

ON GENERALIZED RICCI-RECURRENT TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. Generalized Ricci-recurrent trans-Sasakian manifolds are studied. Among others, it is proved that a generalized Ricci-recurrent cosymplectic manifold is always recurrent. Generalized Ricci-recurrent trans-Sasakian manifolds of dimension ≥ 5 are locally classified. It is also proved that if M is one of Sasakian, α -Sasakian, Kenmotsu or β -Kenmotsu manifolds, which is generalized Ricci-recurrent with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere; then M is an Einstein manifold.

1. Introduction

A non-flat Riemannian manifold M is called a *generalized Ricci-recurrent manifold* ([3]) if its Ricci tensor S satisfies the condition

$$(1) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$

where ∇ is Levi-Civita connection of the Riemannian metric g , and A , B are 1-forms on M . In particular, if the 1-form B vanishes identically, then M reduces to the well known *Ricci-recurrent manifold* ([14]).

In [15], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c . He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu ([8])

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characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian ([8]).

In the Gray-Hervella classification of almost Hermitian manifolds ([6]), there appears a class, \mathcal{W}_4 , of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds ([5]). An almost contact metric structure on a manifold M is called a *trans-Sasakian structure* ([13]) if the product manifold $M \times \mathbb{R}$ belongs to the class \mathcal{W}_4 . The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ ([9]) coincides with the class of trans-Sasakian structures of type (α, β) . We note that trans-Sasakian structures include cosymplectic ([1]), α -Sasakian ([7]), Sasakian, β -Kenmotsu ([7]), Kenmotsu and normal locally conformal almost cosymplectic ([10]) or f -Kenmotsu ([11]) structures. In ([16]), it is proved that trans-Sasakian manifolds are always *generalized quasi-Sasakian* ([12]); thus these structures provide a large class of generalized quasi-Sasakian structures also.

Thus motivated sufficiently, in this paper we study generalized Ricci-recurrent trans-Sasakian manifolds. Section 2 contains necessary details about trans-Sasakian manifolds. In Section 3, for generalized Ricci-recurrent trans-Sasakian manifolds, a relation between the 1-forms A and B is established. Among others, it is proved that a generalized Ricci-recurrent cosymplectic manifold is always Ricci-recurrent. Generalized Ricci-recurrent trans-Sasakian manifolds of dimension ≥ 5 are also classified. In the last section, an expression for Ricci-tensor for a generalized Ricci-recurrent trans-Sasakian manifold with cyclic Ricci tensor is obtained. It is also proved that if M is one of Sasakian, α -Sasakian, Kenmotsu or β -Kenmotsu manifolds which is generalized Ricci-recurrent manifold with cyclic Ricci tensor and non-zero $A(\xi)$ everywhere, then M is an Einstein manifold.

2. Trans-Sasakian manifolds

Let M be an almost contact metric manifold ([1]) with an almost contact metric structure (φ, ξ, η, g) , that is, φ is a $(1, 1)$ tensor field, ξ is a vector field; η is a 1-form and g is a compatible Riemannian metric such that

$$(2) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$$

$$(3) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(4) \quad g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in TM$.

An almost contact metric structure (φ, ξ, η, g) in M is called a *trans-Sasakian structure* ([13]) if $(M \times \mathbb{R}, J, G)$ belongs to the class \mathcal{W}_4 ([6]), where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, \lambda d/dt) = (\varphi X - \lambda \xi, \eta(X)d/dt)$$

for all vector fields X on M and smooth functions λ on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition ([2])

$$(5) \quad (\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X)$$

for some smooth functions α and β on M , and we say that the trans-Sasakian structure is of type (α, β) .

Let (x, y, z) be Cartesian coordinates in \mathbb{R}^3 , then (φ, ξ, η, g) given by

$$\begin{aligned} \xi &= \partial/\partial z, & \eta &= dz - ydx, \\ \varphi &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, & g &= \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix} \end{aligned}$$

is a trans-Sasakian structure of type $(-1/(2e^z), 1/2)$ in \mathbb{R}^3 ([2]). In fact, in a 3-dimensional K -contact manifold with structure tensors (φ, ξ, η, g) , for a non-constant function f , defining $g' = fg + (1 - f)\eta \otimes \eta$; (φ, ξ, η, g') is a trans-Sasakian structure of type $(1/f, (1/2)\xi(\ln f))$ ([9]).

3. Generalized Ricci-recurrent trans-Sasakian manifolds

Let M be a $(2n + 1)$ -dimensional trans-Sasakian manifold. From (5) it is easy to see that

$$(6) \quad \nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi),$$

$$(7) \quad (\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y).$$

In view of (5), (6) and (7), we are able to state the following Lemma.

LEMMA 3.1. ([4]) *In a $(2n + 1)$ -dimensional trans-Sasakian manifold, we have*

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &\quad + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - (X\alpha)\varphi Y \\ (8) \quad &\quad + (Y\alpha)\varphi X - (X\beta)\varphi^2 Y + (Y\beta)\varphi^2 X, \end{aligned}$$

$$(9) \quad S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X) - (2n-1)X\beta - \eta(X)(\xi\beta) - (\varphi X)\alpha,$$

$$(10) \quad Q\xi = 2n(\alpha^2 - \beta^2)\xi - (2n-1)\text{grad}\beta - (\xi\beta)\xi + \varphi(\text{grad}\alpha),$$

where R and S are curvature and Ricci curvature tensors, while Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$. In particular, we have

$$(11) \quad S(\xi, \xi) = 2n(\alpha^2 - \beta^2 - \xi\beta).$$

Now, we prove the following

THEOREM 3.2. *Let M be a $(2n+1)$ -dimensional generalized Ricci-recurrent trans-Sasakian manifold. Then, the 1-forms A and B are related by*

$$(12) \quad \begin{aligned} B(X) &= 2n\{X(\alpha^2 - \beta^2 - \xi\beta) - (\alpha^2 - \beta^2 - \xi\beta)A(X)\} \\ &\quad - 2(2n-1)(\alpha\varphi X + \beta\varphi^2 X)\beta - 2(\alpha\varphi^2 X - \beta\varphi X)\alpha. \end{aligned}$$

In particular, we get

$$(13) \quad B(\xi) = 2n(\xi(\alpha^2 - \beta^2 - \xi\beta) - (\alpha^2 - \beta^2 - \xi\beta)A(\xi)).$$

Proof. Using (1) in

$$(14) \quad (\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

we get

$$(15) \quad A(X)S(Y, Z) + B(X)g(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = \xi$, in the above equation we obtain

$$S(\xi, \xi)A(X) + B(X) = XS(\xi, \xi) - 2S(\nabla_X \xi, \xi),$$

which in view of (11), (4) and (6) yields (12). The equation (13) is obvious from (12). \square

Let A^* and B^* be the associated vector fields of A and B , that is, $g(X, A^*) = A(X)$ and $g(X, B^*) = B(X)$.

COROLLARY 3.3. *In a $(2n+1)$ -dimensional generalized Ricci-recurrent α -Sasakian (resp. Sasakian) manifold, we have*

$$(16) \quad B = -2n\alpha^2 A \quad (\text{resp. } B = -2nA).$$

Thus, the associated vector fields A^* and B^* are in opposite directions.

Proof. A trans-Sasakian manifold of type $(\alpha, 0)$ is α -Sasakian ([7]). In this case α becomes a constant. If $\alpha = 1$, then α -Sasakian manifold is Sasakian. Thus, from the equation (12), the proof follows immediately. \square

COROLLARY 3.4. *In a $(2n + 1)$ -dimensional generalized Ricci-recurrent normal almost cosymplectic f -structure (or f -Kenmotsu) manifold we have*

$$(17) \quad B(X) = 2n((f^2 + \xi f)A(X) - X(f^2 + \xi f)) - 2(2n - 1)f(\varphi^2 X)f.$$

Proof. A trans-Sasakian structure with $\alpha = 0$ and $\beta \equiv f$ is a normal almost cosymplectic f -structure ([10]) (or f -Kenmotsu structure [11]). Thus, putting $\alpha = 0$ and $\beta \equiv f$ in the equation (12), we get (17). \square

COROLLARY 3.5. *For a $(2n + 1)$ -dimensional generalized Ricci-recurrent β -Kenmotsu (resp. Kenmotsu) manifold, we have*

$$(18) \quad B = 2n\beta^2 A \quad (\text{resp. } B = 2nA).$$

Thus, the associated vector fields A^ and B^* are in same direction.*

Proof. A trans-Sasakian structure is β -Kenmotsu ([7]) if $\alpha = 0$ and $\beta = \text{constant}$. In particular, 1-Kenmotsu structure is a Kenmotsu structure. Putting $f = \beta = \text{constant}$ (resp. $f = 1$) in (17), we obtain (18). \square

A trans-Sasakian structures of type $(0, 0)$ is cosymplectic ([1]). Thus, putting $\alpha = 0 = \beta$ in (12), we get $B = 0$. Hence, we have the following

THEOREM 3.6. *A generalized Ricci-recurrent cosymplectic manifold M is always Ricci-recurrent.*

Now, we give the following classification for generalized Ricci-recurrent trans-Sasakian manifold of dimension ≥ 5 locally.

THEOREM 3.7. *Let M be a generalized Ricci-recurrent trans-Sasakian manifold of dimension $(2n + 1) \geq 5$. Then*

1. *either M is Ricci-recurrent,*
2. *or $B + 2n\alpha^2 A = 0$,*
3. *or $B - 2n\beta^2 A = 0$,*

where α and β are non-zero constant.

Proof. We know that locally a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic, or α -Sasakian or β -Kenmotsu manifold ([9]). Hence, in view of Corollaries 3.3, 3.5 and Theorem 3.6, the proof is complete. \square

4. Generalized Ricci-recurrent trans-Sasakian manifolds with cyclic Ricci tensor

A Riemannian manifold is said to admit cyclic Ricci tensor if

$$(19) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

Now, we prove the following.

THEOREM 4.1. *In a $(2n + 1)$ -dimensional generalized Ricci-recurrent trans-Sasakian manifolds with cyclic Ricci tensor, the Ricci tensor satisfies*

$$\begin{aligned} & A(\xi)S(X, Y) \\ &= 2n\{(\alpha^2 - \beta^2 - \xi\beta)A(\xi) - \xi(\alpha^2 - \beta^2 - \xi\beta)\}g(X, Y) \\ & \quad + (2n - 1)\{A(X)Y\beta + A(Y)X\beta\} \\ & \quad - (2n - 1)(\xi\beta)\{\eta(Y)A(X) + \eta(X)A(Y)\} \\ (20) \quad & + A(X)(\varphi Y)\alpha + A(Y)(\varphi X)\alpha \\ & \quad - 2n\{\eta(X)Y(\alpha^2 - \beta^2 - \xi\beta) + \eta(Y)X(\alpha^2 - \beta^2 - \xi\beta)\} \\ & \quad + 2(2n - 1)\{\eta(X)(\alpha\varphi Y + \beta\varphi^2 Y)\beta + \eta(Y)(\alpha\varphi X + \beta\varphi^2 X)\beta\} \\ & \quad + 2\{\eta(X)(\alpha\varphi^2 Y - \beta\varphi Y)\alpha + \eta(Y)(\alpha\varphi^2 X - \beta\varphi X)\alpha\}. \end{aligned}$$

Proof. Suppose that M is a generalized Ricci symmetric manifold admitting cyclic Ricci tensor. Then in view of (1) and (19), we get

$$\begin{aligned} 0 &= A(X)S(Y, Z) + A(Y)S(Z, X) + A(Z)S(X, Y) \\ & \quad + B(X)g(Y, Z) + B(Y)g(Z, X) + B(Z)g(X, Y). \end{aligned}$$

Moreover, if M is trans-Saskian manifold, putting $Z = \xi$, in the above equation we get,

$$\begin{aligned} A(\xi)S(X, Y) &= -B(\xi)g(X, Y) - A(X)S(Y, \xi) - A(Y)S(X, \xi) \\ & \quad - B(X)\eta(Y) - B(Y)\eta(X), \end{aligned}$$

which in view of (13) and (9) gives (20). \square

COROLLARY 4.2. For a $(2n + 1)$ -dimensional generalized Ricci-recurrent manifold M with cyclic Ricci tensor, we have the following statements:

1. If M is an α -Sasakian manifold, then

$$A(\xi) S(X, Y) = 2n\alpha^2 A(\xi) g(X, Y).$$

2. If M is a Sasakian manifold, then

$$A(\xi) S(X, Y) = 2nA(\xi) g(X, Y).$$

3. If M is a f -Kenmotsu manifold, then

$$\begin{aligned} & A(\xi) S(X, Y) \\ = & 2n \{ \xi (f^2 + \xi f) - A(\xi) (f^2 + \xi f) \} g(X, Y) \\ & + (2n - 1) \{ A(X) Y f + A(Y) X f \} \\ & - (2n - 1) (\xi f) \{ \eta(Y) A(X) + \eta(X) A(Y) \} \\ & + 2n \{ \eta(X) Y (f^2 + \xi f) + \eta(Y) X (f^2 + \xi f) \} \\ & + 2(2n - 1) \{ \eta(X) (f\varphi^2 Y) f + \eta(Y) (f\varphi^2 X) f \}. \end{aligned}$$

4. If M is a β -Kenmotsu manifold, then

$$A(\xi) S(X, Y) = -2n\beta^2 A(\xi) g(X, Y).$$

5. If M is a Kenmotsu manifold, then

$$A(\xi) S(X, Y) = -2nA(\xi) g(X, Y).$$

6. If M is a cosymplectic manifold, then

$$A(\xi) S(X, Y) = 0.$$

A Riemannian manifold is an Einstein manifold if

$$S(X, Y) = \rho g(X, Y).$$

Therefore, in view of Corollary 4.2, we are able to state the following

THEOREM 4.3. Let M be generalized Ricci-recurrent manifold with cyclic Ricci tensor. If M is one of Sasakian, α -Sasakian, Kenmotsu and β -Kenmotsu manifolds with non-zero $A(\xi)$ everywhere, then M is Einstein.

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