^{-1}(0) = \{0\},$$

(9.2) $$A = Aa + a_{-1}(0) \quad \text{with } Aa \cap a_{-1}(0) = \{0\}.$$

Proof. If $a = aba$ with $ba = ab$ and also $a = ab'a$ with $b'a = ab'$ then

$$ab' = abab' = bab'a = ba,$$

so that the projection $p = ba = ab$ is uniquely determined, and hence also by Theorem 1 the (normalized) commuting generalized inverse $b = bab$. We can—at the cost of normalization—convert b to an invertible generalized inverse:

$$c = b + (1-p),\ c' = a + (1-p) \Rightarrow a = aca \text{ and } c'c = 1 = cc'.$$

The decomposition (9.1) is accomplished by writing $x = px + (1-p)x$ for each $x \in A$, where $p = ab = ba$, and similarly for (9.2). ■

The conditions (9.1) and (9.2) are not together sufficient for $a \in aAa$ to be simply polar: for example ([2], Example 4.7; [3], (7.3.6.8)) take

$$a = T = \begin{pmatrix} W & -I \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} c_{00} \\ c_{00} \end{pmatrix} \to \begin{pmatrix} c_{00} \\ c_{00} \end{pmatrix}$$

with $(Wx)_n = (1/n)x_n$ $(n \in \mathbb{N})$ for each $x \in c_{00}$, the terminating sequences.

When $a \in A$ is simply polar then [2], [3] its "Drazin inverse" $a^\times = b$ and its *support projection* $p = ab = ba$ lie in the *double commutant* $\operatorname{comm}^2(a)$ of a in A, and all the powers a^n of a are regular: $a^n = a^nb^na^n$ for each $n \in \mathbb{N}$.

A normal element in a C^*-algebra with a generalized inverse has all these properties:

10. THEOREM. *The Moore–Penrose inverse a^+ of a regular normal element a in a C^*-algebra A is also normal and commutes with a:*

$$a \in aAa \text{ and } aa^* = a^*a \Rightarrow a^+a^{+*} = a^{+*}a^+ \text{ and } aa^+ = a^+a\,. \tag{10.1}$$

In particular, normal regular elements are simply polar.

Proof. If $a \in A$ is normal then $c = a$ lies in the double commutant of a and a^*, and therefore by (5.1) commutes with the Moore–Penrose inverse a^+. This gives the second part of (10.1); for the normality recall (5.5) and (7.2):

$$(a^*a)^+ = a^+a^{*+} = a^+a^{+*} \quad \text{and} \quad (aa^*)^+ = a^{*+}a^+ = a^{+*}a^+\,. \ \blacksquare$$

We are indebted to the referee for the argument (5.4), which enables us to deduce (5.1) without using the B^*-condition.

The special case of Theorem 8, in which $A = BL(X, X)$ is the algebra of all bounded operators on a Hilbert space X, tells us that an operator T has closed range in X if and only if the multiplication L_T has closed range in $BL(X, X)$. This can be seen directly: if, more generally, X, Y and $Z \neq \{0\}$ are normed spaces and $T \in BL(X, Y)$ then with the help of "rank one" operators $h \odot y : z \to h(z)y$ there is implication

$$L_T BL(Z, X) \text{ closed in } BL(Z, Y) \Rightarrow T(X) \text{ closed in } Y\,,$$

for if $y = \lim Tx_n$ and if $z \in Z$ and $h \in Z^\dagger$ satisfy $h(z) = 1$ then there must be $U \in BL(Z, X)$ for which

$$L_T(h \odot x_n) = h \odot Tx_n \to h \odot y = L_T(U) = TU\,,$$

giving $y = T(Uz) \in T(X)$. Conversely, if the spaces X and Y are complete then $T(X)$ is closed if and only if $T^\wedge : X/T^{-1}(0) \to Y$ is bounded below, and the factorization $T = T^\wedge \circ \pi$ gives inclusion

$$\operatorname{Range} L_T \subseteq \operatorname{Range} L_{T^\wedge}\,. \tag{10.2}$$

If also the null space $T^{-1}(0)$ is complemented then there exists $\omega : X/T^{-1}(0) \to X$ such that $T^\wedge = T \circ \omega$, giving equality in (10.2). The same sort of argument shows that the right multiplication R_T has closed range if and only if the *dual* operator $T^\dagger : Y^\dagger \to X^\dagger$ has closed range, provided the closure of the range of T is complemented.

References

[1] K. R. Goodearl, *Notes on Real and Complex C^*-Algebras*, Shiva, 1982.
[2] R. E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Royal Irish Acad. 85A (1985), 151–176.

[3] R. E. Harte, *Invertibility and Singularity*, Dekker, 1988.
[4] R. E. Harte and M. Ó Searcóid, *Positive elements and the B^* condition*, Math. Z. 193 (1986), 1–9.
[5] I. Kaplansky, *Rings of Operators*, Benjamin, 1968.
[6] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1966.
[7] Z. V. Kovarik, *Similarity and interpolation between projectors*, Acta Sci. Math. (Szeged) 39 (1977), 341–351.

QUEEN'S UNIVERSITY
BELFAST, BT7 1NN
NORTHERN IRELAND, U.K.

UNIVERSITÉ DE LILLE
59655 VILLENEUVE D'ASCQ CEDEX
FRANCE

Received September 17, 1991
Revised version January 22, 1992
(2840)