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On Generalized Quasi Einstein Manifolds

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Abstract. Quasi Einstein manifold is a simple and natural generalization of an Einstein manifold. The object of the present paper is to study some geometric properties of generalized quasi Einstein manifolds. Two non-trivial examples have been constructed to prove the existence of a generalized quasi Einstein manifold.

1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n}g, \tag{1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), (1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation :

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{2}$$

where a, b are smooth functions and A is a non-zero 1-form such that

$$g(X, U) = A(X), \tag{3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [3] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (2). We shall call A the associated 1-form and the unit vector field U is called the generator of the manifold. Such a manifold is denoted by $(QE)_n$.

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Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds. Also quasi Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity[7]. So quasi Einstein manifolds have some importance in the general theory of relativity.

The study of quasi Einstein manifolds was continued by M.C.Chaki [3], S.Guha [11], U.C.De and G.C.Ghosh ([5], [6]), P.Debnath and A.Konar [9], Özgür and Sular [21], Özgür [18] and many others. In a recent paper [25] Shaikh, Kim and Hui studied Lorentzian quasi Einstein manifolds

Several authors have generalized the notion of quasi Einstein manifold such as generalized quasi Einstein manifolds ([4], [20]), nearly quasi Einstein manifolds [8], generalized Einstein manifolds[2], super quasi Einstein manifolds [19], pseudo quasi Einstein manifolds [24] and $N(k)$-quasi Einstein manifolds ([17], [21], [18], [27], [13]).

In 2001, Chaki [4] introduced the notion of generalized quasi Einstein manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + c(A(X)B(Y) + A(Y)B(X)), \tag{4}$$

where a, b, c are certian non-zero scalars and A, B are two non-zero 1-form. The unit vector fields U and V corresponding to the 1-forms A and B respectively, defined by

$$g(X, U) = A(X), \qquad g(X, V) = B(X),$$

for every vector field X are orthogonal, that is, $g(U, V) = 0$. Such as n-dimensional manifold is denoted by $G(QE)_n$. The vector fields U and V are called the generators of the manifold and a, b, c are called the associated scalars. If $c = 0$, then the manifold reduces to a quasi Einstein manifold $(QE)_n$. It may be mentioned that De and Ghosh [5] introduced the same notion in another way. In 2008, De and Gazi [8] introduced nearly quasi Einstein manifolds $N(QE)_n$ and prove the existence of such a manifold by several examples.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a nearly quasi Einstein manifold if the Ricci tensor S is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bE(X, Y),$$

where E is a symmetric tensor of type $(0, 2)$.
In a Riemannian manifold (M^n, g) $(n > 3)$ the Weyl conformal curvature tensor C of type $(1, 3)$ is defined by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[&g(Y, Z)QX - g(X, Z)QY \\ &+S(Y, Z)X - S(X, Z)Y] \\ +\frac{r}{(n-1)(n-2)}[&g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where R, S, r denotes the Riemannian curvature tensor, the Ricci tensor of type $(0, 2)$ and the scalar curvature of the manifold respectively and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX, Y) = S(X, Y)$. If the dimension $n = 3$, then the conformal curvature tensor vanishes identically. The conformal curvature tensor have been studied by several authors in several ways such as ([12], [14], [15], [16], [26]) and many others.

The importance of a $G(QE)_n$ lies in the fact that a four-dimensional semi-Riemannian manifold is relevant to study of a general relativistic fluid spacetime admitting heat flux [23], where U is taken as the velocity vector of the fluid and V is taken as the heat flux vector field.
In the present paper we have studied $G(QE)_n$. The paper is organized as follows:
After introduction in Section 2, we study some basic results of $G(QE)_n$. We prove that if the generator U or V is a parallel vector field, then $G(QE)_n$ reduces to a $(QE)_n$. A necessary condition is obtained for a $G(QE)_n$ to be conformally conservative. Section 3 is devoted to study Ricci-semisymmetric $G(QE)_n$. In the next section we consider Ricci-recurrent $G(QE)_n$. Finally, we construct two non-trivial examples of a $G(QE)_n$.

2. Basic results

Suppose the generator U is a parallel vector field, then $R(X,Y)U = 0$. Hence

$$S(X,U) = 0. \tag{5}$$

Putting $Y = U$ in (4) gives

$$\begin{aligned} S(X,U) &= aA(X) + bA(X) + cB(X) \\ &= (a+b)g(X,U) + cg(X,V). \end{aligned} \tag{6}$$

Using (5) in (6) we get

$$(a+b)g(X,U) + cg(X,V) = 0. \tag{7}$$

Putting $X = V$ in (7) yields $c = 0$. That is, $G(QE)_n$ reduces to a $(QE)_n$. Again if V is a parallel vector field, then $S(X,V) = 0$. Setting $Y = V$ in (4), we obtain

$$\begin{aligned} S(X,V) &= ag(X,V) + bA(X)A(V) + c(A(X)B(V) + A(V)B(X)) \\ &= aB(X) + cA(X), \text{ since } A(V) = g(U,V) = 0. \end{aligned} \tag{8}$$

Putting $X = U$ in (8) gives

$$aB(U) + cA(V) = 0$$

which implies $c = 0$, since $B(U) = g(U,V) = 0$. In this case also $G(QE)_n$ reduces to a $(QE)_n$.
This leads to the following :

Theorem 2.1. *In a $G(QE)_n$ if either of the generators U, V is parallel, then the manifold reduces to a quasi Einstein manifold.*

Corollary 2.1. *If the generator U of a $G(QE)_n$ is a parallel vector field, then $a + b = 0$.*

Theorem 2.2. *In a $G(QE)_n$, QU is orthogonal to U iff $a + b = 0$.*

Proof. In the equation (5) let us set $Y = U$. Then we get

$$S(X,U) = ag(X,U) + bA(X)A(U) + c(A(X)B(U) + A(U)B(X)).$$

Again putting $X = U$, we obtain $S(U,U) = a + b$ and hence $g(QU,U) = a + b$, which implies that QU is orthogonal to U if and only if $a + b = 0$. □

Theorem 2.3. *A necessary condition for a $G(QE)_n$ to be conformally conservative is*

$$2(n-1)dc(U) = (n-2)da(U) + (2n+1)db(U).$$

Proof. A Riemannian manifold of dimension > 3 is said to be of conservative conformal curvature tensor if $divC = 0$ where 'div' denotes divergence. It is known[10] that $divC = 0$ implies

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}[d\tau(X)g(Y,Z) - d\tau(Z)g(X,Y)]. \tag{9}$$

Putting $X = Y = U$ and $Z = V$ in (9) we get

$$(\nabla_U S)(U,V) - (\nabla_V S)(U,U) = \frac{1}{2(n-1)}[d\tau(U)g(U,V) - d\tau(V)g(U,U)]. \tag{10}$$

From (4) we obtain

$$r = an + b \tag{11}$$

and

$$S(U, V) = c. \tag{12}$$

Using (11) and (12) in (10), we get

$$\nabla_U c - \nabla_V(a + b) = \frac{1}{2(n-1)}[-nda(U) - db(U)].$$

That is,

$$2(n-1)dc(U) - (n-2)da(U) - (2n+1)db(U) = 0.$$

This completes the proof. □

3. Ricci-semisymmetric $G(QE)_n$

A Riemannian manifold is said to be Ricci-semisymmetric if $R \cdot S = 0$ holds. In this section we study Ricci-semisymmetric $G(QE)_n$ and prove the following theorem:

Theorem 3.1. *A Ricci-semisymmetric $G(QE)_n$ is either nearly quasi Einstein manifold $N(QE)_n$ or, $A(R(X, Y)V) = 0$.*

Proof. Suppose that $R \cdot S = 0$. Then we get

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0.$$

Now using (4) we get

$$\begin{aligned} &ag(R(X, Y)Z, W) + bA(R(X, Y)Z)A(W) + c\{A(R(X, Y)Z)B(W) \\ &+A(W)B(R(X, Y)Z)\} + ag(Z, R(X, Y)W) + bA(Z)A(R(X, Y)W) \\ &+c\{A(Z)B(R(X, Y)W) + A(R(X, Y)W)B(Z)\} = 0. \end{aligned} \tag{13}$$

Taking $W = U$ and $Z = V$ in (13), we obtain

$$bA(R(X, Y)V) = 0, \text{since } B(R(X, Y)V) = g(R(X, Y)V, V) = 0.$$

Then either $b = 0$ or, $A(R(X, Y)V) = 0$.
If $b = 0$, from (4) we get

$$S(X, Y) = ag(X, Y) + c\{A(X)B(Y) + A(Y)B(X)\} = ag(X, Y) + cE(X, Y),$$

where $E(X, Y) = A(X)B(Y) + A(Y)B(X)$ is a symmetric tensor. Hence either the manifold is a nearly quasi Einstein manifold $N(QE)_n$ or, $A(R(X, Y)V) = 0$. □

4. Nature of the associated 1-forms of a $G(QE)_n$

In this section, we assume that the associated scalars a, b, c are constants and we enquire under what conditions the associated 1-forms A, B to be closed. Let us suppose that the manifold $G(QE)_n$ satisfies Codazzi type of Ricci tensor, that is, the Ricci tensor satisfies

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{14}$$

Using (4) in (14) we get

$$\begin{aligned} & b[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z] + c[(\nabla_X A)YB(Z) \\ & \quad +A(Y)(\nabla_X B)Z + (\nabla_X A)ZB(Y) + A(Z)(\nabla_X B)Y] \\ = & \ b[(\nabla_Y A)XA(Z) + A(X)(\nabla_Y A)Z] + c[(\nabla_Y A)XB(Z) \\ & \quad +A(X)(\nabla_Y B)Z + (\nabla_Y A)ZB(X) + A(Z)(\nabla_Y B)X]. \end{aligned} \tag{15}$$

Putting $Z = U$ in (15) and using $(\nabla_X A)U = 0$, since U is a unit vector, we obtain

$$\begin{aligned} b[(\nabla_X A)Y - (\nabla_Y A)X] = c[& A(X)(\nabla_Y B)U + (\nabla_Y B)X \\ & -A(Y)(\nabla_X B)U - (\nabla_X B)Y]. \end{aligned} \tag{16}$$

Now suppose $\nabla_Y U \perp V$, then

$$(\nabla_X B)U = 0. \tag{17}$$

Using (17) in (16), we get

$$b(dA)(X, Y) = -c(dB)(X, Y).$$

Hence we can state the following :

Theorem 4.1. *If a $G(QE)_n$ with associated scalars as constants satisfies Codazzi type of Ricci tensor, then the associated 1-form A is closed if and only if B is closed, provided $\nabla_Y U \perp V$.*

Next suppose the 1-form A is closed. Then

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

which implies

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0, \tag{18}$$

Hence the vector field U is irrotational. Putting $X = U$ in (18), we get

$$g(\nabla_U U, Y) + g(\nabla_Y U, U) = 0.$$

Since U is a unit vector, $g(\nabla_Y U, U) = 0$. Hence

$$g(\nabla_U U, Y) = 0$$

which implies $\nabla_U U = 0$, that is, the integral curves of the vector field U are geodesic.

Thus we can state the following :

Corollary 4.1. *If a $G(QE)_n$ with associated scalars as constants satisfies Codazzi type of Ricci tensor, then the vector field U is irrotational and the integral curves of the vector field U are geodesic provided 1-form B is closed and $\nabla_Y U \perp V$.*

5. Ricci-recurrent $G(QE)_n$

A Riemannian manifold is said to be Ricci-recurrent [22] if the Ricci tensor is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = D(X)S(Y,Z),$$

where D is a non-zero 1-form.
Let (M^n, g) be a $G(QE)_n$ manifold. If U is a parallel vector field, then $\nabla_X U = 0$, from which it follows that $R(X,Y)U = 0$. Therefore $S(Y,U) = 0$. Then from Theorem 1 and Corollary 1, we get $c = 0$ and $a + b = 0$. Therefore we can rewrite the equation (4) in the following form:

$$S(X,Y) = a[g(X,Y) - A(X)A(Y)].$$

Taking the covariant derivative of the above equation with respect to Z, we obtain

$$(\nabla_Z S)(X,Y) = da(Z)[g(X,Y) - A(X)A(Y)],$$

since $\nabla_X U = 0$ implies that $(\nabla_Z A)(X) = 0$. Therefore $(\nabla_Z S)(X,Y) = \frac{da(Z)}{a}S(X,Y)$, i.e., the manifold (M^n, g) is Ricci-recurrent.

Conversely, suppose that $G(QE)_n$ is Ricci-recurrent. Then

$$(\nabla_X S)(Y,Z) = D(X)S(Y,Z),\ D(X) \neq 0.$$

But

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Therefore

$$D(X)S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{19}$$

Putting $Y = Z = U$ in (19), we obtain

$$D(X)(a+b) = X(a+b) - S(\nabla_X U, U) - S(U, \nabla_X U). \tag{20}$$

From the equation (4), we obtain

$$\begin{aligned} S(\nabla_X U, U) &= ag(\nabla_X U, U) + bA(\nabla_X U) + cB(\nabla_X U) \\ &= (a+b)A(\nabla_X U) + cB(\nabla_X U) \end{aligned}$$

Hence from (20), we get

$$X(a+b) - D(X)(a+b) = 2(a+b)A(\nabla_X U) + 2cB(\nabla_X U). \tag{21}$$

Since $A(U) = 1$ implies $g(\nabla_X U, U) = 0$, i.e., $A(\nabla_X U) = 0$, therefore from (21) $B(\nabla_X U) = 0$ if and only if $d(a+b)(X) = (a+b)D(X)$. But $B(\nabla_X U) = 0$ implies that either U is a parallel vector field or $\nabla_X U \perp V$.

Thus we can state the following:

Theorem 5.1. *A $G(QE)_n$ is a Ricci-recurrent manifold provided the generator U is a parallel vector field. Conversely, if a $G(QE)_n$ is a Ricci-recurrent manifold, then either the vector field U is parallel or, $\nabla_X U \perp V$.*

6. Examples of generalized quasi Einstein manifolds

Example 6.1. *We consider a Riemannian manifold* $(\mathbb{R}^4, g)$ *endowed with the metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\ ]$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant and $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\begin{aligned}
\Gamma^1_{11} &= \frac{q}{1+2q},\ \Gamma^1_{22} = -\frac{q}{1+2q},\ \Gamma^1_{33} = -\frac{q}{1+2q}, \\
\Gamma^1_{44} &= -\frac{q}{1+2q},\ \Gamma^2_{12} = \frac{q}{1+2q},\ \Gamma^3_{13} = \frac{q}{1+2q}, \\
\Gamma^4_{14} &= \frac{q}{1+2q},
\end{aligned}$$

$$\begin{aligned}
R_{1221} &= R_{1331} = R_{1441} = \frac{q}{1+2q}, \\
R_{2332} &= R_{2442} = R_{3443} = \frac{q^2}{1+2q},
\end{aligned}$$

$$\begin{aligned}
R_{11} &= \frac{3q}{(1+2q)^2}, \\
R_{22} &= R_{33} = R_{44} = \frac{q}{1+2q}.
\end{aligned}$$

The scalar curvature is $\frac{6q(1+q)}{(1+2q)^3}$ which is non-zero and non-constant. We take scalars a, b and c as follows :

$$a = \frac{q}{(1+2q)^2},\ b = \frac{3q}{(1+2q)^3} - \frac{q}{(1+2q)^2},\ c = \frac{q}{1+2q}.$$

We choose the 1-forms as follows :

$$A_i(x) = \begin{cases} \sqrt{1+2q}, & \text{for } i=1 \\ 0, & \text{for } i=2,3,4 \end{cases}$$

and

$$B_i(x) = \begin{cases} \sqrt{\frac{1+2q}{3}}, & \text{for } i=2,3,4 \\ 0, & \text{for } i=1 \end{cases}$$

We have,

$$R_{11} = ag_{11} + bA_1A_1 + c(A_1B_1 + A_1B_1), \tag{22}$$

$$R_{22} = ag_{22} + bA_2A_2 + c(A_2B_2 + A_2B_2), \tag{23}$$

$$R_{33} = ag_{33} + bA_3A_3 + c(A_3B_3 + A_3B_3), \tag{24}$$

$$R_{44} = ag_{44} + bA_4A_4 + c(A_4B_4 + A_4B_4). \tag{25}$$

R.H.S. of (22) is $\frac{3q}{(1+2q)^2} = R_{11} = L.H.S$ of (22).
R.H.S. of (23) is $\frac{q}{(1+2q)} = R_{22} = L.H.S$ of (23).
Similarly we can show that the (24) and (25) are also true. We shall now show that the 1-forms are unit and orthogonal.

$$g^{ij}A_iA_j = g^{11}A_1A_1 + g^{22}A_2A_2 + g^{33}A_3A_3 + g^{44}A_4A_4 = 1,$$

$$g^{ij}B_iB_j = g^{11}B_1B_1 + g^{22}B_2B_2 + g^{33}B_3B_3 + g^{44}B_4B_4 = 1$$

and

$$g^{ij}A_iB_j = g^{11}A_1B_1 + g^{22}A_2B_2 + g^{33}A_3B_3 + g^{44}A_4B_4 = 0.$$

So, the manifold under consideration is a generalized quasi Einstein manifold.

Example 2. We consider the 3-dimensional manifold $M = \left\{(x, y, z) \in R^3\right\}$, where (x, y, z) are the standart coordinates in R^3. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = e_3, [e_1, e_3] = 0, [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{26}$$

which is known as Koszul's formula. This formula yields

$$\begin{aligned} \nabla_{e_1}e_1 &= 0,\ \nabla_{e_1}e_2 = \frac{1}{2}e_3,\ \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_2}e_1 &= -\frac{1}{2}e_3,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3}e_1 &= -\frac{1}{2}e_2,\ \nabla_{e_3}e_2 = \frac{1}{2}e_1,\ \nabla_{e_3}e_3 = 0. \end{aligned}$$

It is known that

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{27}$$

With the help of the above results and using (27), we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned} R(e_2, e_3)e_3 &= \frac{1}{4}e_2,\ R(e_1, e_3)e_3 = \frac{1}{4}e_1,\ \ R(e_1, e_2)e_2 = -\frac{3}{4}e_1, \\ R(e_2, e_3)e_2 &= -\frac{1}{4}e_3,\ R(e_1, e_3)e_1 = -\frac{1}{4}e_3,\ \ R(e_1, e_2)e_1 = \frac{3}{4}e_2, \end{aligned}$$

and the components which can be obtained from these by the symmetric properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(e_1,e_1) = -\frac{1}{2},\ S(e_2,e_2) = -\frac{1}{2},\ S(e_3,e_3) = \frac{1}{2},$$

and the scalar curvature is $-\frac{1}{2}$. Since $\{e_1,e_2,e_3\}$ is a frame field, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1'e_1 + b_1'e_2 + c_1'e_3,$$

and

$$Y = a_2'e_1 + b_2'e_2 + c_2'e_3,$$

where $a_i', b_i', c_i' \in R^+$ such that $a_1'a_2' + b_1'b_2' + c_1'c_2' \neq 0$. Hence

$$\begin{aligned} S(X,Y) &= -\frac{1}{2}(a_1'a_2' + b_1'b_2' - c_1'c_2') \\ g(X,Y) &= a_1'a_2' + b_1'b_2' + c_1'c_2' \end{aligned}$$

We choose the associated scalars as follows:

$$a = 1,\ b = -\frac{3}{2}\ \ and\ \ c = -\frac{1}{2}.$$

We also choose two associated 1-forms as follows:

$$\begin{aligned} A(X) &= \left(a_1'a_2' + b_1'b_2'\right)^{\frac{1}{2}},\ \ \forall X. \\ B(X) &= \frac{c_1'c_2'}{2\left(a_1'a_2' + b_1'b_2'\right)^{\frac{1}{2}}},\ \ \forall X. \end{aligned}$$

By virtue of the definition and chosen of two scalars and 1-forms, we can say that $\left(M^3, g\right)$ is a generalized quasi Einstein manifold whose associated scalars are constants.

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