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On generalized strongly modified h -convex functions

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Abstract

We derive some properties and results for a new extended class of convex functions, generalized strongly modified h -convex functions. Moreover, we discuss Schur-type, Hermite–Hadamard-type, and Fejér-type inequalities for this class. The crucial fact is that this extended class has awesome properties similar to those of convex functions.

Keywords: h -convex function; Modified h -convex function; Schur-type inequality; Hermite–Hadamard inequality; Fejér-type inequality

1 Introduction

Nowadays, in science and modern analysis the convexity plays an important role in economics, statistics, management science, engineering, and optimization theory. For instance, Barani et al. [1] presented the Hermite–Hadamard inequality for functions with preinvex absolute values of derivatives. Characterizations of convexity via Hadamard's inequality has been studied in [2]. In 2003, Dragomir and Pearce [3] proposed some applications of Hermite–Hadamard inequalities. In 2015, Dragomir [4] presented inequalities of Hermite–Hadamard type for h -convex functions on linear spaces. Some other interesting results can be found in books [5, 6] and research papers [7, 8]. In the recent years, generalizations and extensions were made rapidly for convex functions; for a recent generalization, see [9–11].

Convexity in the classical sense for a function $g : L = [a_1, a_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$g(ta_1 + (1-t)a_2) \leq tg(a_1) + (1-t)g(a_2),$$

where $a_1, a_2 \in L$ and $t \in [0, 1]$.

The work on the convexity is extended day by day by using some techniques; see [12–14]. The strongly extended convexity is widely used in optimization, economics, and nonlinear programming.

Convex functions satisfy several inequalities in which famous inequalities are of Schur type, Hermite–Hadamard-type, and Fejér-type inequalities. The Hermite–Hadamard-type inequality introduced by Jacques Hadamard for classical convex functions $g : L =$

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$[a_1, a_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ as

$$g\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx \leq \frac{g(a_1) + g(a_2)}{2}.$$

For extended versions of this inequality, see [12] and [13]. For further reading, see [15–19].

Lipót Fejér presented an extended version of the Hermite–Hadamard inequality, known as the Fejér inequality or a weighted version of the Hermite–Hadamard inequality. If $f : I \rightarrow \mathbb{R}$ is a convex function, then

$$g\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} w(x) dx \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} w(x)g(x) dx \leq \frac{g(a_1) + g(a_2)}{2} \int_{a_1}^{a_2} w(x) dx,$$

where $a_1 \leq a_2$, and $w : I \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. For further extended versions and development, see [20] and [8].

In this paper, we first present some preliminaries and basic results. In the next section, we investigate Schur-type, Hermite–Hadamard-type, and Fejér-type inequalities for the newly introduced class of functions.

2 Preliminaries

In this section, we investigate a new class of convexity by using a basic result. There is no loss of generality in the extended version of convexity. To get asymptotic results, it is necessary to put some restrictions: L is an interval in \mathbb{R} , and $\eta : A \times A \rightarrow B \subseteq \mathbb{R}$ is a bifunction.

Definition 1 (*h*-convex function [21]) Let $g, h : L \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then g is called an *h*-convex function if

$$g(ta_1 + (1 - t)a_2) \leq h(t)g(a_1) + h(1 - t)g(a_2)$$

for all $a_1, a_2 \in L$ and $t \in [0, 1]$.

Definition 2 (Modified *h*-convex function [13]) Let $g, h : L \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then g is called a modified *h*-convex function if

$$g(ta_1 + (1 - t)a_2) \leq h(t)g(a_1) + (1 - h(t))g(a_2) \tag{1}$$

for all $a_1, a_2 \in L$ and $t \in [0, 1]$.

Definition 3 (Generalized modified *h*-convex function) Let functions $g, h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then $g : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called a generalized modified *h*-convex function if

$$g(ta_1 + (1 - t)a_2) \leq g(a_2) + h(t)\eta(g(a_1), g(a_2)) \tag{2}$$

for all $a_1, a_2 \in I$ and $t \in [0, 1]$.

Definition 4 (Wright-convex function [20]) A function $g : L \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be Wright-convex if

$$g((1-t)a_1 + ta_2) + g(ta_1 + (1-t)a_2) \leq g(a_1) + g(a_2)$$

for all $a_1, a_2 \in L$ and $t \in [0, 1]$.

Definition 5 (Additivity) A function η is said to be additive if $\eta(x_1, y_1) + \eta(x_2, y_2) = \eta(x_1 + x_2, y_1 + y_2)$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$; see [22] for more detail.

Definition 6 (Nonnegative homogeneity) A function η is said to be nonnegatively homogeneous if $\eta(\lambda a_1, \lambda a_2) = \lambda \eta(a_1, a_2)$ for all $a_1, a_2 \in \mathbb{R}$ and $\lambda \geq 0$.

Definition 7 (Supermultiplicativity [23]) A function $g : L \subset \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be a supermultiplicative function if $g(a_1 a_2) \geq g(a_1)g(a_2)$ for all $a_1, a_2 \in L, t \in [0, 1]$.

Definition 8 (Similar-order functions [24]) Functions f and g are said to be of similar order on $L \subseteq \mathbb{R}$ if $(f(x) - f(y), g(x) - g(y)) \geq 0$ for all $x, y \in L$.

Now we are going to introduce a new extended definition of convexity.

Definition 9 (Generalized strongly modified h -convex function) Let $g, h : L \subset \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative functions. Then g is called a generalized strongly modified h -convex function if

$$g(ta_1 + (1-t)a_2) \leq g(a_2) + h(t)\eta(g(a_1), g(a_2)) - \mu t(1-t)(a_1 - a_2)^2 \tag{3}$$

for all $a_1, a_2 \in L$ and $t \in [0, 1]$.

Remark 1

1. Inequality (3) reduces to inequality (1) if $\mu = 0$ and $\eta(x, y) = x - y$.
2. Definition (9) becomes the definition of a classical convex function when $\mu = 0, \eta(x, y) = x - y$, and $h(t) = t$.
3. Inequality(3) reduces to inequality (2) when $\mu = 0$.
4. If $h(t) = t$, then definition (9) reduces to the definition of a strongly generalized convex function [12].

Example 1 A function $g : L = [a_1, a_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ is defined by $g(x) = x^2, \eta(a_1, a_2) = 2a_1 + a_2$, and $h(t) \geq t$, then g is a generalized strongly modified h -convex function.

3 Main results

This section contains some basic and straightforward results. The following proposition shows the linearity of the class of generalized strongly modified h -convex functions.

Proposition 1 Let f and g be generalized strongly modified h -convex functions where η is additive and nonnegatively homogeneous. Then for all $a, b \in \mathbb{R}, af + bg$ is also a generalized strongly modified h -convex function.

Proposition 2 *Let h_1, h_2 be nonnegative functions on L such that $h_2(t) \leq h_1(t)$. If g is a generalized strongly modified h_2 -convex function, then g is also a generalized strongly modified h_1 -convex function.*

Proof As g is generalized strongly modified h -convex function, for all $a_1, a_2 \in L$ and $t \in [0, 1]$, we have

$$\begin{aligned} g(ta_1 + (1 - t)a_2) &\leq g(a_2) + h_2(t)\eta(g(a_1), g(a_2)) - \mu t(1 - t)(a_1 - a_2)^2 \\ &\leq g(a_2) + h_1(t)\eta(g(a_1), g(a_2)) - \mu t(1 - t)(a_1 - a_2)^2. \end{aligned}$$

This completes the proof. □

Remark 2 If g is a generalized strongly modified h_1 -convex and $h_1(t) \leq h_2(t)$, then g is a generalized strongly modified h_2 -convex function.

Proposition 3 *Let f be a linear function such that $f(x) - f(y) = x - y$, and let g be a generalized strongly modified h -convex function. Then $g \circ f$ is also a generalized strongly modified h -convex function.*

Proof As f is a linear function such that $f(x) - f(y) = x - y$ and g is a generalized strongly modified h -convex function, for all $a_1, a_2 \in L$ and $t \in [0, 1]$, we get

$$\begin{aligned} (g \circ f)(ta_1 + (1 - t)a_2) &= g(tf(a_1) + (1 - t)f(a_2)) \\ &\leq (g \circ f)(a_2) + h(t)\eta((g \circ f)(a_1), (g \circ f)(a_2)) \\ &\quad - \mu t(1 - t)(f(a_1) - f(a_2))^2 \\ &= (g \circ f)(a_2) + h(t)\eta((g \circ f)(a_1), (g \circ f)(a_2)) \\ &\quad - \mu t(1 - t)(a_1 - a_2)^2. \end{aligned}$$

This shows that $g \circ f$ is a generalized strongly modified h -convex function. □

Proposition 4 *Let functions $g_j : L \subset \mathbb{R} \rightarrow \mathbb{R}$ be generalized strongly modified h -convex functions, $\sum_{j=1}^m \lambda_j = 1$, and let η be additive non-negatively homogeneous function. Then their linear combination $f : \mathbb{R} \rightarrow \mathbb{R}$ is also a generalized strongly modified h -convex function.*

Proof As $g_j : L \subset \mathbb{R} \rightarrow \mathbb{R}$ be generalized strongly modified h -convex functions, for $a_1, a_2 \in L$ and $t \in [0, 1]$, let

$$f(x) = \sum_{j=1}^m \lambda_j g_j(x).$$

Set $x = (ta_1 + (1 - t)a_2)$. Then

$$\begin{aligned} f(ta_1 + (1 - t)a_2) &= \sum_{j=1}^m \lambda_j g_j(ta_1 + (1 - t)a_2) \\ &\leq \sum_{j=1}^m \lambda_j g_j(a_2) + h(t) \sum_{j=1}^m \lambda_j \eta(g_i(a_1), g_i(a_2)) \\ &\quad - \mu t(1 - t)(a_1 - a_2)^2 \sum_{j=1}^m \lambda_j \\ &= f(a_2) + h(t) \eta\left(\sum_{j=1}^m \lambda_j g_i(a_1), \sum_{j=1}^m \lambda_j g_i(a_2)\right) \\ &\quad - \mu t(1 - t)(a_1 - a_2)^2 \\ &= f(a_2) + h(t) \eta(f(a_1), f(a_2)) - \mu t(1 - t)(a_1 - a_2)^2. \end{aligned}$$

This completes the proof. □

Corollary 1 *Every generalized strongly modified h -convex function is a generalized modified convex function.*

Proof Let g be a generalized modified h -convex function. Then

$$\begin{aligned} g(ta_1 + (1 - t)a_2) &\leq g(a_2) + h(t) \eta(g(a_1), g(a_2)) - \mu t(1 - t)(a_1 - a_2)^2 \\ &\leq g(a_2) + h(t) \eta(g(a_1), g(a_2)) \end{aligned}$$

for all $a_1, a_2 \in L \subset \mathbb{R}$. □

Corollary 2 *If g is generalized strongly convex function and $t \leq h(t)$, then g is a generalized strongly modified h -convex function.*

Theorem 1 (Schur-type inequality) *Let $g : L \rightarrow \mathbb{R}$ be a generalized strongly modified h -convex function, let h be a supermultiplicative function, and let $\eta : N \times N \rightarrow M$ be a bifunction for appropriate $A, B \subseteq \mathbb{R}$. Then for $a_1, a_2, a_3 \in L$ such that $a_1 < a_2 < a_3$ and $a_3 - a_1, a_3 - a_2, a_2 - a_1 \in L$, we have the inequality*

$$\begin{aligned} h(a_3 - a_1)g(a_2) &\leq h(a_3 - a_1)g(a_3) + h(a_3 - a_2) \eta(g(a_1), g(a_2)) \\ &\quad - \mu(a_3 - a_2)(a_2 - a_1)h(a_3 - a_1) \end{aligned} \tag{4}$$

if and only if g is a generalized strongly modified h -convex function.

Proof Let $a_1, a_2, a_3 \in L \subset \mathbb{R}$ be such that $\frac{(a_3 - a_2)}{(a_3 - a_1)} \in (0, 1) \subseteq L$, $\frac{(a_2 - a_1)}{(a_3 - a_1)} \in (0, 1) \subseteq L$, and $\frac{(a_3 - a_2)}{(a_3 - a_1)} + \frac{(a_2 - a_1)}{(a_3 - a_1)} = 1$. Then

$$h(a_3 - a_1) = h\left(\frac{a_3 - a_1}{a_3 - a_2}(a_3 - a_2)\right) \geq h\left(\frac{a_3 - a_1}{a_3 - a_2}\right)h(a_3 - a_2)$$

as h is supermultiplicative.

Suppose $h(a_3 - a_2) \geq 0$. Then by the definition of g we have

$$g(tx + (1 - t)y) \leq g(y) + h(t)\eta(g(x), g(y)) - \mu t(1 - t)(x - y)^2. \tag{5}$$

Inserting $\frac{(a_3 - a_2)}{(a_3 - a_1)} = t$, $x = a_1$, and $y = a_3$ into inequality (5), we obtain

$$\begin{aligned} g\left(\frac{(a_3 - a_2)}{(a_3 - a_1)}a_1 + \left(1 - \frac{(a_3 - a_2)}{(a_3 - a_1)}\right)a_3\right) &\leq g(a_3) + h\left(\frac{(a_3 - a_2)}{(a_3 - a_1)}\right)\eta(g(a_1), g(a_3)) \\ &\quad - \mu(a_3 - a_2)(a_2 - a_1) \\ &\leq g(a_3) + \frac{h(a_3 - a_2)}{h(a_3 - a_1)}\eta(g(a_1), g(a_3)) \\ &\quad - \mu(a_3 - a_2)(a_2 - a_1), \end{aligned} \tag{6}$$

$$\begin{aligned} g(a_2)h(a_3 - a_1) &\leq h(a_3 - a_1)g(a_3) \\ &\quad + h(a_3 - a_2)\eta(g(a_1), g(a_3)) \\ &\quad - \mu(a_3 - a_2)(a_2 - a_1)h(a_3 - a_1). \end{aligned}$$

Conversely, suppose inequality (4) holds and insert $a_1 = x$, $a_2 = tx + (1 - t)y$, and $a_3 = y$ into inequality (4). Then we get

$$\begin{aligned} h(y - x)g(tx + (1 - t)y) &\leq h(y - x)g(y) + h(y - x)h(t)\eta(g(x), g(y)) \\ &\quad - \mu h(y - x)t(y - x)(1 - t)(y - x), \\ g(tx + (1 - t)y) &\leq g(y) + h(t)\eta(g(x), g(y)) - \mu t(1 - t)(x - y)^2. \end{aligned}$$

This completes the proof. □

Remark 3

1. By taking $h(t) = t$ in (4) it is reduced to a Schur-type inequality for generalized strongly convex functions.
2. If $\mu = 0$ and $\eta(x, y) = x - y$, then (4) is reduced to a Schur-type inequality for modified h -convex functions; see [13].

Further, we will discuss the Hermite–Hadamard-type inequality for generalized strongly modified h -convex functions.

Theorem 2 (Hermit–Hadamard-type inequality) *Let function $g : L \rightarrow \mathbb{R}$ be a generalized strongly modified h -convex function on $[a_1, a_2]$ with $a_1 < a_2$. Then*

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2 - a_1)^2 &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx \\ &\leq g(a_2) + N_\eta - \frac{\mu}{6}(a_2 - a_1)^2. \end{aligned} \tag{7}$$

Proof Choosing $w = ta_1 + (1 - t)a_2$ and $z = (1 - t)a_1 + ta_2$, we have

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) &= g\left(\frac{w + z}{2}\right) \\ &= g\left(\frac{ta_1 + (1 - t)a_2 + (1 - t)a_1 + ta_2}{2}\right). \end{aligned}$$

Now by the definition of g we have

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) &\leq g((1 - t)a_1 + ta_2) + h\left(\frac{1}{2}\right)\eta(g(ta_1 + (1 - t)a_2), g((1 - t)a_1 + ta_2)) \\ &\quad - \mu\frac{1}{2}\left(1 - \frac{1}{2}\right)(a_2 - a_1)^2(2t - 1)^2. \end{aligned}$$

Integrating with respect to t on $[0, 1]$, we get

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) &\leq \int_0^1 g((1 - t)a_1 + ta_2) dt \\ &\quad + h\left(\frac{1}{2}\right) \int_0^1 \eta(g(ta_1 + (1 - t)a_2), g((1 - t)a_1 + ta_2)) dt \\ &\quad - \frac{\mu}{4}(a_2 - a_1)^2 \int_0^1 (2t - 1)^2 dt. \end{aligned}$$

Putting $x = (1 - t)a_1 + ta_2$, we get

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx + h\left(\frac{1}{2}\right)M_\eta - \frac{\mu}{12}(a_2 - a_1)^2, \\ g\left(\frac{a_1 + a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2 - a_1)^2 &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx. \end{aligned} \tag{8}$$

In the right-hand side of inequality (8), we set $x = ta_1 + (1 - t)a_2$, and using the definition of g , we get

$$\begin{aligned} \int_{a_1}^{a_2} g(x) dx &\leq (a_2 - a_1)g(a_2) + (a_2 - a_1) \int_0^1 h(t)\eta(g(a_1, g(a_2))) dt - \frac{\mu}{6}(a_2 - a_1)^2, \\ \frac{1}{(a_2 - a_1)} \int_{a_1}^{a_2} g(x) dx &\leq g(a_2) + N_\eta - \frac{\mu}{6}(a_2 - a_1)^2. \end{aligned} \tag{9}$$

Now from inequalities (8) and (9) we get

$$\begin{aligned} g\left(\frac{a_1 + a_2}{2}\right) - h\left(\frac{1}{2}\right)M_\eta + \frac{\mu}{12}(a_2 - a_1)^2 &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x) dx \\ &\leq g(a_2) + N_\eta - \frac{\mu}{6}(a_2 - a_1)^2. \end{aligned} \tag{10}$$

This completes the proof. □

Remark 4

1. If we take $\mu = 0$ and $\eta(x, y) = x - y$, then the Hermite–Hadamard-type inequality (10) is reduced to Hermite–Hadamard-type inequality for modified h -convex functions; for details, see [13].
2. If we put $h(t) = t$ in (10), then we get a Hermite–Hadamard-type inequality for generalized strongly convex functions; see [12].
3. If we take $\mu = 0$, $\eta(x, y) = x - y$ and $h(t) = t$, then inequality (10) is reduced to a Hermite–Hadamard-type inequality for classical convex functions.

Now we prove the following lemma by using technique of [25]. This lemma has the crucial fact that generalized strongly modified h -convex functions behave like classic convex functions.

Lemma 1 *Let g be a generalized modified h -convex function, and suppose that $\eta(x, y) = -\eta(y, x)$. Then*

$$g(a_1 + a_2 - x) \leq g(a_1) + g(a_2) - g(x) \quad \forall x \in [a_1, a_2],$$

where $x = ta_1 + (1 - t)a_2$ and $t \in [0, 1]$.

Proof As g is generalized modified h -convex function, for $x = ta_1 + (1 - t)a_2$, we get

$$\begin{aligned} g(a_1 + a_2 - x) &= g((1 - t)a_1 + ta_2) \\ &\leq g(a_1) + h(t)\eta(g(a_2), g(a_1)) \\ &= g(a_1) + g(a_2) - g(a_2) - h(t)\eta(g(a_1), g(a_2)) \\ &= g(a_1) + g(a_2) - [g(a_2) + h(t)\eta(g(a_1), g(a_2))] \\ &\leq g(a_1) + g(a_2) - g(x). \end{aligned}$$

This completes the proof. □

Lemma 2 *Let g be q the generalized strongly modified h -convex function, and suppose that $\eta(x, y) = -\eta(y, x)$. Then*

$$g(a_1 + a_2 - x) \leq g(a_1) + g(a_2) - g(x) \quad \forall x \in [a_1, a_2], \tag{11}$$

where $x = ta_1 + (1 - t)a_2$ and $t \in [0, 1]$.

Proof Let g be a generalized strongly modified h -convex function. Then for $x = ta_1 + (1 - t)a_2$, we get

$$\begin{aligned} g(a_1 + a_2 - x) &= g((1 - t)a_1 + ta_2) \\ &\leq g(a_1) + h(t)\eta(g(a_2), g(a_1)) - \mu t(1 - t)(a_1 - a_2)^2 \\ &\leq g(a_1) + g(a_2) - g(a_2) - h(t)\eta(g(a_1), g(a_2)) \\ &\quad - \mu t(1 - t)(a_1 - a_2)^2 + 2\mu t(1 - t)(a_1 - a_2)^2 \end{aligned}$$

$$\begin{aligned} &\leq g(a_1) + g(a_2) - [g(a_2) + h(t)\eta(g(a_1),g(a_2)) - \mu t(1-t)(a_1 - a_2)^2] \\ &\leq g(a_1) + g(a_2) - g(x). \end{aligned}$$

This completes the proof. □

It is very interesting that when g is a modified h -convex function [13], generalized modified h -convex, or generalized strongly modified h -convex function, then inequality (11) holds.

Theorem 3 (Fejér-type inequality) *Let $g : [a_1, a_2] \rightarrow \mathbb{R}$ be a generalized strongly modified h -convex, and let $w : [a_1, a_2] \rightarrow \mathbb{R}$ be nonnegative, integrable, and symmetric with respect to $\frac{a_1+a_2}{2}$. Then*

$$\begin{aligned} &g\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} w(x) dx + \frac{\mu}{4} \int_{a_1}^{a_2} (a_1 + a_2 - 2x)w(x) dx - N_\eta(a_1, a_2) \\ &\leq \int_{a_1}^{a_2} g(x)w(x) dx \\ &\leq \frac{g(a_1) + g(a_2)}{2} \int_{a_1}^{a_2} w(x) dx + T_\eta(a_1, a_2) - \mu \int_{a_1}^{a_2} (x - a_2)(a_1 - x)w(x) dx, \end{aligned} \tag{12}$$

where

$$\begin{aligned} N_\eta(a_1, a_2) &= h\left(\frac{1}{2}\right) \int_{a_1}^{a_2} \eta(g(a_1 + a_2 - x), g(x))w(x) dx, \\ T_\eta(a_1, a_2) &= \frac{\eta(g(a_1), g(a_2))}{2} \int_{a_1}^{a_2} h\left(\frac{x - a_2}{a_1 - a_2}\right)w(x) dx. \end{aligned}$$

Proof Let g be a generalized strongly modified h -convex function. Then

$$\begin{aligned} &g\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} w(x) dx = \int_{a_1}^{a_2} g\left(\frac{a_1 + a_2 - x + x}{2}\right)w(x) dx \\ &\leq \int_{a_1}^{a_2} g(x)w(x) dx \\ &\quad + h\left(\frac{1}{2}\right) \int_{a_1}^{a_2} \eta(g(a_1 + a_2 - x), g(x))w(x) dx \\ &\quad - \int_{a_1}^{a_2} \mu \frac{1}{2} \left(1 - \frac{1}{2}\right) (2x - a_1 - a_2)^2 w(x) dx, \end{aligned} \tag{13}$$

$$\begin{aligned} &g\left(\frac{a_1 + a_2}{2}\right) \int_{a_1}^{a_2} w(x) dx + \frac{\mu}{4} \int_{a_1}^{a_2} (a_1 + a_2 - 2x)^2 w(x) dx - N_\eta(a_1, a_2) \\ &\leq \int_{a_1}^{a_2} g(x)w(x) dx. \end{aligned}$$

In the right hand-side of inequality (13), put $x = ta_1 + (1 - t)a_2$. Then

$$\begin{aligned} \int_{a_1}^{a_2} g(x)w(x) dx &= (a_2 - a_1) \int_0^1 g(ta_1 + (1 - t)a_2)w(ta_1 + (1 - t)a_2) dt, \\ \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x)w(x) dx &\leq \int_0^1 g(a_2)w(ta_1 + (1 - t)a_2) dt \\ &\quad + \eta(g(a_1), g(a_2)) \int_0^1 h(t)w(ta_1 + (1 - t)a_2) dt \\ &\quad - \mu(a_2 - a_1)^2 \int_0^1 t(1 - t)w(ta_1 + (1 - t)a_2) dt. \end{aligned} \tag{14}$$

Similarly, if we put $x = ta_2 + (1 - t)a_1$ in the right-hand side of inequality (13), then we get the inequality

$$\begin{aligned} \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} g(x)w(x) dx &\leq \int_0^1 g(a_1)w(ta_2 + (1 - t)a_1) dt \\ &\quad + \eta(g(a_2), g(a_1)) \int_0^1 h(t)w(ta_2 + (1 - t)a_1) dt \\ &\quad - \mu(a_2 - a_1)^2 \int_0^1 t(1 - t)w(ta_2 + (1 - t)a_1) dt. \end{aligned} \tag{15}$$

Adding inequalities (14) and (15), where w is symmetric, we get

$$\begin{aligned} \frac{2}{a_2 - a_1} \int_{a_1}^{a_2} g(x)w(x) dx &\leq (g(a_1) + g(a_2)) \int_0^1 w(ta_1 + (1 - t)a_2) dt \\ &\quad + [\eta(g(a_1), g(a_2)) + \eta(g(a_2), g(a_1))] \int_0^1 h(t)w(ta_1 + (1 - t)a_2) dt \\ &\quad - 2\mu(a_2 - a_1)^2 \int_0^1 t(1 - t)w(ta_1 + (1 - t)a_2) dt. \end{aligned} \tag{16}$$

Putting $x = ta_1 + (1 - t)a_2$ in the right-hand side of inequality (16), we have

$$\begin{aligned} \int_{a_1}^{a_2} g(x)w(x) dx &\leq \frac{(g(a_1) + g(a_2))}{2} \int_{a_1}^{a_2} w(x) dx \\ &\quad + \frac{[\eta(g(a_1), g(a_2)) + \eta(g(a_2), g(a_1))]}{2} \int_{a_1}^{a_2} h\left(\frac{x - a_2}{a_1 - a_2}\right)w(x) dx \\ &\quad - \mu \int_{a_1}^{a_2} (x - a_2)(a_1 - x)w(x) dx, \\ \int_{a_1}^{a_2} g(x)w(x) dx &\leq \frac{(g(a_1) + g(a_2))}{2} \int_{a_1}^{a_2} w(x) dx + T_\eta(a_1, a_2) \\ &\quad - \mu \int_{a_1}^{a_2} (x - a_2)(a_1 - x)w(x) dx. \end{aligned} \tag{17}$$

Now from inequalities (13) and (17) we get Fejér-type inequality (12) for generalized strongly modified h -convex functions. \square

Remark 5

1. If $h(t) = t$, then inequality (12) reduced to Fejér type inequality for generalized strongly convex functions, see [12].
2. If we put $\mu = 0$ and $\eta(x, y) = x - y$ then inequality (12) becomes a Fejér-type inequality for modified h -convex functions; see [13].
3. If we put $\mu = 0$, $\eta(x, y) = x - y$, and $h(t) = t$, then inequality (12) is reduced to a Fejér-type inequality for classical convex functions.

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References

1. Barani, A., Ghazanfari, A.G., Dragomir, S.S.: Hermite–Hadamard inequality for functions whose derivatives absolute values are preinvex. *J. Inequal. Appl.* **2012**(1), 247 (2012)
2. Bessenyei, M., Páles, Z.: Characterizations of convexity via Hadamard's inequality. *Math. Inequal. Appl.* **9**(1), 53–62 (2006)
3. Dragomir, S.S., Pearce, C.: Selected topics on Hermite–Hadamard inequalities and applications. *Math. Prepr. Arch.* **2003**(3), 463–817 (2003)
4. Dragomir, S.S.: Inequalities of Hermite–Hadamard type for h -convex functions on linear spaces. *Proyecciones* **34**(4), 323–341 (2015)
5. Hardy, G.H., Littlewood, J.E., Polya, G.: *Inequalities*. Cambridge University Press, Cambridge (1952) (1988)
6. Kuczma, M.: *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*. Springer, Berlin (2009)
7. Merentes, N., Nikodem, K.: Remarks on strongly convex functions. *Aequ. Math.* **80**(1–2), 193–199 (2010)
8. Tseng, K.L., Hwang, S.R., Dragomir, S.S.: Fejér-type inequalities (I). *J. Inequal. Appl.* **2010**(1), 531976 (2010)
9. Niculescu, C., Persson, L.E.: *Convex Functions and Their Applications*. Springer, New York (2006)
10. Peajcariac, J.E., Tong, Y.L.: *Convex Functions, Partial Orderings, and Statistical Applications*. Academic Press, San Diego (1992)
11. Zălinescu, C.: *Convex Analysis in General Vector Spaces*. World Scientific, Singapore (2002)
12. Awan, M.U.: On strongly generalized convex functions. *Filomat* **31**(18), 5783–5790 (2017)
13. Noor, M., Noor, K., Awan, U.: Hermite–Hadamard type inequalities for modified h -convex functions. *Transylv. J. Math. Mech.* **2014**, 6 (2014)
14. Noor, M.A., Noor, K.I., Iftikhar, S., Awan, M.U.: Strongly generalized harmonic convex functions and integral inequalities. *J. Math. Anal.* **7**(3), 66–71 (2016)
15. Bessenyei, M., Páles, Z.: Hermite–Hadamard inequalities for generalized convex functions. *Aequ. Math.* **69**(1), 32–40 (2005)
16. Mitrinović, D.S., Vasić, P.M.: *Analytic Inequalities (Vol. 1)*. Springer, Berlin (1970)
17. Mordukhovich, B.S., Nam, N.M.: An easy path to convex analysis and applications. *Synth. Lect. Math. Stat.* **6**(2), 1–218 (2013)
18. Rockafellar, R.T.: *Convex Analysis (Vol. 28)*. Princeton University press, Princeton (1970)
19. Zabandan, G., Bodaghi, A., Kiliçman, A.: The Hermite–Hadamard inequality for r -convex functions. *Journal of Inequalities and Applications* **2012**(1), 215 (2012)

20. Ciobotariu-Boer, V.: Hermite–Hadamard and Fejér inequalities for Wright-convex functions. *Octagon Math. Mag.* **17**(1), 53–69 (2009)
21. Varosanec, S.: On h -convexity. *J. Math. Anal. Appl.* **326**(1), 303–311 (2007)
22. Shaikh, A.A., Iqbal, A., Mondal, C.K.: Some results on φ -convex functions and geodesic φ -convex functions. *Differ. Geom. Dyn. Syst.* **20**, 159–170 (2018)
23. Finol, C.E., Wojtowitz, M.: Multiplicative properties of real functions with applications to classical functions. *Aequ. Math.* **59**(1–2), 134–149 (2000)
24. Peajcariac, J.E., Tong, Y.L.: *Convex Functions, Partial Orderings, and Statistical Applications*. Academic Press, San Diego (1992)
25. Sarikaya, M.Z., Set, E., Ozdemir, M.E.: On some new inequalities of Hadamard type involving h -convex functions. *Acta Math. Univ. Comen.* **79**(2), 265–272 (2010)

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