ON GENERALIZED WALSH FOURIER SERIES

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1. Introduction. The main purpose of the present paper is to prove the theorems on the generalized Walsh Fourier series which we announced in the previous paper [11].

Let $\{\alpha(n)\}$ be a sequence of integers not less than 2, and put

$$A(n) = \alpha(0)\alpha(1)\ldots\alpha(n-1), \qquad A(-n) = 1/A(n)$$

empty product being considered to be equal to 1.[1)]

The generalized Rademacher functions $\phi_n(t)$ $(n = 0, 1, 2, \ldots\ldots)$ are defined as

$$\phi_n(t) = \exp(2\pi ik/\alpha(n)) \qquad (i = \sqrt{-1})$$

for t belonging to the left-semiclosed intervals

$$(1.1) \qquad [kA(-n-1),\ (k+1)\,A(-n-1)),\ k = 0, 1, 2, \ldots, A(n+1)-1$$

and $$\phi_n(t+1) = \phi_n(t) \qquad \text{for all } t.$$

An elementary consideration shows that these functions $\phi_n(t)$ $(n = 0, 1, 2 \ldots)$ are orthonormal over the interval $(0, 1)$, or

$$\int_0^1 \phi_m(t)\,\overline{\phi}_n(t)\,dt = \begin{cases} 0 & (m \neq n, \text{ say, e.g. } m > n) \\ 1 & (m = n). \end{cases}$$

It is worth observing that this orthogonality is a consequence of the following fact:

$\phi_m(t)$ has mean 0 over each of the intervals (1.1) where $\phi_n(t)$ takes a constant value.

Now we can define the generalized Walsh functions $\psi_n(t)$ $(n = 0, 1, 2 \ldots)$ as follows:

$$\psi_0(t) \equiv 1,$$
$$\psi_n(t) = \phi_{n(1)}^{a(1)}(t)\phi_{n(2)}^{a(2)}(t)\ldots\phi_{n(r)}^{a(r)}(t)$$

provided that n is expressed in the form

$$(1.2) \qquad n = a(1)A(n(1)) + a(2)A(n(2)) + \ldots + a(r)A(n(r)) \geqq 1$$

where

$$(1.3) \qquad n(1) > n(2) > \ldots > n(r) \geqq 0; \quad 0 < a(j) < \alpha(j) \qquad (j = 1, 2, \ldots r).$$

It is easily seen from the above remark on $\phi_n(t)$ that the functions $\psi_n(t)$ thus defined form an orthonormal system over the unit interval. Moreover, this system is complete, as we shall see in §3.

If $\alpha(n) = 2$ $(n = 0, 1, 2, \ldots)$, our functions reduce to those of Walsh

1) And similarly, we consider that the empty sum is equal to 0.

himself, and the case $\alpha(n) = \alpha$ $(n = 0, 1, 2, \ldots.)$ was studied by H. E. Chrestenson [1]. The general definition seems to have been given by J. J. Price (cf. [8]). We shall assume, in the latter half of §3 and thereafter, unless the contrary is stated explicitly, that the sequence $\{\alpha(n)\}$ is bounded, say $\alpha(n) \leqq \alpha$ $(n = 0, 1, 2, \ldots.)$.

In § 2 we consider some properties of the "A-group", whose characters are essentially the generalized Walsh functions defined above. The first consideration in this direction was done by N. J. Fine [2] who defined the "dyadic group" in regard to the case of the "proper" Walsh functions, in which $\alpha(n) = 2$ $(n = 0, 1, 2, \ldots.)$.

§ 3 is dedicated to the proof of the completeness of our system $\{\psi\}$ and a concise treatise of Haar functions, generalized a little more than in our preceding note [10].

In § 4 we generalize an inequality of R. E. A. C. Paley [7], which is fundamental to the $L^p(p > 1)$ theory of Walsh Fourier series, and then apply it to prove the mean convergence of Generalized Walsh Fourier series.

§ 5 is a generalization of § 4, done in such a way as I. I. Hirschman [6] generalized Paley's results.

In §6 we give two examples which show that the boundedness of the sequence $\{\alpha(n)\}$ is indispensable to the truth of Paley's inequality.

The final section deals with summability factors and convergence factors.

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2. The A-group. Let g_n $(n = 0, 1, 2, \ldots.)$ be cyclic groups of orders $\alpha(n)$, which are understood to be the remainder groups of the division modulo $\alpha(n)$, respectively. Let G be their direct product, so that its elements are sequences $\bar{t} = \{t_n\}$, $t_n \in g_n$.

Clearly, G is an Abelian group which is compact with respect to the weak topology. The group operation in G is the termwise addition modulo $\alpha(n)$, denoted by $\dot{+}$, and the inverse element of $\bar{t} \in G$ is denoted by $\dot{-}\bar{t}$. We write simply $\bar{t} \dot{-} \bar{u}$ for $\bar{t} \dot{+} (\dot{-}\bar{u})$.

To every element $\bar{t} \in G$ corresponds a number $t \in [0, 1]$ defined by

$$t = \lambda(\bar{t}) = \sum_{n=1}^{\infty} t_n A(-n).$$

The inverse mapping μ, of λ. is determined uniquely, except for those t's $\in [0, 1]$ which are "A-rationals" (by which we mean those t's of the form $kA(-n)$). It is easily seen that the group character (which is a continuous representation having absolute value 1) of G and the generalized Walsh functions pass into one another by these mappings, except for at most a countable set of arguments. We abbreviate $\lambda(\mu(t) \dot{+} \mu(u))$ resp. $\lambda(\mu(t) \dot{-} \mu(u))$ into $t \dot{+} u$ resp. $t \dot{-} u$, provided that they are determined uniquely. These yield for every $t,\ u \in [0,1]$

$$(2.1)\qquad \psi_n(t \dotplus u) = \psi_n(t)\psi_n(u),\quad \psi_n(t \dot{-} u) = \psi_n(t)\overline{\psi}_n(u)$$

except for u's belonging to a certain countable set.

We have following propositions which are easily verified:

LEMMA 1. *Let* $0 \leqq t < 1, 0 \leqq u < 1$ *and the A-expansion of* u *be* 0 *in the first* n *places* $(n \geqq 0)$, *then we have*

$$(2.2)\qquad t - (\alpha(n) - 1)u \leqq t \dotplus u \leqq t + u$$

$$(2.3)\qquad t - u \leqq t \dot{-} u \leqq t - (\alpha(n) - 1)\, u.$$

LEMMA 2. *Let* $0 \leqq t < 1,\ 0 \leqq u < 1$ *and their A-expansion coincide in the first* n *places*, $n \geqq 0$, *then we have*

$$(2.4)\qquad \min(t \dot{-} u, u \dot{-} t) \geqq |t - u|/\alpha(n).$$

If the sequence $\{\alpha(n)\}$ is bounded, say $2 \leqq \alpha(n) \leqq \alpha$, we have, as a corollary of Lemma 1:

LEMMA 3. *Let* $f(t) \in L(0,1)$,[2] *then, for almost every* $t \in (0,1)$ *we have*

$$I_1 = \int_0^x |f(t \dotplus u) - f(t)|\, du = o(x)$$

$$(as\ x \to +0).$$

$$I_2 = \int_0^x |f(t \dot{-} u) - f(t)|\, du = o(x)$$

PROOF. We have only to prove the first half. Putting

$$E = \{t \dotplus u : 0 \leqq u \leqq x\}$$

we see by Lemma 1, that $E \subset [t - (\alpha - 1)x,\ t - x]$. Since the transformation $T_t : u \to t \dotplus u$ is measure-preserving, we have

$$I_1 = \int_E |f(u) - f(t)|\ du \leqq \int_0^x |f(t + u) - f(t)|\, du$$

$$+ \int_{-(\alpha-1)x}^0 |f(t + u) - f(t)|\ du = o(x) + o(x) = o(x) \qquad \text{a.e.}$$

3. We have already seen that the functions $\psi_n(t)$, $n = 0, 1, 2, \ldots.$ form an orthonormal system over the unit interval.

Let $f(t) \in L(0,1)$ and write

$$(3.1)\qquad f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(t)$$

where c_ν is given by

2) We consider, here and in the sequel, only those functions which have period 1.

(3.2) $$c_\nu = \int_0^1 f(u)\,\overline{\psi_\nu(u)}\,du.$$

The n-th partial sum $s_n(t)$ of (3.1) is then expressed as

(3.3) $$\begin{aligned} s_n(t) = \sum_{\nu=0}^{n-1} c_\nu \psi_\nu(t) &= \sum_{\nu=0}^{n-1} \psi_\nu(t) \int_0^1 f(u)\,\overline{\psi_\nu(u)}\,du \\ &= \int_0^1 f(u) \sum_{\nu=0}^{n-1} \psi_\nu(t)\,\overline{\psi_\nu(u)}\,du \\ &= \int_0^1 f(u) \sum_{\nu=0}^{n-1} \psi_\nu(t \dot{-} u)\,du \\ &= \int_0^1 f(u)\,D_n(t \dot{-} u)\,du \end{aligned}$$

where $D_n(t)$ is the Dirichlet Kernel:

(3.4) $$D_n(t) = \sum_{\nu=0}^{n-1} \psi_\nu(t).$$

Since the values assumed by $\phi_n(t)$ are either 1 or one of the $\alpha(n)$-th roots of 1, we see that

(3.5) $$\sum_{j=0}^{\alpha(n)-1} \phi_n^j(t) = \begin{cases} \alpha(n) & \text{if } \phi_n(t) = 1 \\ 0 & \text{if } \phi_n(t) \neq 1. \end{cases}$$

On the other hand, we have

(3.6) $$\begin{aligned} D_{A(n+1)}(t) = \sum_{\nu=0}^{A(n+1)-1} \psi_\nu(t) &= \sum_{j=0}^{\alpha(n)-1} \sum_{\nu=0}^{A(n)-1} \psi_{jA(n)+\nu}(t) \\ &= \sum_{j=0}^{\alpha(n)-1} \phi_n^j(t) \sum_{\nu=0}^{A(n)-1} \psi_\nu(t) = D_{A(n)}(t) \sum_{j=0}^{\alpha(n)-1} \phi_n^j(t). \end{aligned}$$

By induction, we can infer from (3.5) and (3.6) that

(3.7) $$D_{A(n)}(t) = \begin{cases} A(n) & 0 \leqq t < A(-n) \\ 0 & A(-n) \leqq t < 1. \end{cases}$$

Substituting this into (3.3), we obtain

(3.8) $$s_{A(n)}(t) = A(n) \int_{I(n,t)} f(u)\,du,$$

where $I(n,t)$ is the interval of the form $[kA(-n),\ (k+1)A(-n))$ containing t. Thus we have proved the following proposition:

THEOREM K. *At every point where $f(t)$ is equal to the derivative of its indefinite integral, we have*

(3.9) $$\lim_{n\to\infty} s_{A(n)}(t) = f(t).$$

Moreover, by the well-known maximal theorems of Hardy and Littlewood (*cf. e. g.* [14; pp. 244-245]) *we have*[3)]

$$(3.10)\quad \begin{aligned} &\int_0^1 \sup_n |s_{A(n)}(t)|^p\,dt \leqq B_p \int_0^1 |f(t)|^p\,dt && (p>1);\\ &\int_0^1 \sup_n |s_{A(n)}(t)|\,dt \leqq B \int_0^1 |f(t)|\log^+|f(t)|\,dt + B;\\ &\int_0^1 \sup_n |s_{A(n)}(t)|^r\,dt \leqq B_r \left(\int_0^1 |f(t)|\,dt\right)^r && (0<r<1);\\ &\sup_{n\ t} |s_{A(n)}(t)| \leqq \operatorname{ess\,sup} |f(t)|. \end{aligned}$$

provided that the right-hand side exists.

Now it is evident that our system $\{\psi\}$ is complete in $L(0,1)$: for, if all of the Fourier coefficients of a function $f(t)\in L(0,1)$ are equal to 0, $f(t)$ has its $A(n)$-th partial sums vanishing identically, so that does the limit of these partial sums, which is equal to $f(t)$ itself almost everwhere, vanish identically.

Let us pass to the study of generalized Haar functions: put

$$\varphi_0(t) = \varphi_{0,0}(t) \equiv 1 \qquad 0 \leqq t < 1$$
$$\varphi_1(t) = \varphi_{1,0}(t) = \phi_0(t)$$

and generally

$$\varphi_{l,m}(t) = \begin{cases} \phi_{l-1}(t) & (mA(-l+1) \leqq t < (m+1)A(-l+1)) \\ 0 & \text{elsewhere} \end{cases}$$

$$m = 0,1,\dots,A(l-1)-1; \qquad l = 2,3,\dots.$$

$$\chi_0(t) = \varphi_0(t)$$

$$\chi^{(j)}_{l,m}(t) = \varphi^j_{l,m}(t)\sqrt{A(l-1)} \quad \begin{pmatrix} j = 1,2,\dots, & \alpha(l-1)-1\ ; \\ m = 0,1,\dots\dots & A(l-1)-1\ ; \\ l = 1,2,\dots, & \end{pmatrix}$$

and we rearrange $\{\chi^{(j)}_{l,m}\}$ into a sequence $\{\chi_n\}$ $(n = 0,1,2,\dots.)$ lexicographically with respect to l, m, j, so that $\chi_n (n \geqq 1)$ is the $\chi^{(j)}_{l\,m}$ where n is expressed in the form

$$(3.11)\qquad \begin{aligned} n &= \sum_{\lambda=1}^{l-1} (\alpha(\lambda-1)-1)A(\lambda-1) - m(\alpha(l-1)-1) + j \\ &= A(l-1) - m(\alpha(l-1)-1) + j - 1 \end{aligned}$$

We call the functions χ_n (or $\chi^{(j)}_{l,m}$) the generalized Haar functions. The remark given in § 1 subsists here too, and the system $\{\chi_n\}$ is orthonormal over the unit interval. Moreover, it is verified without difficulty that (a proof is given in a moment) this system is also complete in $L(0,1)$. For this system, the

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3) We use, here and in the sequel, the letter B with or without subscripts to denote a constant (which need not the same in different contexts) depending only on parameters disposed explicitly.

following theorem is valid:

THEOREM H. *Let $f(t) \in L(0,1)$ and the sequence $\{\alpha(n)\}$ be bounded, say $\alpha(n) \leqq \alpha$. Then the generalized Haar Fourier series of $f(t)$ converges almost everywhere to $f(t)$. In particular, the series converges at every point of contiuuity of f: the convergence is uniform in $t \in [a, b] \subset [0, 1)$ or for all t, when $f(t)$ is known to be continuous in the designated place respectively.*

PROOF. Let t be fixed and the intervals $I(l-1, i, t)$ and $I(l, k, t)$ have the same meaning as above. Then it is easy to see that

$$\sum_{j=1}^{\alpha(l-1)-1} \chi_{l,m}^{(j)}(t)\,\overline{\chi_{l,m}^{(j)}(u)} = \begin{cases} A(l) - A(l-1) & u \in I(l, k, t) \\ -A(l-1) & u \in I(l-1, i, t) - I(l, k, t) \\ 0 & \text{elsewhere} \end{cases}$$

and consequently

$$K_{A(l-1)-(m+1)(\alpha(l-1)-1)}(t, u) = \begin{cases} A(l) \\ A(l-1) \\ 0 \end{cases} \tag{3.12}$$

according to

$u \in I(l, k, t)$ with $0 \leqq k \leqq (m+1)\alpha(l-1) - 1$,

$u \in I(l-1, i, t)$ with $m + 1 \leqq i \leqq A(l-1) - 1$

ortherwise respectively.

In particular, taking $m = A(l-1) - 1$ we have

$$K_{A(l)}(t, u) = \begin{cases} A(l) & u \in I(l, k, t) \\ 0 & \text{elsewhere} \end{cases} \tag{3.13}$$

of which we made use above.

The formulas (3.12) and (3.13) together show that

$$K_{A(l-1)+m(\alpha(l-1)-1)}(t, u) = \begin{cases} \text{either} & K_{A(l)}(t, u) = D_{A(l)}(t \dot{-} u). \\ \text{or} & K_{A(l-1)}(t, u) = D_{A(l-1)}(t \dot{-} u). \end{cases}$$

This facts and a consideration similar to what led us to Theorem K yield, n and l, m, j being related by (3.11)

$$\begin{aligned} \left| \int_0^1 f(u) K_n(t, u)\, du \right| &\leqq \int_0^1 |f(u)| \; |K_n(t, u)|\, du \\ &\leqq \int_0^1 |f(u)|\, K_{A(l-1)+m\alpha(l-1)-1}(t, u)\, du \\ &\quad + \int_0^1 |f(u)| \sum_{i=1}^{j} \left| \chi_{l,m}^{(i)}(u)\, \overline{\chi_{l,m}^{(i)}(u)} \right| du \end{aligned} \tag{3.14}$$

$$\leqq A(\lambda) \int_{I(\lambda, \kappa, t)} |f(u)|\, du + j\; A(l-1) \int_{I(l-1, m, \cdot)} |f(u)|\, du$$

$$\leqq B_\alpha \sup_{h>0} \frac{1}{2h} \int_{t-h}^{t+h} |f(u)|\, du.$$

Thus we have, by a maximal theorem of Hardy and Littlewood,

$$\int_0^1 \sup_n \left| \int_0^1 f(u)\, K_n(t,u)\, du \right|^r dt \leqq B_{r,\alpha} \left(\int_0^1 |f(t)|\, dt \right)^r \qquad (0 < r < 1),$$

from which our first assertion follows.

In order to see the last half of the theorem, we take $f(u) = 1$ in (3.14) obtaining

$$\int_0^1 |K_n(t,u)|\, du \leqq B_\alpha.$$

As a moment's inspection of $K_n(x,t)$ shows that this is a quasi-positive kernel, our theorem is now established completely.

4. We are now in a position to prove a generalization of the fundamental inequality of Paley. It should be remembered that we have been assuming the boundedness of $\{\alpha(n)\}$, say $\alpha(n) \leqq \alpha$. Paley's result reads as follows:

THEORFM P. *Let $\psi_n(t)$ $(n = 0,1,2,\ldots.)$ be the "proper" Walsh functions corresponding to the sequence $(2,2,2,\ldots.)$ and let $f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(t) \in L^p(0,1)$ $p > 1$. Putting*

$$f_n(t) = \sum_{\nu=2^n}^{2^{n+1}-1} c_\nu \psi_\nu(t) \qquad (n = 0,1,2,\ldots.)$$

one has

$$B_p \int_0^1 |f(t)|^p\, dt \leqq \int_0^1 \left(|c_0|^2 + \sum_{n=0}^{\infty} |f_n(t)|^2 \right)^{p/2} dt \leqq B_p \int_0^1 |f(t)|^p\, dt.$$

This can be brought into our case "formally", that is, we can prove the following proposition:

THEOREM P′. *Let $\psi_n(t)$ $(n = 0,1,2,\ldots.)$ be the generalized Walsh functions and let $f(t) \sim \sum_{\nu=}^{\infty} c_\nu \psi_\nu(t) \in L^p(0,1)$, $p > 1$. Then, putting $\Delta_n(t) = \sum_{\nu=A(n)}^{A(n+1)-1} c_\nu \psi_\nu(t)$ $(n = 0,1,2,\ldots.)$ we have*

$$B_{p,\alpha} \int_0^1 |f(t)|^p\, dt \leqq \int_0^1 \left(|c_0|^2 + \sum_{n=0}^{\infty} |\Delta_n(t)|^2 \right)^{p/2} dt \tag{4.1}$$
$$\leqq B_{m\,\alpha} \int_0^1 |f(t)|^p\, dt.$$

However, Theorem P′ is not so effective in applications as Theorem P in the theory of "proper" Walsh functions; a "finer" decomposition of Fourier series would be needed, as we are going to see.

THEOREM 1. *Let* $f(t) \in L^p(0,1)$ $(p>1)$, $f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi(t)$ *and put*

$$\delta_{n,j}(t;f) \equiv \delta_{n,j}(t) = \sum_{\nu=jA(n)}^{(j+1)A(n)-1} c_\nu \psi_\nu(t) \qquad \begin{pmatrix} j=1,2,\ldots, \alpha(n)-1; \\ n=0,1,2.\ldots \end{pmatrix}.$$

Then we have

$$B_{p,\alpha} \int_0^1 |f(t)|^p dt \leqq \int_0^1 \Big(|c_0|^2 + \sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |\delta_{n,j}(t)|^2 \Big)^{p/2} dt \tag{4.2}$$
$$\leqq B_{p,\alpha} \int_0^1 |f(t)|^p dt.$$

PROOF. Clearly we may suppose that $c_0=0$, and $f(t)$ is real-valued; because if this case is proved, the general case then follows by Minkowski's inequality. Assuming first p is an even integer $2k$, we prove Theorem 1 and Theorem P′ together in three steps, of which the second is trivial:

$$\int_0^1 |f(t)|^p dt \leqq B_{p,\alpha} \int_0^1 \Big(\sum_{\nu=0}^{\infty} |\Delta_n(t)|^2 \Big)^{p/2} dt, \tag{4.3}$$

$$\int_0^1 \Big(\sum_{n=0}^{\infty} |\Delta_n(t)| \Big)^{p/2} dt \leqq B_{p,\alpha} \int_0^1 \Big(\sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |\delta_{n,j}(t)|^2 \Big)^{p/2} dt, \tag{4.4}$$

$$\int_0^1 \Big(\sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |\delta_{n,j}(t)|^2 \Big)^{p/2} dt \leqq B_{p,\alpha} \int_0^1 |f(t)|^p dt. \tag{4.5}$$

We begin with the proof of (4.3); write $S_n(t)$ for $s_{A(n)}(t)$, then $S_n(t)$ is real-valued and

$$S_{n+1}(t) = S_n(t) + \Delta_n(t),$$

so that we have

$$0 \leqq S_{n+1}^{2k} = (S_n + \Delta_n)^{2k} = S_n^{2k} + \sum_{l=1}^{2k} \binom{2k}{l} S_n^{2k-l} \Delta_n^l$$

Subtracting S_n^{2k} and integrating over the unit interval we see

$$\Big| \int_0^1 (S_{n+1}^{2k} - S_n^{2k}) dt \Big| = \Big| \sum_{l=1}^{2k} \binom{2k}{l} \int_0^1 S_n^{2k-l} \Delta_n^l dt \Big|$$
$$= \Big| \sum_{l=2}^{2k} \binom{2k}{l} \int_0^1 S_n^{2k-l} \Delta_n^l dt \Big| \tag{4.6}$$

$\Big($observe that by the remark of §1, $\int_0^1 S_n^{2k} \Delta_n \, dt = 0\Big)$. A trivial computation including Hölder's inequality and

$$a^{\varepsilon} b^{1-\varepsilon} \leqq \max(a, b) \leqq a + b \quad (0 \leqq a,\ 0 \leqq b,\ 0 < \varepsilon < 1)$$

shows that the right-hand side does not exceed

$$\sum_{l=2}^{2k} \binom{2k}{l} \left(\int_0^1 S_n^{2k-2} \Delta_n^2 \, dt + \int_0^1 \Delta_n^{2k} dt \right).$$

Summing up (4.6) for $n = 0, 1, \ldots . N$ we have

$$\int_0^1 S_{N+1}^{2k} \, dt \leqq \sum_{n=0}^{N} \left| \int_0^1 (S_{n+1}^{2k} - S_n^{2k}) \, dt \right|$$

$$\leqq 2^{2k} \left\{ \int_0^1 \left(\max_{0 \leqq n \leqq N} S_n^{2k-2} \right) \sum_{n=0}^{N} \Delta_n^2 \, dt + \int_0^1 \sum_{n=0}^{N} \Delta_n^{2k} \, dt \right\}$$

$$\leqq 2^{2k} \left\{ \left(\int_0^1 \max_{0 \leqq n \leqq N} S_n^{2k} \, dt \right)^{1-1/k} \left(\int_0^1 \left(\sum_{n=0}^{N} \Delta_n^2 \right)^k dt \right)^{1/k} + \int_0^1 \left(\sum_{n=0}^{N} \Delta_n^2 \right)^k dt \right\}$$

$$\leqq 2^{2k} \left\{ B_p \left(\int_0^1 S_{N+1}^{2k} \, dt \right)^{1-1/k} \left(\int_0^1 \left(\sum_{n=0}^{N} \Delta_n^2 \right)^k dt \right)^{1/k} + \int_0^1 \left(\sum_{n=0}^{N} \Delta_n^2 \right)^k dt \right\},$$

where the first inequality of (3.10) was used. Consequently we have

$$\int_0^1 S_{N+1}^{2k} \, dt \leqq B_p \int_0^1 \left(\sum_{n=0}^{N} \Delta_n^2 \right)^k dt \leqq B_p \int_0^1 \left(\sum_{n=2}^{\infty} \Delta_n^2 \right)^k dt.$$

An application of Fatou's lemma yields (4.3).

For the proof of (4.5) we rearrange $\{\delta_{n,j}\}$ lexicographically with respect to n, j into a sequence $\{d_m\}$ $m = 0, 1, 2, \ldots .$ so that $d_0 = \delta_{0,1}$, $d_0 = \delta_{0,2}, \ldots ,$ $d_{\alpha(0)-1} = \delta_{1,1} \cdots$ We need two lemmas:

LEMMA 4. *Let* $m \neq n$, $\max(m, n) \geqq \max(n(1), \ldots .. \ n(k-1))$[3]
Then

$$\int_0^1 |d_{n(1)}(t)|^2 \ldots . |d_{n(k-1)}(t)|^2 \, d_m(t) \, \overline{d}_n(t) \, dt = 0.$$

PROOF. Considering the complex conjugate if necessary, we may assume

4) Here the $n(i)$'s are *not* the "exponents" of A-expansion. Since no confusion will arise, we may use this notation.

that $m > n$. Write

$$d_m(t) = \delta_{\lambda^1\iota}(t) = \sum_{\nu=\iota A(\lambda)\nu}^{(\iota+1)A(\lambda)-1} c_\nu \psi(t) = \phi_\lambda^\iota(t) \sum_{\nu=}^{A(\lambda)-1} c_{\nu+\iota A(\lambda)} \psi_\nu(t) = \phi_\lambda^\iota(t)\gamma_{\lambda,\iota}(t),$$

$$\bar{d}_n(t) = \bar{\delta}_{l,j}(t) = \sum_{\nu=jA(l)}^{(j+1)A(l)-1} \bar{c}_\nu \psi_\nu(t) = \phi_l^{-j}(t) \sum_{\nu=0}^{A(l)-1} \bar{c}_{\nu+jA(l)} \psi_\nu(t) = \phi_l^{-j}(t) g_{l,j}(t).$$

There are two posibilities, in both of which the assertion is easily inferred from the remark of § 1.

(i) If $\lambda > l$ then the function ϕ_λ^ι has mean 0 over each of the intervals $[\mu A(-\lambda), (\mu+1) A(-\lambda))$ $\mu = 0, 1, \ldots, A(\lambda) - 1$, where the product of the rest

$$|d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \gamma_{\lambda,\iota} \bar{d}_n$$

is a constant.

(ii) If $\lambda = l$ and $\iota > j$, the same is said about $\phi_\lambda^{\iota-j}$ and

$$|d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \gamma_{\lambda,\iota} \bar{g}_{\lambda,j}.$$

LEMMA 5. *For $q \geqq 2$, we have*

$$\left(\sum_{m=0}^{\infty} \int_0^1 |d_m|^q \, dt\right)^{1/q} \leqq \left(\int_0^1 |f(t)|^q \, dt\right)^{1/q} \tag{4.7}$$

PROOF. (4.7) holds for $q = 2$, when it reduces to the Parseval relation. It holds also for $q = \infty$, since $d_m(t)$ being a $\delta_{n,j}(t)$,

$$|d_m(t)| = \left| \int_0^1 f(u) \sum_{\nu=jA(n)}^{(j+1)A(n)-1} \psi_\nu(t \dot{-} u) \, du \right|$$

$$= \left| \int_0^1 f(u) \phi_n^j (t \dot{-} u) \; D_{A(n)} (t \dot{-} u) \, du \right|$$

$$\leqq \int_0^1 |f(u)| \; D_{A(n)} (t \dot{-} u) \, du \leqq \operatorname{ess\,sup} |f(u)|$$

yields that $\sup_{m,t} |d_m(t)| \leqq \operatorname{ess\,sup} |f(t)|$. To obtain (4.7) for general $q \geqq 2$, we have only to interpolate these extremal cases by means of the well-known convexity theorem of M. Riesz.

Now let us return to the proof of (4.5): what we must prove is ($p = 2k$ is an evern integer)

$$\int_0^1 \left(\sum_{m=0}^{\infty} |d_m(t)|^2\right)^k dt \leqq B_{p,\alpha} \int_0^1 |f(t)|^{2k} \, dt. \tag{4.5$'$}$$

Put $\qquad F_n(t) = \sum_{m=0}^{n} d_m(t). \qquad$ Then, for $N > n$,

$$
\begin{aligned}
F_N^2 &= \left| F_n + \sum_{m=n+1}^{N} d_m \right|^2 \\
&= |F_n|^2 + \sum_{m=n+1}^{N} |d_m|^s + F_n \sum_{m=n+1}^{N} \bar{d}_m + \bar{F}_n \sum_{m=n+1}^{N} d_m + \sum_{l,m=n+1}^{N}{}' d_l \bar{d}_m,
\end{aligned} \tag{4.8}
$$

where Σ' means that the terms with $l = m$ are omitted in the summation. Take a pair of $k-1$ non-negative integers $n(1), \ldots, n(k-1)$ with $\max(n(1), \ldots, n(k-1)) = n$. Multiplying both sides of (4.8) by $|d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2$ and integrating over the unit interval, we see by lemma 4 that

$$
\int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \, |F_N|^2 \, dt
$$

$$
= \int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \, |F_n|^2 \, dt + \sum_{m=n+1}^{N} \int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \, |d_m|^2 \, dt,
$$

so that we have

$$
\sum_{m=n+1}^{N} \int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 |d_m|^2 \, dt \leqq \int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \, |F_N|^2 \, dt.
$$

Letting the pair $(n(1), \ldots, n(k-1))$ run over all lattice points in the $(k-1)$ dimensional cube Q: $\max(n(1), \ldots n(k-1)) = n \leqq N-1$, we have

$$
\sum_{Q} \sum_{m=n+1}^{N} \int_0^1 |d_{n(1)}|^2 \ldots |d_{n(k-1)}|^2 \, |d_m|^2 \, dt \leqq \int_0^1 |F_N|^2 \left(\sum_{n=0}^{N-l} |d_n|^2 \right)^{k-1} dt,
$$

or, a fortiori, we obtain

$$
\sum_{m=1}^{N} \int_0^1 |d_m|^2 \left(\sum_{n=0}^{m-1} |d_n|^2 \right)^{k-1} dt \leqq \int_0^1 |F_N|^2 \left(\sum_{n=0}^{N-1} |d_n|^2 \right)^{k-1} dt. \tag{4.9}
$$

Now, summing up the inequalities

$$
\begin{aligned}
&\int_0^1 \left(\sum_{n=0}^{m} |d_n|^2 \right)^k dt - \int_0^1 \left(\sum_{n=0}^{m-1} |d_n|^2 \right)^k dt \\
&= \sum_{l=1}^{k} \binom{k}{l} \int_0^1 |d_m|^{2l} \left(\sum_{n=0}^{m-1} |d_n|^2 \right)^{k-l} dt \\
&\leqq \sum_{=l}^{k} \binom{k}{l} \left(\int_0^1 |d_m|^{2k} \, dt \right)^{1-\epsilon(l)} \left(\int_0^1 |d_m|^2 \left(\sum_{n=0}^{m-1} |d_n|^2 \right)^{k-1} dt \right)^{\epsilon(l)} \\
&\leqq \sum_{l=1}^{k} \binom{k}{l} \left(\int_0^1 |d_m|^{2k} \, dt + \int_0^1 |d_m|^2 \left(\sum_{n=0}^{m-1} |d_n|^2 \right)^{k-1} dt \right)
\end{aligned}
$$

$$\leqq 2^k\left(\int_0^1 |d_m|^{2k}dt + \int_0^1 |d_m|^2\left(\sum_{n=0}^{m-1}|d_n|^2\right)^{k-1} dt\right)$$

for $m = 0, 1, \ldots, N$ we have

$$\int_0^1 \left(\sum_{n=0}^{N}|d_n|^2\right)^k dt \tag{4.10}$$

$$\leqq 2^k\left(\sum_{m=0}^{N}\int_0^1 |d_m|^{2k} dt + \sum_{m=1}^{N}\int_0^1 |d_m|^2\left(\sum_{n=0}^{m-1}|d_n|^2\right)^{k-1} dt\right)$$

(4.9), (4.10) and Lemma 5 yield

$$\int_0^1 \left(\sum_{n=0}^{N}|d_n|^2\right)^k dt$$

$$\leqq 2^k\int_0^1 |f|^{2k} dt + 2^k\int_0^1 |F_N|^2\left(\sum_{n=1}^{N-1}|d_n|^2\right)^{k-1} dt$$

$$\leqq 2^{k+1}\max\left\{\int_0^1 |f|^{2k} dt,\ \int_0^1 |F_N|^2\left(\sum_{n=0}^{N-1}|d_n|^2\right)^{k-1} dt\right\}$$

An application of Hölder's inequality shows

$$\int_0^1 \left(\sum_{n=0}^{N}|d_n|^2\right)^k dt \tag{4.11}$$

$$\leqq \max\left(2^{k+1}\int_0^1 |f|^{2k} dt,\ 2^{k(k+1)}\int_0^1 |F_N|^{2k} dt\right).$$

Since $F_N(t)$ is of the form $s_{\iota(n)}(t) + \sum_{j=1}^{l}\delta_{n,j}(t)$ for some n and l, ($l \leqq \alpha(n) - 1 \leqq \alpha - 1$) it is easily majorated by $f(t)$:

$$\int_0^1 |F_N|^{2k} dt \leqq B_{p,\alpha}\left(\int_0^1 |s_{\iota(n)}|^{2k} dt + \sum_{j=1}^{l}\int_0^1 |\delta_{n,j}|^{2k} dt\right)$$

$$\leqq B_{p,\alpha}\int_0^1 |f|^{2k} dt.$$

Substituting this into (4.11) we have (4.5)′, which was to be proved.

In order to prove Theorem P′ and Theorem 1 for general $p > 1$, we may argue as follows.

(4.1) and (4.2) have their equivalent forms which are convenient for interpolation: that is

$$B_{p,\alpha}\int_0^1 |f(t)|^p dt \leqq \int_0^1 \left|\sum_{n=0}^{\infty} \Delta_n(t) r_n(\theta)\right|^p dt \leqq B_{p,\alpha}\int_0^1 |f(t)|^p dt \tag{4.1$'$}$$

$$B_{p,\alpha}\int_0^1 |f(t)|^p dt \leqq \int_0^1 \left|\sum_{n=0}^{\infty} d_n(t)\, r_n(\theta)\right|^p dt \leqq B_{p,\alpha}\int_0^1 |f(t)|^p dt \qquad \text{for every } \theta \tag{4.2$'$}$$

where $r_n(\theta)$ are the "proper" Rademacher functions. Observing that $r_n^2(\theta)=1$ for every θ and n, (4.1)′ and (4.2)′ are easily deduced from (4.1) or (4.2) respectively: while the opposite implication is a consequence of the Khintchine inequality (integrating with respect to θ over the unit interval). Thus we have (4.1)′ and (4.2) for p even integers, and by interpolating between two consequtive even integers, it is seen that they are also true for $p \geqq 2$. The case $1 < p \leqq 2$ is reduced, by the conjugacy argument, to the case $2 \leqq q < \infty$, where q is the conjugate exponent of p. Thus (4.1)′ and (4.2′) hold for $p > 1$ and so are (4.1) and (4.2).

Considering a special case in which each of the $\delta_{n,j}(t)$'s consists of a single term, we have the following corollary to Theorem 1:

Corollary. *Let* $p > 0$, $f(t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} c_{n,j}\, \phi_n^j(t)$. *Then we have*

$$B_{p,\alpha}\int_0^1 |f(t)|^p dt \leqq \left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |c_{n,j}|^2\right)^{p/2} \leqq B_{p,\alpha}\int_0^1 |f(t)|^p dt. \tag{4.12}$$

In fact the first inequality follows directly from Theorem 1 and Hölder's inequality. The second is deduced from the first by observing

$$\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |c_{n,j}|^2 = \int_0^1 |f(t)|^2 dt = \int_0^1 |f|^{2p/3}\, |f|^{2-2p/3} dt$$
$$\leqq \left(\int_0^1 |f|^p dt\right)^{2/3} \left(\int_0^1 |f|^{6-2p} dt\right)^{1/3}$$
$$\leqq \left(\int_0^1 |f|^p dt\right)^{2/3} \cdot B_{p,\alpha}\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |c_{n,j}|^2\right)^{1-p/3},$$

where we may and do suppose that $0 < p < 3$.

We now proceed to the proof of the "mean convergence".

THEOREM 2. *Let $f(t) \in L^p(0,1)$ $(p > 1)$, $f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(t)$ and put $s_n(t) = \sum_{\nu=0}^{n-1} c_\nu \psi_\nu(t)$. Then we have*

$$(4.13) \qquad \int_0^1 |s_n(t)|^p dt \leqq B_{p,\alpha} \int_0^1 |f(t)|^p dt;$$

$$(4.14) \qquad \int_0^1 |f(t) - s_n(t)|^p dt \to 0 \qquad as\ n \to \infty.$$

PROOF. We have only to prove (4.13) and with an additional assumption $c_0 = 0$, for (4.14) follows from (4.13) by means of an approximation with (generalized Walsh) polynomial[5], and the case $c_0 \neq 0$ is easily reduced to that of $c_0 = 0$.

Let N be given and write $N = a(1)A(n(1)) + \ldots + a(r)A(n(r))$. We have

$$s_N(t) = \int_0^1 f(u)\, D_N(t \dot{-} u)\, du = \int_0^1 f(u) \sum_{\nu=0}^{n-1} \psi_\nu(t \dot{-} u)\, du$$

and so

$$s_N(t)\, \phi_{n(1)}^{-a(1)}(t) \ldots \phi_{n(r)}^{-a(r)}(t)$$
$$= \int_0^1 g(u)\, \phi_{n(1)}^{-a(1)}(t \dot{-} u) \ldots \phi_{n(r)}^{-a(r)}(t \dot{-} u) \sum_{\nu=0}^{N-1} \psi_\nu(t \dot{-} u)\, du$$
$$= \int_0^1 g(u)\, K(t \dot{-} u)\, du,$$

where

$$g(u) = f(u)\, \phi_{n(1)}^{-a(1)}(u) \ldots \phi_{n(r)}^{-a(r)}(u)$$

and

$$K(u) = \sum_{l=1}^{r} \sum_{\nu=(\alpha(t)-a(l))A(n(l))}^{A(n(l)+1)-1} \psi_\nu(u)$$

As it is easily seen that, by Theorem 1, for a bounded "sequence"

$$\{\lambda_{n,j}\}, \quad |\lambda_{n,j}| \leqq M \qquad (j = 1, 2, \ldots, \alpha(n) - 1;\ n = 0, 1, 2, \ldots)$$

$$\int_0^1 \left| \sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} \lambda_{n,j}\, \delta_{n,j}(t) \right|^p dt \leqq B_{p,\alpha} M^p \int_0^1 |f(t)|^p dt$$

(cf. [6]) we have

5) We shall say in the sequel simply "polynomial" instead of "generalized Walsh polynomial".

$$(4.15)\quad \int_0^1 \left|\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} \lambda_{n,j}\,\delta_{n,j}(t)\right|^p dt \leqq B_{p,\alpha}\int_0^1 |g(t)|^p\,dt = B_{p,\alpha}\int_0^1 |f(t)|^p\,dt,$$

where $\delta_{n,j}(t;\, g)$ is a $\delta_{n,j}(t)$ made regarding $g(t)$, and we put

$$\lambda_{n,j} = \begin{cases} 1 & \text{for those } (n,j) \text{ for which } \delta_{n,j}(t) \text{ has } \psi_\nu(t) \text{ in common with } K(t) \\ 0 & \text{otherwise.}\end{cases}$$

But the left hand side of (4.15) is equal to $\int_0^1 |s_N(t)|^p\,dt$, (4.13) is proved.

5. THEOREM 3. *Let $p > 1$, $-1/p < \gamma < 1 - 1/p$ and suppose*

$$\int_0^1 |f(t)|^p t^p dt_\gamma < \infty,\quad f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(t).$$

Then we have

$$\text{(i)}\quad B_{p,\alpha\,\gamma}\int_0^1 |f(t)|^p\, t^{p\gamma}\,dt \leqq \int_0^1 \Big(|c_0|^2 + \sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |\delta_{p,j}(t)^2|\Big)^{p/2} t^{p\gamma}\,dt$$
$$\leqq B_{p,\alpha,\gamma}\int_0^1 |f(t)|^2\, t^{p\gamma}\,dt;$$

$$\text{(ii)}\quad \int_0^1 |s_N(t)|^p\, t^{p\gamma}\,dt \leqq B_{p,\alpha,\gamma}\int_0^1 |f(t)|^p\, t^{p\gamma}\,dt^{6)}:$$

$$\text{(iii)}\quad \int_0^1 |f(t) - s_N(t)|^p\, t^{p\gamma}\,dt \to 0 \quad as \quad N \to \infty.$$

This was proved, when $\alpha(n) = 2$ for all n, by I. I. Hirschman [6]. His proof is applicable to our case, with a few slight modifications, the first of wnich is the following

LEMMA 6. *Let $x_n \geqq 0$, $r > 0$, $s > 0$ and let $\{w_n\}$ be a sequence of positive numbers for which*

$$w_n/w_{n-1} \leqq q < 1 \qquad (n = 1, 2, \dots.)$$

holds for some q independent of n. Putting

$$X_n = \Big(\sum_{k=0}^{n} x_k^s\Big)^{1/s}$$

we have

6) We wish to correct an erratum which took place in [11]. On the right-hand side of the inequality (ii) of Theorem 3, the weight $t^{p\gamma}$ should be inserted, as is the case here.

$$(5.1) \qquad \sum_{n=0}^{\infty} X_n^r w_n \leqq B_{q\,r,s} \sum_{n=0}^{\infty} x_n^r w_n$$

PROOF. We have two cases:

(a) $\qquad 0 < r/s \leqq 1$

(b) $\qquad 1 < r/s.$

The proof of the case (a) is very simple, indeed, since it is easily seen that

$$X_n^r = \left(\sum_{k=0}^{n} x_k^s\right)^{r/s} \leqq \sum_{k=0}^{n} x_k^r$$

we have only to invert the order of summations:

$$\sum_{n=0}^{\infty} X_n^r w_n \leqq \sum_{n=0}^{\infty} w_n \sum_{k=0}^{n} x_k^r = \sum_{n=0}^{\infty} x_k^r \sum_{n=k}^{\infty} w_n$$

$$\leqq \sum_{k=n}^{\infty} x_k^r w_k \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \sum_{k=0}^{\infty} x_k^r w_k.$$

The case (b) is less simple and may be proved as follows.

Write

$$y_n = x_n^s w_n^{s/r}, \qquad Y_n = X_n^s w_n^{s/r}.$$

Then (5.1) would follow if we have proved that

$$(5.2) \qquad \sum_{n=0}^{\infty} Y_n Z_n \leqq B_{q,r,s} \left(\sum_{n=0}^{\infty} y_n^{r/s}\right)^{s/r}$$

for all non-negative sequence $\{Z_n\}$ such that $\sum_{n=0}^{\infty} Z_n^{r/(r-s)} = 1$. But we have

$$Y_n = \sum_{k=0}^{n} Y_k w_n^{s/r} w_k^{-s/r},$$

$$\sum_{n=k}^{\infty} w_n^{s/r} w_k^{-s/r} \leqq \sum_{n=0}^{\infty} q^{ns/r} = \frac{1}{1-q^{s/r}},$$

$$\sum_{k=0}^{n} w_n^{s/r} w_k^{-s/r} \leqq \sum_{k=0}^{n} q^{ks/r} < \frac{1}{1-q^{s/r}},$$

so that [5: Theorem 275] yields (5.2).

PROOF OF THEOREM 3. We may assume that $c_0 = 0$ and prove the theorem with the weight $t^{p\gamma}$ replaced by its "approximant" $\omega^{p\gamma}(t)$, where $\omega(t)$ is defined by

$$\omega(0) = 0, \qquad \omega(t) = A(-n) \qquad (A(-n) < t \leqq A(-n+1), n = 1, 2, \dots).$$

Write for $j = 1, 2, \dots, \alpha(n) - 1$; $n = 0, 1, 2, \dots$

$$\mu_{n,j} = \sum_{\nu = jA(n)}^{(j+1)A(n)-1} c_\nu$$

and

$$g_{n,j}(t) = A(-n)\mu_{n,j} \sum_{\nu=jA(n)}^{(j+1)A(n)-1} \Psi_\nu(t) = A(-n)\mu_{n,j}\phi_n^j(t) D_{A(n)}(t).$$

By (3.7), we have

$$(5.3) \qquad |g_{n,j}(t)| = \begin{cases} |\mu_{n,j}| & 0 \leqq t < A(-n) \\ 0 & A(-n) \leqq t < 1, \end{cases}$$

so, for every $t \neq 0$, the summation

$$g(t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} g_{n,j}(t)$$

is finite. Let us put

$$R_N = \sum_{n=0}^{N} \sum_{j=1}^{\alpha(n)-1} |\mu_{n,j}| \qquad \text{and} \qquad S_N^2 = \sum_{n=0}^{N} \sum_{j=1}^{\alpha(n)-1} |\mu_{n,j}|^2.$$

From (5.3) we have, for $A(-N) \leqq t < A(-N+1)$,

$$|g(t)| \leqq R_N$$

and

$$\sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |g_{n,j}(t)|^2 = \sum_{n=0}^{N} \sum_{j=1}^{\alpha(n)-1} |\mu_{n,j}|^2 = S_N^2.$$

Thus

$$(5.4) \qquad \int_0^1 \omega^{p\gamma}(t)|g(t)|^p\,dt = \sum_{m=1}^{\infty} \int_{A(-m)}^{A(-m+1)} \omega^{p\gamma}(t)|g(t)|^p\,dt$$

$$= \sum_{m=1}^{\infty} A^{p\gamma}(-m) \int_{A(-m)}^{A(-m+1)} |g(t)|^p\,dt$$

$$\leqq \sum_{m=1}^{\infty} A^{p\gamma}(-m) R_m^p (A(-m+1) - A(-m))$$

$$\leqq (\alpha - 1) \sum_{m=1}^{\infty} R_m^p A^{1+p\gamma}(-m)$$

and, by (5.3)

$$(5.5) \qquad \int_0^1 \omega^{p\gamma}(t) \Big(\sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |g_{n,j}(t)|^2 \Big)^{p/2} dt$$

$$= \sum_{m=1}^{\infty} \int_{A(-m)}^{A(-m+1)} \omega^{p\gamma}(t) \Big(\sum_{n=0}^{\infty} \sum_{j=1}^{\alpha(n)-1} |g_{n,j}(t)|^2 \Big)^{p/2} dt$$

$$= \sum_{m=1}^{\infty} A^{p\gamma}(-m) S_m^p (A(-m+1) - A(-m))$$

$$\leqq (\alpha-1)\sum_{m=1}^{\infty} S_m^p\, A^{1+p\gamma}(-m).$$

On the other hand, the equality

$$\mu_{n,j}=\sum_{\nu=jA(n)}^{(j+1)A(n)-1} c_\nu=\int_0^1 f(t)\sum_{\nu=jA(n)}^{(j+1)A(n)-1}\Psi_\nu(t)\,dt$$

$$=\int_0^1 f(t)\,\phi_n^{-j}(t)\,D_{A(n)}(t)\,dt$$

yields, for a fixed β satisfying $\gamma<\beta<1-1/p$,

$$|\mu_{n,j}|^p\leqq\left(\int_0^1 |f(t)|\,D_{A(n)}(t)\,dt\right)^p$$

$$\leqq A^p(n)\int_0^{A(-n)}|f(t)|^p\,\omega^{p\beta}(t)\,dt\cdot\left(\int_0^{A(-n)}\omega^{-q\beta}(t)\,dt\right)^{p/q}$$

$$\leqq B_{p,\alpha,\gamma}A^{1+p\beta}(n)\int_0^{A(-n)}|f(t)|^p\omega^{p\beta}(t)\,dt,$$

where q is the conjugate exponent of p. Denoting the characteristic function of the interval $[0,A(-n))$ by $\chi(n,t)$, we have

$$|\mu_{n,j}|^p\;A^{1+p\gamma}(-n)\leqq B_{p,\alpha,\gamma}\;A^{p(\beta-\gamma)}(n)\int_0^1|f(t)|^p\,\omega^{p\beta}(t)\,\chi(n,t)\,dt$$

and, summing up this inequality,

$$(5.6)\qquad \sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\mu_{n,j}|^p A^{1+p\gamma}(-n)$$

$$\leqq B_{p,\alpha,\gamma}\int_0^1|f(t)|^p\,\omega^{p\beta}(t)\sum_{n=0}^{\infty}A^{p(\beta-\gamma)}(n)\,\chi(n,t)\,dt.$$

Since it is easily seen that there is a constant $B_{p,\alpha,\beta,\gamma}=B_{p,\alpha,\gamma}$ such that

$$\sum_{n=0}^{\infty}A^{p(\beta-\gamma)}(n)\,\chi(n,t)\leqq B_{p,\alpha,\gamma}\,\omega^{-p(\beta-\gamma)}(t),$$

(5.6) can be written in the form of

$$(5.7)\qquad \sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\mu_{n,j}|^p A^{1+p\gamma}(-n)\leqq B_{p,\alpha,\gamma}\int_0^1|f(t)|^p\,\omega^{p\gamma}(t)\,dt.$$

(5.4), (5.5), (5.7) and Lemma 6 give

$$(5.8)\qquad \int_0^1|g(t)|^p\,\omega^{p\gamma}(t)\,dt\leqq B_{p,\alpha,\gamma}\int_0^1|f(t)|^p\,\omega^{p\gamma}(t)\,dt,$$

(5.9) $$\int_0^1 \left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |g_{n,j}(t)|^2\right)^{p/2} \omega^{p\gamma}(t)\,dt \leqq B_{p,\alpha,\gamma}\int_0^1 |f(t)|^p\,\omega^{p\gamma}(t)\,dt.$$

On the other hand, in order that a function $h(t)$ have the form

$$\sum_{\nu=jA(n)}^{(j+1)A(n)-1} c_\nu\,\psi_\nu(t)$$

it is necessary and sufficient that

(a) $h(t)$ is constant on each of the intervals

$$[mA(-n-1),\ (m+1)A(-n-1)) \qquad m=0,1,\dots,A(n+1)-1$$

and

(b) $h(t)\,\phi_n^{-k}(t)$ $(k=0,1,\dots,j-1,j+1,\dots,\alpha(n)-1)$
has mean 0 over each of the intervals

$$[mA(-n),\ )m+1)\,A(-n)),\ m=0,1,\dots,\ A(n)-1.$$

This fact shows

$$\omega^\gamma(t)\ (\delta_{n,j}(t)-g_{n,j}(t)) = \omega(t)) = \omega^\gamma(t)\ (\delta_{n,j}(t\,;f)-\delta_{n,j}(t\,;\,g)) = \delta_{n,j}(t:\ \omega^\gamma f-\omega^\gamma g).$$

since $\delta_{n,j}(t\,;f)=\mu_{n,j}\phi_n^j(t)=g_{n,j}(t)$ for $0\leqq t<A(-n)$.

Now we can appeal to Theorem 1, obtaining

$$\int_0^1 \left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1} |\delta_{n,j}(t)-g_{n,j}(t)|^2\right)^{p/2} \omega^{p\gamma}(t)\,dt \leqq B_{p,\alpha,\gamma}\int_0^1 |f(t)-g(t)|^p\,\omega^{p\gamma}(t)\,dt$$

This, combined with (5.8) and (5.9), gives the second half of (i).

The first half is proved similarly. We have

$$\mu_{n,j} = \sum_{\nu=jA(n)}^{(j+1)A(n)-1} c = \int_0^1 f(t)\sum_{\nu=jA(n)}^{(j+1)A(n)-1}\bar{\psi}_\nu(t)\,dt = \int_0^1 \delta_{n,j}(t)\sum_{\nu=jA(n)}^{(j+1)A(n)-1}\bar{\psi}_\nu(t)\,dt = \int_0^1 \delta_{n,j}(t)\,\phi_n^{-j}(t)\,D_{A(n)}(t)\,dt = A(n)\int_0^{A(-n)} \delta_{n,j}(t)\,\phi_n^{-j}(t)\,dt,$$

and consequently

$$|\mu_{n,j}|^p \leqq B_{p,\alpha,\gamma}\,A^{1+p\beta}(n)\int_0^{A(-n)} |\delta_{n,j}(t)|^p\,\omega^{p\beta}(t)\,dt$$

$$\leqq B_{p,\alpha,\gamma}\ A^{1+p\beta}(n)\int_0^{1(-n)}\left(\sum_{m=0}^{\infty}\sum_{i=1}^{\alpha(m)-1}|\delta_{m,i}(t)|^2\right)^{p/2}\omega^{p\beta}(t)\,dt.$$

By an argument similar to one that led to (5.7), we see

$$(5.10)\quad \sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\mu_{i,j}|^p A^{1+p\gamma}(-n)$$

$$\leqq B_{p,\alpha,\gamma}\int_0^1\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\right)^{p/2}\omega^{p\gamma}(t)\,dt.$$

(5.4), (5.5), (5.10) and Lemma 6 together show

$$(5.11)\quad \int_0^1|g(t)|^p\omega^{p\gamma}(t)\,dt\leqq B_{p,\alpha,\gamma}\int_0^1\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\right)^{p/2}\omega^{p\gamma}(t)\,dt,$$

$$(5.12)\quad \int_0^1\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|g_{n,j}(t)|^2\right)^{p/2}\omega^{p\gamma}(t)\,dt$$

$$\leqq B_{p,\alpha,\gamma}\int_0^1\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\right)^{p/2}\omega^{p\gamma}(t)\,dt.$$

By Theorem 1,

$$(5.13)\quad \int_0^1|f(t)-g(t)|^p\omega^{p\gamma}(t)dt\leqq B_{p,\alpha,\gamma}\int_0^1\left(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\right)^{p/2}\omega^{p\gamma}(t)\,dt.$$

Combining (5.11), (5.12), (5.13) we obtain the first half of (i).

Part (ii) of the theorem is proved in the same line as Theorem 2 is deduced from Theorem 1. The only thing to be observed is that

$$\phi_{n(1)}^{-a(1)}(u)\cdots\phi_{n(r)}^{-a(r)}(u)\,D_n(u)=\sum_{i=1}^{r}\sum_{\nu=(\alpha(n(i))-a(i))A(n(i))}^{A(n(i)+1)-1}\psi_\nu(u)$$

where the $n(i)$'s and $a(i)$'s are related to n by (1.2) and (1.3). Part (iii) is an immediate consequence of part (ii), because it is easily seen that the polynomials are dense in our space of all functions $f(t)$ for which $\int_0^1|f(t)|^p t^{p\gamma}\,dt<\infty$, the norm being taken as the $1/p$ th power of that integral.

Since the latter half of § 3 we have constantly supposed that the sequence $\{\alpha(n)\}$ is bounded, $\alpha(n)\leqq\alpha$. If we remove this restriction, our fundamental Theorem 1 ceases to be true. That is, we can say as follows:

THEOREM 4. *Let the sequence* $\{\alpha(n)\}$ *be unbounded. Then* (i) *there is a function* $f(t)$, *belonging to every Lebesgue class* $L^p(0,1)$, $0<p<2$ *for which*

$$\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2=\infty$$

for all t;

(ii) *there is a function* $g(t)$, *belonging to none of Lebesgue classes* $L^p(0,1)$, $p>2$, *and for which*

$$\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\leqq M$$

for all t.

PROOF. Since the following proof depends on the mutual independence of $\phi_n(t)$'s and the relation (3.5) only, we may, extracting a subsequence if necessary, suppose that

(5.14) $$\alpha(n+1)/\alpha(n)\geqq\lambda>1. \qquad (n=0,1,2,\ldots)$$

(i) Let $c_{n,j}=c_n=\dfrac{1}{\sqrt{\alpha(n)}}$, $\quad C=\sum_{n=1}^{\infty}c_n \quad$ and put

$$f(t)=\sum_{n=1}^{\infty}\sum_{j=1}^{\alpha(n)-1}c_{n\,j}\,\phi_n^j(t)+C$$

Then, by (3.5), we have

(5.15) $$f(t)=\sum_{n=1}^{\infty}c_n\sum_{j=0}^{\alpha(n)-1}\phi_n^j(t)=\sum_{n=1}^{\infty}{}'\,c_n\,\alpha_n=\sum_{n=1}^{\infty}{}'\sqrt{\alpha(n)}$$

where Σ' denotes the summation over those n's for which $\phi_n(t)=1$. Observe that this summation is finite a.e., by the well-known Borel-Cantelli lemma.

Now let us define the sets $E(n)$, $n=0,1,2,\ldots\ldots$, by

$$E(0)=\{t:\phi_n(t)\neq1 \quad \text{for all } n\geqq1\}$$

$$E(n)=\{t:\phi_n(t)=1, \text{ and } \phi_m(t)\neq1 \text{ for all } m\geqq n+1\}.$$

These sets are mutually disjoint, together fill up the interval $(0,1)$ and their measures are respectively

$$\text{meas}\,E(0)=\prod_{n=1}^{\infty}\left(1-\frac{1}{\alpha(n)}\right)$$

$$\text{meas}\,E(n)=\frac{1}{\alpha(n)}\prod_{m=1}^{\infty}\left(1-\frac{1}{\alpha(m)}\right).$$

(5.15) shows that $0\leqq f(t)\leqq\sum_{k=1}^{n}\sqrt{\alpha(k)}$ for $t\in E(n)$: consequently for $0<p<2$,

$$\int_0^1 f^p(t)dt=\sum_{n=1}^{\infty}\int_{E(n)}f^p(t)\,dt$$

$$\leqq\sum_{n=1}^{\infty}\frac{1}{\alpha(n)}\left(\sum_{k=1}^{n}\sqrt{\alpha(k)}\right)^p\leqq B_{p,\lambda}\sum_{n=1}^{\infty}(\alpha(n))^{-1+p/2}<\infty$$

by (5.14). But it is evident that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2=\sum_{n=1}^{\infty}c_n^2(\alpha(n)-1)=\sum_{n=1}^{\infty}\left(1-\frac{1}{\alpha(n)}\right)=\infty.$$

(ii) Let us now take $c_{n,j}=c_n=1/n\sqrt{\alpha(n)}$ and consider

$$g(t)=\sum_{n=1}^{\infty}c_n+\sum_{n=1}^{\infty}c_n\sum_{j=1}^{\alpha(n)-1}\phi_n^j(t)=\sum_{n=1}^{\infty}{}'c_n\alpha(n)=\sum_{n=1}^{\infty}{}'\frac{\sqrt{\alpha(n)}}{n}$$

Now

$$\delta_{n,j}(t)=\delta_{n,j}(t\,;g)=\phi_n^j(t)/n\sqrt{\alpha(n)}$$

and for every t,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\leqq\sum_{n=1}^{\infty}c_n^2\alpha(n)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

But, for $p>2$

$$\int_0^1 g^p(t)\,dt=\sum_{n=1}^{\infty}\int_{E(n)}g^p(t)\geqq\sum_{n=1}^{\infty}c_n^p\alpha^p(n)\text{ meas }E(n)$$

$$>B_{p,\lambda}\sum_{n=1}^{\infty}\frac{\alpha^{p/2-1}(n)}{n^p}>B_{p,\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{(p/2-1)n}}{n^p}=\infty.$$

6. The Cesàro summability of the "proper" Walsh Fourier series was proved by N. J. Fine [3]. Recently, S. Yano [13] sharpened this result into a maximal theorem and brought to the case of $\alpha(n)=\alpha$ with arbitrary α. In this connection we prove two theorems, the one concerning Cesàro summability factors, the other convergence factors.

THEOREM 5. *Let* $f(t)\in L(0,1)$, $f(t)\sim\sum_{\nu=0}^{\infty}c_\nu\psi_\nu(t)$. *Then putting the* $(C,-\eta)$ *means of the series* $\sum_{\nu=0}^{\infty}\frac{c_\nu\psi_\nu(t)}{(\nu+1)^\eta}$ *by* $N_n^{(\eta)}(t\,;f)$, *we have*

(i) $$\int_0^1\sup_n|N_n^{(\eta)}(t\,;f)|\,dt\leqq B_{\alpha,\eta}\int_0^1|f(t)|\,dt\,;$$

(ii) *the sequence* $\{N_n^{(\eta)}(t\,;f)\}$ *converges almost everywhere.*

For the case of $\alpha(n)=2$ $(n=0,1,2,\ldots.)$ this theorem was proved by S. Yano [12], his proof is also applicable to our case, as we are going to see. We begin by proving several lemmas:

LEMMA 7. *For* $0<t<1$ *and* $n\geqq1$, *we have*

(6.1) $$|D_n(t)|\leqq\min(n,\alpha/t).$$

This is almost known; we prove it for the sake of completeness only.

PROOF. For a given t, choose N so that

$$A(-N) \leqq t < A(-N+1)$$

and write

$$n = qA(N) + r \qquad 0 \leqq r < A(N).$$

Then

$$D_n(t) = \sum_{\nu=0}^{n-1} \psi_\nu(t) = \sum_{l=0}^{q-1} \sum_{\nu=0}^{A(N)-1} \psi_{\nu+lA(N)}(t) + \sum_{\nu=0}^{r-1} \psi_{\nu+qA(N)}(t)$$

$$= D_{A(N)}(t) \sum_{l=0}^{q-1} \psi_{lA(N)}(t) + \sum_{\nu=0}^{r-1} \psi_{\nu+qA(N)}(t).$$

Since $D_{A(N)}(t) = 0$ for $A(-N) \leqq t < 1$, we have

$$|D_n(t)| \leqq r < A(N) \leqq \alpha(N-1)/t \leqq \alpha/t$$

as was to be proved.

REMARK. From (6.1) we obtain

$$\int_0^1 |D_n(t)|\, dt = \int_0^{1/n} |D_n(t)|\, dt + \int_{1/n}^1 |D_n(t)|\, dt$$

(6.2)

$$\leqq n \int_0^{1/n} dt + \alpha \int_{1/n}^1 \frac{dt}{t} \leqq B_\alpha \log(n+1)$$

and an appeal to Lemma 3 shows, for $f(t) \in L(0, 1)$,

(6.3) $\qquad s_n(t) = o(\log n) \qquad$ a.e.

where $s_n(t)$ denotes the n-th partial sum of the Fourier series of $f(t)$.

LEMMA 8. *Let* $0 < \eta < 1$ *and put* $H_n^{(\eta)}(t) = \sum_{\nu=0}^{n-1} \frac{\psi_\nu(t)}{(\nu+1)^\eta}$. *Then we have*

(6.4) $\qquad |H_n^{(\eta)}(t)| \leqq B_{\alpha,\eta}/t^{1-\eta} \qquad 0 < t < 1.$

PROOF. If $0 < t \leqq 1/n$, the assertion is almost trivial:

$$|H_n^{(\eta)}(t)| \leqq \sum_{\nu=0}^{n-1} \frac{1}{(\nu+1)^\eta} \leqq 1 + \frac{n^{1-\eta}}{1-\eta} \leqq \frac{2}{(1-\eta)t^{1-\eta}}.$$

Suppose now $1/n < t < 1$. Writing $m = [1/t]$, we have

$$|H_n^{(\eta)}(t)| = \left| \sum_{\nu=0}^{n-1} \frac{\psi(t)}{(\nu+1)^\eta} \right|$$

$$\leqq \left| \sum_{\nu=0}^{m-1} \frac{\psi_\nu(t)}{(\nu+1)^\eta} \right| + \left| \sum_{\nu=m}^{n-1} \frac{\psi_\nu(t)}{(\nu+1)^\eta} \right| \equiv S_1 + S_2,$$

say. That $S_1 \leqq B_\eta/t^{1-\eta}$ has already been shown. As to S_2, Abel's transformation shows

$$S_2 = \sum_{\nu=m}^{n-2} \left(\frac{1}{(\nu+1)^\eta} - \frac{1}{(\nu+2)^\eta} \right) (D_{\nu+1}(t) - D_m(t)) + \frac{D_n(t) - D_m(t)}{(n-1)^\eta}$$

$$\leqq \frac{2\alpha}{t} \cdot \frac{1}{(m+1)^\eta} + \frac{2}{(n-1)^\eta} \cdot \frac{\alpha}{t} \leqq \frac{4\alpha}{t^{-\eta}} \qquad \text{q. e. d.}$$

LEMMA 9. *We have, for* $0 < \eta < 1$, $0 < m \leqq n$, $0 < t < 1$,

$$\left| \sum_{\nu=n-m}^{n-1} A_{n-\nu-1}^{(-\eta)} \psi_\nu(t) \right| < \frac{B_{\alpha,\eta}}{t^{1-\eta}}$$

where $A_m^{(k)} = \binom{m+k}{k} = \frac{\Gamma(m+k+1)}{\Gamma(k+1)\Gamma(m+1)} \sim \frac{m^k}{\Gamma(k+1)}$ $\qquad (k \neq -1, -2, \ldots\ldots)$.

PROOF. For $0 < t \leqq 1/m$, we have

$$\left| \sum_{\nu=n-m}^{n-1} A_{n-\nu-1}^{(-\eta)} \psi_\nu(t) \right| \leqq \sum_{\nu=n-1}^{n-1} A_{n-\nu-1}^{(-\eta)} \leqq \sum_{\nu=0}^{m} A_\nu^{(\eta)} \leqq B_\eta \sum_{\nu=0}^{m} \frac{1}{(\nu+1)^\eta}$$

$$\leqq B_\eta m^{1-\eta} \leqq B_\eta / t^{1-\eta}.$$

For $1/m < t < 1$, we have, putting $p = [1/t]$.

$$\left| \sum_{\nu=n-m}^{n-1} A_{n-\nu-1}^{(\eta)} \psi_\nu(t) \right| \leqq \left| \sum_{\nu=n-m}^{n-p-1} A_{n-\nu-1}^{(\eta)} \psi_\nu(t) \right| + \left| \sum_{\nu=n-p}^{n-1} A_{n-\nu-1}^{(-\eta)} \psi_\nu(t) \right|$$

$$\equiv T_1 + T_2,$$

say. It is sufficient to estimate T_1. By Abel's transformation, we see

$$T_1 \leqq \sum_{\nu=n-m}^{n-p-1} |A_{n-\nu-1}^{(-\eta-1)}| \, |D_\nu(t)| + A_p^{(-\eta)} |D_{n-p}(t)| + A_m^{(-\eta)} |D_{n-m}(t)|$$

$$\leqq \frac{\alpha}{t} \left(\sum_{\nu=n-m}^{n-p-2} |A_{n-\nu-1}^{(-\eta-1)}| + A_p^{(\eta)} + A_m^{(-\eta)} \right)$$

$$\sim \frac{\alpha}{t} \left(\sum_{\nu=p}^{m-1} \frac{\nu^{-\eta-1}}{|\Gamma(-\eta-1)|} + \frac{p^{-\eta}}{\Gamma(1-\eta)} + \frac{m^{-\eta}}{\Gamma(1-\eta)} \right)$$

$$\leqq \frac{B_{\alpha,\eta}}{t} \left(\sum_{\nu=p}^{\infty} \frac{1}{\nu^{1+\eta}} + \frac{1}{p^\eta} \right) \leqq \frac{B_{\alpha,\eta}}{t} \cdot \frac{1}{p^\eta} \leqq \frac{B_{\alpha,\eta}}{t^{1-\eta}} \qquad \text{q. e. d.}$$

Now we put

$$K_n^{(-\eta)}(t) = \frac{1}{A_{n-1}^{(-\eta)}} \sum_{\nu=0}^{n-1} \frac{A_{n-\nu-1}^{(-\eta)} \psi_\nu(t)}{(\nu+1)^\eta}$$

and

$$N_n^{(\eta)}(t; f) = \int_0^1 f(u) K_n^{(-\eta)} (t \dotplus u) \, du,$$

so that in particular if $p_k(t)$ is a polynomial $\sum_{\nu=0}^{k-1} b_\nu \psi_\nu(t)$, we have

(6.5) $$N_n^{(\eta)}(t\,;p_k) = \sum_{\nu=0}^{k-1} \frac{b_\nu \psi_\nu(t)}{(\nu+1)^\eta} \frac{A^{(-\eta)}_{n-\nu-1}}{A^{(-\eta)}_{n-1}} \to \sum_{\nu=0}^{k-1} \frac{b_\nu \psi_\nu(t)}{(\nu+1)^\eta} = p_k^*(t).$$

LEMMA 10. *We have, for $0 < \eta < 1$ and $0 < t < 1$.*

$$|K_n^{(-\eta)}(t)| \leqq B_{\alpha,\eta}/t^{1-\eta}$$

PROOF.
$$K_n^{(-\eta)}(t) = \frac{1}{A^{(-\eta)}_{n-1}} \sum_{\nu=0}^{n-1} A^{(-\eta)}_{n-\nu-1} \frac{\psi_\nu(t)}{(\nu+1)^\eta}$$
$$= \frac{1}{A^{(-\eta)}_{n-1}} \left(\sum_{\nu=0}^{[n/2]-1} + \sum_{\nu=[n/2]}^{n-1} \right) = P_n + Q_n,$$

say. By Abel's transformation, we have

$$P_n = \frac{1}{A^{(1-\eta)}_{n-1}} \left\{ \sum_{\nu=0}^{p-2} H^{(\eta)}_{\nu+1}(t)\, A^{(-\eta-1)}_{n-\nu-1} + H_p(t)\, A^{(-\eta)}_{n-p} \right\}$$

where we write p for $[n/2]$, from which it follows that, by Lemma 8.

$$|P_n| \leqq \frac{1}{A^{(-\eta)}_{n-1}} B_{\alpha,\eta} t^{\eta-1} \left(\sum_{\nu=0}^{p-2} |A^{(-\eta-1)}_{n-\nu-1}| + A^{(-\eta)}_{n-p} \right)$$
$$\leqq B_{\alpha,\eta} n^\eta t^{\eta-1} \left(\sum_{\nu=p}^{n-1} \frac{1}{\nu^{1+\eta}} + p^{-\eta} \right)$$
$$\leqq B_{\alpha,\eta} n^\eta t^{\eta-1} p^{-\eta} \leqq B_{\alpha,\eta} t^{\eta-1}.$$

As to Q_n, a similar argument (using Lemma 9 instead of Lemma 8) shows

$$|Q_n| \leqq \frac{1}{A^{(-\eta)}_{n-1}} \sum_{\nu=p}^{n-2} \left(\frac{1}{(\nu+1)^\eta} - \frac{1}{(\nu+2)^\eta} \right) \sum_{j=p}^{\nu} A^{(-\eta)}_{n-j-1} \psi_j(t)$$
$$+ \frac{n^{-\eta}}{A^{(-\eta)}_{n-1}} \left| \sum_{\nu=p}^{n-1} A^{(-\eta)}_{n-\nu-1} \psi_\nu(t) \right|$$
$$\leqq \frac{B_{\alpha,\eta}}{A^{(-\eta)}_{n-1}} \sum_{\nu=p}^{n-1} \left\{ \left(\frac{1}{\nu^{1+\eta}} + \frac{1}{n^\eta} \right) \max_{p \leqq i \leqq n} \sum_{j=p}^{\nu} A^{(-\eta)}_{n-j-1} \psi_j(t) \right.$$
$$\leqq B_{\alpha,\eta} \max_{p \leqq i \leqq n} \left| \sum_{j=p}^{\nu} A^{(-\eta)}_{n-j-1} \psi_j(t) \right| \leqq \frac{B_{\alpha,\eta}}{t^{1-\eta}}$$

q. e. d.

PROOF OF THEOREM 5. By Lemma 10, we have

(6.6) $$|N_n^{(\eta)}(t\,;f)| \leqq \int_0^1 |f(u)|\, |K_n^{(-\eta)}(t \dot{-} u)|\, du$$
$$= \int_0^1 |f(t \dot{+} u)|\, |K_n^{(-\eta)}(u)|\, du \leqq B_{\alpha,\eta} \int_0^1 |f(t \dot{+} u)|\, u^{\eta-1}\, du$$

Since the right-hand side of (6.6) is independent of n, taking the supremum with respect to n and integrating with respect to t over the unit interval, we obtain

$$\int_0^1 \sup_n |N_n^{(\eta)}(t:f)|\, dt \leqq B_{\alpha,\eta}\int_0^1 dt \int_0^1 |f(t+u)|u^{\eta-1}\, du$$

$$= B_{\alpha,\eta}\int_0^1 u^{\eta-1}\, du \int_0^1 |f(t+u)|\, dt$$

$$= B_{\alpha,\eta}\int_0^1 |f(t)|\, dt$$

which is the part (i) of our theorem.

To infer (ii) from this maximal inequality, we may argue as follows. Let $f^*(t)$ be the sum of the series $\sum_{\nu=0}^{\infty} \frac{c_\nu \psi_\nu(t)}{(\nu+1)^\eta}$. This series converges almost everywhere by (6.3), and in L^1-norm by Lemma 8, and $N_n^{(\eta)}(t;f)$ converges in L^1-norm by (i) already proved. Thus we have, by "consistency" of $(C, -\eta)$ summability,

$$(6.7) \qquad \int_0^1 |f^*(t) - N_n^{(\eta)}(t;f)|\, dt \to 0 \qquad \text{as } n\to\infty.$$

For a given $\varepsilon > 0$ let us choose a polynomial $p(t) = p_k(t)$ so that

$$\int_0^1 |f(t) - p_k(t)|\, dt < \frac{\varepsilon^2}{2B_{\alpha,\eta}}$$

Our assertion (i), applied to the function $f(t) - p(t)$, yields

$$\int_0^1 \sup_n |N_n^{(\eta)}(t.f-p)|\, dt \leqq B_{\alpha,\eta}\int_0^1 |f(t)-p(t)|\, dt < \frac{\varepsilon^2}{2}$$

and by (6.7)

$$\int_0^1 |f^*(t) - p^*(t)|\, dt \leqq \frac{\varepsilon^2}{2}$$

where $p^*(t)$ is a polynomial expressed by (6.5).

Now define the set $E = E(\varepsilon)$ by

$$E = \{t: \sup_n |N_n^{(\eta)}(t;f-p)| > \varepsilon\} \cap \{t: |f^*(t) - p^*(t)| > \varepsilon\}$$

Then we have

$$\text{meas}\, E < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and, for t belonging to the complement of E,

$$\limsup_{n\to\infty} |N_n^{(\eta)}(t:f) - f^*(t)| \leqq \limsup_{n\to\infty} |N_n^{(\eta)}(f-p)| - |f^*(t)-p^*(t)|$$
$$\leqq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we complete the proof of (ii).

Next we prove a theorem on the convergence factor for the class L^2.

THEOREM 6. *Let* $f(t) \in L^2(0,1)$, $f(t) \sim \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(t)$

Then putting

$$s_n^*(t) = \sum_{\nu=0}^{n-1} \frac{c_\nu \psi_\nu(t)}{\sqrt{\log(\nu+2)}}$$

we have

(i) $$\int_0^1 \sup_n |s_n^*(t)|^2\, dt \leqq B_\alpha \int_0^1 |f(t)|^2\, dt$$

(ii) *the sequence* $\{s_n^*(t)\}$ *converges almost everywhere.*

In case of $\alpha(n) = 2$ $(n = 0, 1, 2, \ldots)$, this theorem was stated by R. E. A. C. Paley [7] and proved by S. Yano [12]. Our proof is different from that of Yano, and done following the line of G. H. Hardy and J. E. Littlewood [5]. We shall need a lemma, which was proved by G. Sunouchi [9] for general $p > 1$, in case of the "proper" Walsh Fourier series, and known, in essence, also for the generalised Walsh Fourier series and general $p > 1$ (Yano [13]). But we supply it with a proof, for the sake of completeness.

LEMMA 11. *Let* $f(t) \in L^2(0,1)$ $.f(t) \sim \sum c_\nu \psi_\nu(t)$. *Then denoting by* $\sigma_n(t)$ *the* $(C,1)$ *means of this series, we have*

$$\int_0^1 \sup_n |\sigma_n(t)|^2\, dt \leqq B_\alpha \int_0^1 |f(t)|^2\, dt.$$

PROOF. The method given in [9] applies with few changes: this is done in two steps.

(a) $$\int_0^1 \sup_n |\sigma_{A(n)}(t)|^2\, dt \leqq B \int_0^1 |f(t)|^2\, dt.$$

Since $$|\sigma_{A(n)}(t)|^2 \leqq 2|s_{A(n)}(t)|^2 + 2|s_{A(n)}(t) - \sigma_{A(n)}(t)|^2,$$
it is sufficient to prove that (cf. the first of the inequalities (3.9))

$$\int_0^1 \sup_n |s_{A(n)}(t) - \sigma_{A(n)}(t)|^2\, dt \leqq \sum_{n=0}^{\infty} \int_0^1 |s_{A(n)}(t) - \sigma_{A(n)}(t)|^2\, dt$$
$$\leqq B_\alpha \int_0^1 |f(t)|^2\, dt,$$

of which the first inequality is trivial. But we have

$$\sum_{n=0}^{\infty}\int_0^1 |s_{A(n)}(t)-\sigma_{A(n)}(t)|^2\,dt$$

$$=\sum_{n=1}^{\infty}\frac{1}{A^2(n)}\sum_{\nu=1}^{A(n)-1}\nu^2|c_\nu|^2$$

$$=\sum_{n=1}^{\infty}\frac{1}{A^2(n)}\sum_{j=1}^{n-1}\sum_{\nu=A(j)}^{A(j+1)-1}\nu^2|c_\nu|^2$$

$$=\sum_{j=0}^{\infty}\sum_{\nu=A(j)}^{A(j+1)-1}\nu^2|c_\nu|^2\sum_{n=j+1}^{\infty}\frac{1}{A^2(n)}$$

$$\leqq 2\sum_{j=0}^{\infty}\sum_{\nu=A(j)}^{A(j+1)-1}|c_\nu|^2=2\int_0^1 |f(t)|^2\,dt,$$

as desired.

(b) $$\int_0^1 \sup_n |\sigma_n(t)|^2\,dt\leqq B_\alpha\int_0^1 |f(t)|^2\,dt$$

For a given n, there is an N such that $A(N)\leqq n<A(N+1)$. We have

$$|\sigma_n(t)|^2\leqq 2|\sigma_n(t)-\sigma_{A(N)}(t)|^2+2|\sigma_{A(N)}(t)|^2$$

and

$$|\sigma_n(t)-\sigma_{A(N)}(t)|^2\leqq\Big(\sum_{j=A(N)}^{A(N+1)-1}|\sigma_{j+1}(t)-\sigma_j(t)|\Big)^2$$

$$\leqq\sum_{j=A(N)}^{A(N+1)-1}\Big(\sqrt{j}\,|\sigma_{j+1}(t)-\sigma_j(t)|\Big)^2\sum_{j=A(N)}^{A(N+1)-1}\Big(\frac{1}{\sqrt{j}}\Big)^2$$

$$\leqq\log\alpha(N)\sum_{j=A(N)}^{A(N+1)-1}j|\sigma_{j+1}(t)-\sigma_j(t)|^2$$

$$\leqq\log\alpha\sum_{j=A(N)}^{A(N+1)-1}j|\sigma_{j+1}(t)-\sigma_j(t)|^2.$$

Thus we have

$$\sup_n|\sigma_n(t)|^2\leqq 2\sup_n|\sigma_{A(N)}(t)|^2+2\log\alpha\sum_{N=0}^{\infty}\sum_{j=A(N)}^{A(N+1)-1}j|\sigma_{j+1}(t)-\sigma_j(t)|^2 \tag{6.8}$$

$$=2\sup_N|\sigma_{A(N)}(t)|^2+2\log\alpha\sum_{j=1}^{\infty}j|\sigma_{j+1}(t)-\sigma_j(t)|^2$$

Integrating both sides of (6.8) and appealing to (a), we have (b). (See also Kaczmarz-Steinhaus, Theorie der Orthogonalreihen, p. 188.)

Now we turn to the proof of Theorem 6. Let us first prove

$$\int_0^1 \frac{|s_{n(t)}(t)|^2}{\log (n(t)+2)}\, dt \leqq B_\alpha \int_0^1 |f(t)|^2\, dt \tag{6.9}$$

for any measurable function $n(t)$ taking non-negative integers as its values. Without loss of generality, we may suppose that $\int_0^1 |f(t)|^2\, dt = 1$, and confine ourselves to those $n(t)$'s which are bounded by a number, say H, arbitrarily fixed. Thus what we must prove is reduced to the following inequality:

$$\sup_g \left| \int_0^1 U_H(t)\overline{g(t)}\, dt \right| = \sup_g |J_g| \leqq B \tag{6.10}$$

where $U_H(t) = s_{n(t)}(t)(\log(n(t)+2))^{-1/2}$ and the supremum is taken for those g's for which $\int_0^1 |g(t)|^2\, dt = 1$. Now

$$J = J_g = \int_0^1 \frac{s_{n(t)}(t)\,\overline{g(t)}}{\sqrt{\log(n(t)+2)}}\, dt$$

$$= \int_0^1 \frac{\overline{g(t)}}{\sqrt{\log(n(t)+2)}}\, dt \int_0^1 f(u)\, D_{n(t)}(t \dot{-} u)\, du$$

$$= \int_0^1 f(u)\, du \int_0^1 \frac{\overline{g(t)}\, D_{n(t)}(t \dot{-} u)}{\sqrt{\log(n(t)+2)}}\, dt$$

and by Schwarz's inequality,

$$|J|^2 \leqq \int_0^1 |f(u)|^2\, du \int_0^1 \left| \int_0^1 \frac{g(t)\, D_{n(t)}(t \dot{-} u)}{\sqrt{\log(n(t)+2)}}\, dt \right|^2 du$$

$$= \int_0^1 du \int_0^1 \frac{\overline{g(t)}\, D_{n(t)}(t \dot{-} u)}{\sqrt{\log(n(t)+2)}}\, dt \int_0^1 \frac{g(x)\, \overline{D_{n(x)}(x \dot{-} u)}}{\sqrt{\log(n(x)+2)}}\, dx \tag{6.11}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{\overline{g(t)}\, D_{n(t)}(t \dot{-} u)\, g(x)\, \overline{D}_{n(x)}(x \dot{-} u)}{\sqrt{\log(n(t)+2)}\,\sqrt{\log(n(x)+2)}}\, dt\, dx\, du.$$

Integrating first by u and observing the fact that

$$\int_0^1 D_{n(t)}(t)(t \dot{-} u)\, \overline{D}_{n(x)}(x \dot{-} u)\, du = D_{n(t,x)}(t \dot{-} x)$$

where $n(t,x) = \min(n(t),\ n(x))$, we see that the last tripple integral in (6.11) takes the form

$$\int_0^1 \int_0^1 \frac{\overline{g(t)}\, g(x)\, D_{n(t,x)}(t \dot{-} x)}{\sqrt{\log(n(t)+2)}\,\sqrt{\log(n(x)+2)}}\, dt\, dx \tag{6.12}$$

$$\leqq \int_0^1 \int_0^1 \frac{|g(t)|}{\sqrt{\log(n(t)+2)}} \cdot \frac{|g(x)|}{\sqrt{\log n(x)+2}} |D_{n(t,x)}(t \dot{-} x)| \, dt \, dx$$

$$\leqq B_\alpha \int_0^1 \int_0^1 \left\{ \frac{|g(t)|^2}{\log(n(t)+2)} + \frac{|g(x)|^2}{\log(n(x)+2)} \right\} \frac{dt \, dx}{|t-x| + (n(t,x)+1)^{-1}}$$

by Lemmas 2 and 7. Here

$$J_1 \equiv \int_0^1 \int_0^1 \frac{|g(t)|^2}{\log(n(t)+2)} \frac{dt \, dx}{|t-x| + (n(t,x)+1)^{-1}} \tag{6.13}$$

$$\leqq \int_0^1 \frac{|g(t)|^2}{\log(n(t)+2)} dt \int_0^1 \frac{dx}{|t-x| + n(t,(x)+1)^{-1}}$$

$$\leqq B \int_0^1 |g(t)|^2 dt = B$$

and similarly

$$J_2 \equiv \int_0^1 \int_0^1 \frac{|g(x)|^2}{\log(n(x)+2)} \frac{dt \, dx}{|t-x| + (n(t,x)+1)^{-1}} \leqq B. \tag{6.14}$$

(6.12), (6.13) and (6.14) show that (6.10) holds, and (6.9) is proved.

Let us now proceed to the proof of our assertion (i). By Abel's transformations repeated twice, we see

$$s_n^*(t) = \sum_{\nu=0}^{n-3} (\nu+1)\sigma_{\nu+1}(t) \, \Delta^2 \frac{1}{\sqrt{\log(\nu+2)}}$$

$$+ (n-1)\sigma_{n-1}(t) \Delta \frac{1}{\sqrt{\log(n-1)}} + \frac{s_n(t)}{\sqrt{\log(n+2)}}$$

$$= P_n + Q_n + R_n,$$

say. Because of the inequality

$$|s_n^*(t)|^2 \leqq 3(|P_n|^2 + |Q_n|^2 + |R_n|^2$$

it is sufficient to prove

$$\int_0^1 \sup_n |P_n|^2 dt \leqq B_\alpha \int_0^1 |f(t)|^2 dt, \tag{6.15}$$
$$\int_0^1 \sup_n |Q_n|^2 dt \leqq B_\alpha \int_0^1 |f(t)|^2 dt$$

(we have already dealt with R_n). But, as is easily seen, we have

$$|P_n| \leqq B \sup_{\nu \leqq n} |\sigma_\nu(t)| \sum_{\nu=0}^{\infty} \frac{1}{(\nu+1)(\log(\nu+2))^{3/2}} \leqq B \sup_\nu |\sigma_\nu(t)|$$

and

$$|Q_n| \leqq B|\sigma_{n-1}(t)| \frac{1}{(\log(n+2))^{3/2}} \leqq B \sup_n |\sigma_n(t)|,$$

so the inequalities (6.15) are deduced directly from Lemma 11.

The assertion (ii) follows from (i).

REFERENCES

[1] H. E. CHRESTENSON, A class of generalized Walsh functions, Pacific Journ. of Math. 5(1955), 17-31.

[2] N. J. FINE, On Walsh functions, Trans. Amer. Math. Soc. 65 (1949), 372 414.

[3] ______, Cesàro summability of Walsh-Fourier series, Proc. Nat. Acad. Sci. U.S. A. 41(1955). 588-591.

[4] G. H. HARDY AND J. E. LITTLEWOOD, On partial sums of Fourier series, Proc. Cambridge Phil. Soc. 40(1944). 103-107.

[5] G. E. HARDY, J. E. LITTLEWOOD and G. PÓLYA, Inequalities, Cambridge, 1952.

[6] I. I. HIRSCHMAN Jr. The decomposition of Walsh and Fourier series, Memoirs Amer. Math. Soc. no. 15(1955).

[7] R. E. A. C. PALEY, A remarkable series of orthonormal functions, Proc. London Math. Soc. 34(1932), 241-279.

[8] J. J. PRICE, Certain classes of orthonormal step functions (Abstract), Bull. Amer. Math. Soc. 62(1956), 388.

[9] G. SUNOUCHI, On the Walsh-Kaczmarz series, Proc. Amer. Math. Soc. 2(1951), 5-11.

[10] C. WATARI, A generalization of Haar functions, Tôhoku Math. Journ. 8(1956), 286 290.

[11] ______, On generalized Walsh Fourier series I. Proc. Japan Acad. 33(1957) 435-438.

[12] S. YANO, On Walsh Fourier series, Tôhoku Math. Journ. 3(1951), 223 242.

[13] ______, Cesàro summability of Walsh Fourier series, Tôhoku Math. Journ. 9(1957), 267-272.

[14] A. ZYGMUND, Trigonometrical series, Warszawa, 1935.

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