International Scholarly Research Network
ISRN Geometry
Volume 2012, Article ID 309486, 12 pages
doi:10.5402/2012/309486

Research Article

On Generalized Sasakian-Space-Forms

H. G. Nagaraja, G. Somashekhara, and Savithri Shashidhar

Department of Mathematics, Bangalore University, Central College Campus, Bangalore, Karnataka 560 001, India

Correspondence should be addressed to H. G. Nagaraja, hgnagaraj1@gmail.com

Received 31 October 2012; Accepted 16 December 2012

Academic Editors: A. Morozov, A. Popov, and E. H. Saidi

The purpose of the present paper is to characterize pseudoprojectively flat and pseudoprojective semisymmetric generalized Sasakian-space-forms.

1. Introduction

Alegre et al. [1] introduced and studied the generalized Sasakian-space-forms. The authors Alegre and Carriazo [2], Somashekhara and Nagaraja [3, 4], and De and Sarkar [5, 6] studied the generalized Sasakian-space-forms. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian-space-form if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of M is given by

$$R(X,Y)Z = f_1 R_1(X,Y)Z + f_2 R_2(X,Y)Z + f_3 R_3(X,Y)Z, \tag{1.1}$$

for any vector fields X, Y, Z on M, where

$$\begin{gathered} R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z,)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi. \end{gathered} \tag{1.2}$$

In this paper, we study the curvature properties like flatness, symmetry, and semisymmetry properties in a generalized Sasakian-space-form by considering a pseudoprojective curvature tensor.

The paper is organized as follows. Section 2 of this paper contains some preliminary results on the generalized Sasakian-space-forms. In Section 3, we study the pseudoprojectively flat generalized Sasakian-space-form and obtain necessary and sufficient conditions for a generalized Sasakian-space-form to be pseudoprojectively flat. In the next section, we deal with pseudoprojectively semisymmetric generalized Sasakian-space-forms, and it is proved that a generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is pseudoprojectively flat and $f_1 = f_3$. The last section is devoted to the study of τ-flat and $\tau-\phi$-semi symmetric generalized Sasakian-space-forms. In this section, we prove that the associated functions f_1, f_2, f_3 are linearly dependent.

In a $(2n+1)$-dimensional almost contact metric manifold, the pseudoprojective curvature tensor $\widetilde{P}$ [7] is defined by

$$\widetilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

where a and b are constants and R, S, and r are the Riemannian curvature tensor of type $(0,2)$, the Ricci tensor, and the scalar curvature of the manifold, respectively. If $a = 1$, $b = -(1/2n)$, then (1.3) takes the form

$$\widetilde{P}(X,Y)Z = P(X,Y)Z, \tag{1.4}$$

where P is the projective curvature tensor. A manifold (M,ϕ,ξ,η,g) shall be called pseudoprojectively flat if the pseudoprojective curvature tensor $\widetilde{P} = 0$. It is known that the pseudoprojectively flat manifold is either projectively flat (if $a \neq 0$) or Einstein (if $a = 0$ and $b \neq 0$).

2. Preliminaries

A $(2n+1)$-dimensional C^∞-differentiable manifold M is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X,\phi Y) = d\eta(X,Y), \tag{2.6}$$

for all vector fields X and Y.

In a generalized Sasakian-space-form, the following hold:

$$\begin{aligned} R(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.9}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.10}$$

3. Pseudoprojectively Flat Generalized Sasakian-Space-Forms

If the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ under consideration is pseudoprojectively flat, then from (1.3) we have

$$\begin{aligned} {}'R(X,Y,Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)] \\ & + \frac{r}{(2n+1)a}\left(\frac{a}{2n} + b\right)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{3.1}$$

where a and b are constants and ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Now taking $Z = \xi$ in (3.1) and using (2.1), (2.2), (2.7), and (2.9), we get

$$\begin{aligned} (f_1 - f_3)[\eta(Y)g(X,W) - \eta(X)g(Y,W)] = {} & \frac{2nb}{a}(f_1 - f_3)(\eta(Y)g(X,W) - \eta(X)g(Y,W)) \\ & + \frac{r}{(2n+1)a}\left(\frac{a}{2n} + b\right) \\ & \times (\eta(Y)g(X,W) - \eta(X)g(Y,W)). \end{aligned} \tag{3.2}$$

Again putting $X = \xi$ in (3.2), we get

$$\left[\left(\frac{a+2nb}{a}\right)\left(\frac{2n(2n+1)(f_1 - f_3) - r}{2n(2n+1)}\right)\right][\eta(Y)\eta(W) - g(Y,W)] = 0. \tag{3.3}$$

The aforementioned equation implies

$$\left(\frac{a+2nb}{a}\right)\left[\frac{2n(2n+1)(f_1-f_3)-r}{2n(2n+1)}\right]=0. \tag{3.4}$$

That is, either

$$(a+2nb)=0 \tag{3.5}$$

or

$$r=2n(2n+1)(f_1-f_3). \tag{3.6}$$

If $a+2nb=0$, $a\neq 0$ and $b\neq 0$, then, from (1.3), it follows that $\widetilde{P}(X,Y)Z=aP(X,Y)Z$. Thus in this case pseudoprojective flatness and projective flatness are equivalent.

If $a+2nb\neq 0$, $a\neq 0$ and $b\neq 0$, then comparing (2.10) and (3.6), we get

$$3f_2+(2n-1)f_3=0. \tag{3.7}$$

Using (3.7) in (2.9), we get

$$S(X,Y)=2n(f_1-f_3)g(X,Y). \tag{3.8}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X=Y=e_i$ and summing over $1\leq i\leq 2n+1$, we obtain

$$r=2n(2n+1)(f_1-f_3). \tag{3.9}$$

This shows that $M(f_1,f_2,f_3)$ is Einstein with a scalar curvature $r=2n(2n+1)(f_1-f_3)$. Thus we state the following.

Theorem 3.1. *A pseudoprojectively flat generalized Sasakian-space-form is either projectively flat or an Einstein manifold with a scalar curvature* $r=[2n(2n+1)(f_1-f_3)]$.

Suppose that (3.7) holds. Then in view of (2.7) and (2.9), we can write (1.3) as

$$\begin{aligned}
{}'\widetilde{P}(X,Y,Z,W)=\ & af_1(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\
& + af_2[g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W)+2g(X,\phi Y)g(\phi Z,W)] \\
& + af_3[\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)+\eta(Y)g(X,Z)g(\xi,W) \\
& \quad -\eta(X)g(Y,Z)g(\xi,W)]+b[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)] \\
& -\left(\frac{r}{2n+1}\right)\left(\frac{a}{2n}+b\right)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)],
\end{aligned} \tag{3.10}$$

where

$$'\widetilde{P}(X,Y,Z,W) = g\left(\widetilde{P}(X,Y)Z,W\right). \tag{3.11}$$

Replacing X by ϕX and Y by ϕY, we get

$$\begin{aligned}\widetilde{P}(\phi X,\phi Y,Z,W) = {} & af_3(g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)) \\ & + af_2\left(g(\phi X,\phi Z)g\left(\phi^2 Y,W\right) - g(\phi Y,\phi Z)g\left(\phi^2 X,W\right)\right. \\ & \left.+2g\left(\phi X,\phi^2 Y\right)g(\phi Z,W)\right).\end{aligned} \tag{3.12}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold.

Taking $Y = W = e_i$ and summation over i, $1 \le i \le 2n+1$, we get

$$\sum_{i=1}^{2n+1}\widetilde{P}(\phi X,\phi e_i,Z,e_i) = af_3(g(\phi X,\phi Z)) + af_2\left(-g(\phi X,\phi Z)g(\phi e_i,\phi e_i) - g\left(\phi^2 X,\phi^2 Z\right)\right). \tag{3.13}$$

Again putting $X = Z = e_i$ and taking summation over i, $1 \le i \le 2n+1$, we get $f_2 = 0$ with $a \neq 0$. In view of (3.7), we get $f_3 = 0$.

Now (2.7) reduces to the form

$$R(X,Y)Z = f_1[g(Y,Z)X - g(X,Z)Y], \tag{3.14}$$

from which we have $S(X,Y) = 2nf_1 g(X,Y)$, and consequently

$$r = 2n(2n+1)f_1. \tag{3.15}$$

By using (3.14) and (3.15) in (1.3), we get $\widetilde{P}(X,Y)Z = 0$. This leads to the following.

Theorem 3.2. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is pseudoprojectively flat if and only if $a+2nb \neq 0$, $a \neq 0$, $b \neq 0$ and $3f_2 + (2n-1)f_3 = 0$.*

Alegre and Carriazo [2] proved that any contact metric generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with a dimension greater than or equal to five is a Sasakian manifold and f_1, f_2, and f_3 must be constants.

Thus from (3.14), we have the following theorem.

Theorem 3.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ with a dimension greater than or equal to 5 is of constant curvature f_1 if and only if $a+2nb \neq 0$, $a \neq 0$, $b \neq 0$, and $3f_2 + (2n-1)f_3 = 0$.*

4. Pseudoprojective Semisymmetric Generalized Sasakian-Space-Form

Definition 4.1. If a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ satisfies

$$R(X,Y)\cdot \widetilde{P} = 0, \tag{4.1}$$

then the manifold is said to be pseudoprojectively semisymmetric manifold.

By using (1.3), (2.1), (2.2), (2.7), and (2.9), we have

$$\begin{aligned}\eta\left(\widetilde{P}(X,Y)Z\right) = {} & \left[a(f_1-f_3)-\left(\frac{r}{2n+1}\right)\left(\frac{a}{2n}+b\right)\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \\ & + b[S(Y,Z)\eta(X)-S(Y,Z)\eta(Y)].\end{aligned} \tag{4.2}$$

Taking $Z=\xi$ in (4.2), we get

$$\eta\left(\widetilde{P}(X,Y)\xi\right) = 0. \tag{4.3}$$

Again putting $X=\xi$ in (4.2), we get

$$\begin{aligned}\left(\eta\left(\widetilde{P}(\xi,Y)Z\right)\right) = {} & \left[a(f_1-f_3)-\left(\frac{r}{2n+1}\right)\left(\frac{a}{2n}+b\right)\right][g(Y,Z)-\eta(Y)\eta(Z)] \\ & + b[S(Y,Z)-2n(f_1-f_3)\eta(Y)\eta(Z)].\end{aligned} \tag{4.4}$$

From (4.1), we have

$$\begin{aligned}&\left(R(X,Y)\widetilde{P}(U,V)W\right)-\widetilde{P}(R(X,Y)U,V)W \\ &\quad -\widetilde{P}(U,R(X,Y)V)W-\widetilde{P}(U,V)R(X,Y)W = 0.\end{aligned} \tag{4.5}$$

Taking $X=\xi$ and contracting the above with respect to ξ, we get

$$\begin{aligned}(f_1-f_3)\Big\{&{}'\overline{P}(U,V,W,Y)-\eta(Y)\eta\left(\overline{P}(U,V)W\right)+\eta(U)\eta\left(\overline{P}(Y,V)W\right) \\ &-g(Y,U)\eta\left(\overline{P}(\xi,V)W\right)+\eta(V)\eta\left(\overline{P}(U,Y)W\right)-g(Y,V)\eta\left(\overline{P}(U,\xi)W\right) \\ &+\eta(W)\eta\left(\overline{P}(U,V)Y\right)-g(Y,W)\eta\left(\overline{P}(U,V)\xi\right)\Big\} = 0.\end{aligned} \tag{4.6}$$

Putting $U=Y$ in (4.6) and with the help of (4.2) and (4.3), we get either

$$f_1 = f_3 \tag{4.7}$$

or

$$
\begin{aligned}
{}'\overline{P}(Y,V,W,Y) - g(Y,Y)\eta\left(\overline{P}(\xi,V)W\right) \\
- g(Y,V)\eta\left(\overline{P}(Y,\xi)W\right) + \eta(W)\eta\left(\overline{P}(Y,V)Y\right) = 0.
\end{aligned}
\tag{4.8}
$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold of the manifold. Putting $Y = e_i$ and taking summation over i, $1 \leq i \leq 2n+1$, and using (4.2) and (4.4), we obtain

$$S(V,W) = Ag(V,W) + B\eta(V)\eta(W), \tag{4.9}$$

where

$$
\begin{aligned}
A &= 2nf_1 + 3f_2 - f_3, \\
B &= (2n+1)\left[-3f_2 - (2n-1)f_3\right].
\end{aligned}
\tag{4.10}
$$

Now contracting (4.9), we obtain

$$r = (2n+1)A + B. \tag{4.11}$$

Using (4.10) in (4.11), we get

$$r = 2n(2n+1)\left(f_1 - f_3\right). \tag{4.12}$$

In view of (2.10), (4.12) yields

$$3f_2 + (2n-1)f_3 = 0. \tag{4.13}$$

From (2.9) and (4.13), we have

$$S(V,W) = 2n\left(f_1 - f_3\right)g(V,W). \tag{4.14}$$

Now using (4.12) and (4.14) in (4.2), we get

$$\eta\left(\widetilde{P}(U,V)W\right) = 0. \tag{4.15}$$

Plugging (4.15) in (4.6), we obtain

$$\widetilde{P}(U,V,W,Y) = 0. \tag{4.16}$$

Therefore by taking into account (4.7) and (4.16), we have either $f_1 = f_3$ or $M(f_1, f_2, f_3)$ is pseudoprojectively flat.

Conversely, suppose that $f_1 = f_3$. Then, from (2.1), (2.2) and (2.7), we have $R(\xi, X)Y = 0$. Hence $R(\xi, U) \cdot \widetilde{P} = 0$. If the space-form is pseudoprojectively flat then clearly it is pseudoprojectively semisymmetric. Hence we can state the following.

Theorem 4.2. *A $2n+1$-dimensional generalized Sasakian-space-form is pseudoprojectively semisymmetric if and only if the space form is either pseudoprojectively flat or $f_1 = f_3$.*

By combining Theorems 3.2 and 4.2, we have the following.

Corollary 4.3. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is pseudoprojectively flat if and only if $f_1 = f_3$ or $a + 2nb \neq 0$ and $3f_2 + (2n-1)f_3 = 0$.*

5. τ-Curvature Tensor in a Generalized Sasakian-Space-Form

In a $(2n+1)$-dimensional Riemannian manifold M, the τ-curvature tensor is given by [8]

$$\begin{aligned}\tau(X, Y)Z = {} & a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y + a_3 S(X, Y)Z \\ & + a_4 g(Y, Z)QX + a_5 g(X, Z)QY + a_6 g(X, Y)QZ \\ & + a_7 r\big(g(Y, Z)X - g(X, Z)Y\big),\end{aligned} \tag{5.1}$$

where $a_0, \ldots, a_7$ are some smooth functions on M. For different values of $a_0, \ldots, a_7$, the τ-curvature tensor reduces to the curvature tensor, quasiconformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, M-projective curvature tensor, W_i-curvature tensors $(i = 0, \ldots 9)$, and W_j^*-curvature tensors $(j = 0, 1)$.

Suppose that $M(f_1, f_2, f_3)$ is τ-flat. Then from (5.1), we have

$$\begin{aligned}-a_0 R(X, Y)Z = {} & a_1 S(Y, Z)X + a_2 S(X, Z)Y + a_3 S(X, Y)Z \\ & + a_4 g(Y, Z)QX + a_5 g(X, Z)QY + a_6 g(X, Y)QZ \\ & + a_7 r\big(g(Y, Z)X - g(X, Z)Y\big).\end{aligned} \tag{5.2}$$

In view of (2.7), (2.8), and (2.9) in (5.2), we have

$$\begin{aligned}& - a_0 \{ f_1 [g(Y, Z)X - g(X, Z)Y] + f_2 [g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ & \quad + f_3 [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi] \} \\ & \qquad = a_1 [(2nf_1 + 3f_2 - f_3)g(Y, Z) - (3f_2 + (2n-1)f_3)\eta(Y)\eta(Z)]X \\ & \qquad \quad + a_2 [(2nf_1 + 3f_2 - f_3)g(X, Z) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Z)]Y \\ & \qquad \quad + a_3 [(2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)]Z\end{aligned}$$

$$
\begin{aligned}
&+ a_4 g(Y,Z)\left[(2nf_1+3f_2-f_3)X-(3f_2+(2n-1)f_3)\eta(X)\xi\right] \\
&+ a_5 g(X,Z)\left[(2nf_1+3f_2-f_3)Y-(3f_2+(2n-1)f_3)\eta(Y)\xi\right] \\
&+ a_6 g(X,Y)\left[(2nf_1+3f_2-f_3)Z-(3f_2+(2n-1)f_3)\eta(Z)\xi\right] \\
&+ a_7 r\left[g(Y,Z)X-g(X,Z)Y\right].
\end{aligned} \tag{5.3}
$$

Putting $X=\phi Y$ in (5.3), we get

$$
\begin{aligned}
&-a_0\Big\{ f_1\left[g(Y,Z)\phi Y-g(\phi Y,Z)Y\right]+f_2\Big[g(\phi Y,\phi Z)\phi Y-g(Y,\phi Z,)\phi^2 Y+2g(\phi Y,\phi Y)\phi Z\Big] \\
&\quad + f_3\left[-\eta(Y)\eta(Z)\phi Y+g(\phi Y,Z)\eta(Y)\xi\right]\Big\} \\
&\qquad = a_1\left[(2nf_1+3f_2-f_3)g(Y,Z)-(3f_2+(2n-1)f_3)\eta(Y)\eta(Z)\right]\phi Y \\
&\qquad + a_2(2nf_1+3f_2-f_3)g(\phi Y,Z)Y+a_4(2nf_1+3f_2-f_3)g(Y,Z)\phi Y \\
&\qquad + a_5 g(\phi Y,Z)\left[(2nf_1+3f_2-f_3)Y-(3f_2+(2n-1)f_3)\eta(Y)\xi\right] \\
&\qquad + a_7 r\left[g(Y,Z)\phi Y-g(\phi Y,Z)Y\right].
\end{aligned} \tag{5.4}
$$

If we choose a unit vector U orthogonal to ξ and taking $Y=U$, then making use of (2.1) and (2.3) in (5.4), we obtain

$$
\begin{aligned}
&\left[-a_0 f_1+(a_2+a_5)(2nf_1+3f_2-f_3)-a_7 r+f_2\right]g(\phi U,Z)U \\
&\quad + \left[a_0(f_1+f_2)+(a_1+a_4)(2nf_1+3f_2-f_3)+a_7 r\right]g(U,Z)\phi U \\
&\quad + 2a_0 f_2 g(U,U)\phi Z=0.
\end{aligned} \tag{5.5}
$$

Putting $Z=U$ in (5.5), we have

$$
\lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3=0, \tag{5.6}
$$

where

$$
\begin{aligned}
\lambda_1 &= a_0+2n(a_1+a_4)+2n(2n+1)a_7, \\
\lambda_2 &= 3(a_0+a_1+a_4+2na_7), \\
\lambda_3 &= -(a_1+a_4+4na_7).
\end{aligned} \tag{5.7}
$$

Thus we have the following.

Theorem 5.1. *If a* $(2n+1)$*-dimensional generalized Sasakian-space-form* $M(f_1,f_2,f_3)$ *is* τ*-flat, then* (5.6) *holds.*

From the above theorem, we discuss the following cases.

Case (i). (1) If $M(f_1, f_2, f_3)$ is quasiconformally flat, then $a_1 = -a_2 = a_4 = -a_5$, $a_3 = a_6 = 0$, $a_7 = (-1/(2n+1))(a_0/2n + 2a_1)$. Putting these in (5.7), we obtain $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(2) If $M(f_1, f_2, f_3)$ is conharmonically flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 0$. Putting these in (5.7), we get $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for W_0^*-flat, W_1-flat, W_3-flat, W_9-flat spaces, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Case (ii). If $M(f_1, f_2, f_3)$ is conformally flat, then $a_0 = 1$, $a_1 = -a_2 = a_4 = -a_5 = -(1/(2n-1))$, $a_3 = a_6 = 0$, $a_7 = 1/2n(2n-1)$.

Putting these in (5.7), we obtain $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. Hence $f_2 = 0$.

Case (iii). (a) If $M(f_1, f_2, f_3)$ is pseudoprojectively flat, then $a_1 = -a_2, a_3 = a_4 = a_5 = a_6 = 0$, $a_7 = -(1/(2n+1))(a_0/2n + a_1)$.

By putting these values in (5.7), we have $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(b) If $M(f_1, f_2, f_3)$ is projectively flat, then $a_0 = 1$, $a_1 = -a_2 = -(1/2n)$, $a_3 = a_4 = a_5 = a_6 = a_7 = 0$.

Making use of the above functional values in (5.7), we get $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

Similarly for concircularly flat, M-projectively flat, W_0-flat, W_1^*-flat, W_2-flat, W_6-flat, and W_8-flat spaces, (5.7) gives $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$.

Case (iv). (a) If $M(f_1, f_2, f_3)$ is W_4-flat, then $a_0 = 1$, $a_5 = -a_6 = 1/2n$, $a_1 = a_2 = a_3 = a_4 = a_7 = 0$.

Putting these in (5.7), we obtain that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

(b) If $M(f_1, f_2, f_3)$ is W_5-flat, then $a_0 = 1$, $a_2 = -a_5 = -(1/2n)$, $a_1 = a_3 = a_4 = a_6 = a_7 = 0$. Putting these in (5.7), we have $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Similarly, for a W_7-flat space, (5.7) gives $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

Summarizing the above cases, we have the following corollaries.

Corollary 5.2. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either quasiconformally flat, conharmonically flat, W_0^*-flat, W_1-flat, W_3-flat, or W_9-flat, then f_1, f_2, and f_3 are linearly dependent.*

Corollary 5.3. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is conformally flat, then $f_2 = 0$.*

The above corollary was already proved by Kim [9] and Sarkar and De [10].

Corollary 5.4. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either pseudoprojectively flat, projectively flat, concircularly flat, M-projectively flat, W_0-flat, W_1^*-flat, W_2-flat, W_6-flat, or W_8-flat, then f_2 and f_3 are linearly dependent.*

Corollary 5.5. *If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is either W_4-flat, W_5-flat, or W_7-flat, then f_1 and f_2 are linearly dependent.*

5.1. $\tau - \phi$-Semisymmetric Generalized Sasakian-Space-Form

Definition 5.6. $M(f_1, f_2, f_3)$ is $\tau - \phi$-semisymmetric if

$$\tau(X, Y) \cdot \phi = 0 \tag{5.8}$$

holds in $M(f_1, f_2, f_3)$.

We know that

$$(\tau(X,Y)\cdot\phi)Z = \tau(X,Y)\phi Z - \phi(\tau(X,Y)Z). \tag{5.9}$$

From (5.8) and (5.9), we have

$$\tau(X,Y)\phi Z - \phi(\tau(X,Y)Z) = 0. \tag{5.10}$$

By using (5.1) in (5.10), we have

$$\begin{aligned} &a_0R(X,Y)\phi Z + a_1S(Y,\phi Z)X + a_2S(X,\phi Z)Y + a_3S(X,Y)\phi Z \\ &\quad + a_4g(Y,\phi Z)QX + a_5g(X,\phi Z)QY + a_6g(X,Y)Q(\phi Z) \\ &\quad + a_7r[g(Y,\phi Z)X - g(X,\phi Z)Y] - a_7r[g(Y,Z)\phi X - g(X,Z)\phi Y] \\ &\quad - \{a_0\phi(R(X,Y)\phi Z) + a_1S(Y,Z)\phi X + a_2S(X,Z)\phi Y + a_3S(X,Y)\phi Z \\ &\qquad + a_4g(Y,Z)\phi(QX) + a_5g(X,Z)\phi(QY) + a_6g(X,Y)\phi(QZ)\} = 0. \end{aligned} \tag{5.11}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Contracting (5.11) with respect to W and putting $Y = W = e_i$, also taking summation over i, $1 \le i \le 2n+1$, and making use of (2.1), (2.4), (2.7), (2.9), and (2.8), we have

$$[2a_1 + (2n+1)a_2]S(X,\phi Z) = Ag(X,\phi Z), \tag{5.12}$$

where

$$\begin{aligned} A = &\left[-(2n-1)a_0 + 4na_4 + 2n(2n+1)a_5 - 2n\left(4n^2-1\right)a_7\right]f_1 \\ &+ [2(n-1)a_0 + 6a_4 + 6na_5 - 6n(2n-1)a_7]f_2 \\ &+ [-2a_4 - 4na_5 + 4n(2n-1)a_7]f_3. \end{aligned} \tag{5.13}$$

Changing Z to ϕZ in (5.12) and also in view of (2.1) and (2.2), (2.9) yields

$$\begin{aligned} S(X,Z) = &\left[\frac{A}{(2a_1 + (2n+1)a_2)}\right]g(X,Z) \\ &+ \left[\frac{2n(2a_1 + (2n+1)a_2)(f_1 - f_3) - A}{(2a_1 + (2n+1)a_2)}\right]\eta(X)\eta(Z). \end{aligned} \tag{5.14}$$

Thus we can state the following.

Theorem 5.7. *A $\tau-\phi$-semisymmetric generalized Sasakian-space-form is η-Einstein provided* $(2a_1 + (2n+1)a_2) \neq 0$.

References

[1] P. Alegre, D. E. Blair, and A. Carriazo, "Generalized Sasakian-space-forms," *Israel Journal of Mathematics*, vol. 141, no. 1, pp. 157–183, 2004.

[2] P. Alegre and A. Carriazo, "Structures on generalized Sasakian-space-forms," *Differential Geometry and its Applications*, vol. 26, no. 6, pp. 656–666, 2008.

[3] G. Somashekhara and H. G. Nagaraja, "Generalized Sasakian-space-forms and trans-Sasakian manifolds," *Acta Mathematical Paedagogicae Academiae Nyiregyhazienis*, vol. 28, no. 2, 2012.

[4] G. Somashekhara and H. G. Nagaraja, "On K-torseforming vector field in a trans-Sasakian generalized Sasakian-space-form," *International Journal of Mathematical Archive*, vol. 3, no. 7, pp. 2583–2588, 2012.

[5] U. C. De and A. Sarkar, "Some results on generalized Sasakian-space-forms," *Thai Journal of Mathematics*, vol. 8, no. 1, pp. 1–10, 2010.

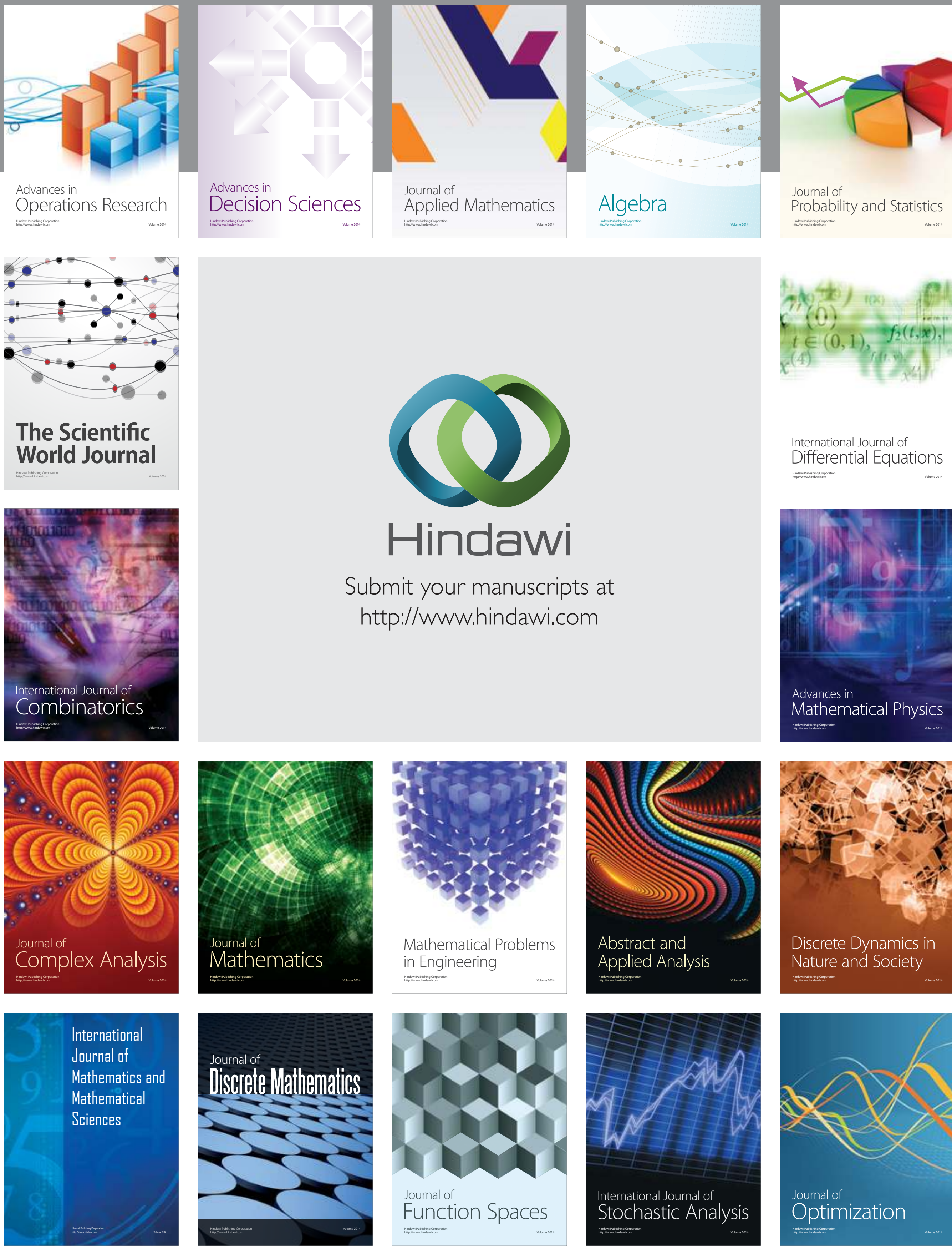
[6] U. C. De and A. Sarkar, "On the projective curvature tensor of generalized Sasakian-space-forms," *Quaestiones Mathematicae*, vol. 33, no. 2, pp. 245–252, 2010.

[7] B. Prasad, "A pseudo projective curvature tensor on a Riemannian manifold," *Bulletin of the Calcutta Mathematical Society*, vol. 94, no. 3, pp. 163–166, 2002.

[8] M. M. Tripathi and P. Gupta, "τ-curvature tensor on a semi-Riemannian manifold," *Journal of Advanced Mathematical Studies*, vol. 4, no. 1, pp. 117–129, 2011.

[9] U. K. Kim, "Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms," *Note di Matematica*, vol. 26, no. 1, pp. 55–67, 2006.

[10] A. Sarkar and U. C. De, "Some curvature properties of generalized Sasakian-space-forms," *Lobachevskii Journal of Mathematics*, vol. 33, no. 1, pp. 22–27, 2012.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization