

ON GENERALIZED WEIGHT NASH EQUILIBRIA FOR GENERALIZED MULTIOBJECTIVE GAMES

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ABSTRACT. In this paper, we will introduce the general concepts of generalized multiobjective game, generalized weight Nash equilibria and generalized Pareto equilibria. Next using the fixed point theorems due to Idzik [5] and Kim-Tan [6], we shall prove the existence theorems of generalized weight Nash equilibria under general hypotheses. And as applications of generalized weight Nash equilibria, we shall prove the existence of generalized Pareto equilibria in non-compact generalized multiobjective game.

1. Introduction

Recently, the study of existence of Pareto equilibria in game theory with vector payoffs has been extensively studied by a number of authors, e.g., see [2-4, 7-12] and the references therein. The motivation for the study of multicriteria models can be found in [2, 7] and the existence of Pareto equilibria is one of the fundamental problem in the game theory. In a recent paper [12], Yu and Yuan proved some existence theorems of Pareto equilibria by using the fixed point theorem and the minimax inequality; and hence they provided an unified study for the existence of Pareto equilibria in multiobjective game under weaker conditions. Those results further generalize the corresponding existence results of Pareto equilibria given in the currently existing literatures. Also, in recent papers [14, 15], Ding obtained some existence of equilibria for

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generalized multiobjective games by using some coercivity conditions and quasi-variational inequalities.

In this paper, we will introduce the general concepts of generalized multiobjective game, generalized weight Nash equilibria and generalized Pareto equilibria. Next using the fixed point theorems due to Idzik [5] and Kim-Tan [6], we shall prove the existence theorems of generalized weight Nash equilibria under general hypotheses. And as applications of generalized weight Nash equilibria, we shall prove the existence of generalized Pareto equilibria in non-compact generalized multiobjective game.

2. Preliminaries

We begin with some notations and definitions. Let A be a subset of a topological space X . We shall denote by 2^A the family of all subsets of A and by \bar{A} the closure of A in X . If A is a subset of a vector space, we shall denote by $co A$ the convex hull of A . If A is a non-empty subset of a topological vector space X and $S, T : A \rightarrow 2^X$ are correspondences, then $co T, \bar{T}, T \cap S : A \rightarrow 2^X$ are correspondences defined by $(co T)(x) = co T(x), \bar{T}(x) = \overline{T(x)}$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let X, Y be non-empty topological spaces and $T : X \rightarrow 2^Y$ be a correspondence. We may call $T(x)$ the *upper section* of T and $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$ the *lower section* of T . Let X be a non-empty convex subset of a vector space E and let $f : X \rightarrow \mathbb{R}$. We say that f is *quasi-convex* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \leq t\}$ is convex; and that f is *quasi-concave* if $-f$ is quasi-convex. A correspondence $T : X \rightarrow 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a correspondence $T : X \rightarrow 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. And we say that T is *continuous* if T is both upper semicontinuous and lower semicontinuous.

Next we recall the following continuity definitions of the real-valued function. Let X be a non-empty subset of a topological space E and $f : X \rightarrow \mathbb{R}$. We say that f is *upper semicontinuous* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is closed in X , and f is *lower semicontinuous* if

$-f$ is upper semicontinuous. Hence if f is upper semicontinuous, then the set $\{x \in X \mid f(x) < t\}$ is open for each $t \in \mathbb{R}$. And we say that f is *continuous* if f is both upper semicontinuous and lower semicontinuous.

We also recall the following: let E be a Hausdorff topological vector space. A set $B \subset E$ is said to be *convexly totally bounded* (simply, *c.t.b.*) whenever for every neighborhood V of $0 \in E$, there exists a finite subset $\{x_i \mid i \in I\} \subset E$ and a finite family of convex sets $\{C_i \mid i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \{x_i + C_i \mid i \in I\}$. Then it is known that every compact subset of a locally convex Hausdorff topological vector space is convexly totally bounded. For details, see Idzik [5].

First, we shall introduce the *generalized game with multicriterion* (or *generalized multiobjective game*) in its strategic form of a finite (or infinite) number of players $G := (X_i, F^i, T_i)_{i \in I}$, where I is a (possibly uncountable) set of players, as follows: For each $i \in I$, X_i is the set of strategies in a Hausdorff topological vector space E_i for the player i , and $F^i : X = \prod_{i \in I} X_i \rightarrow \mathbb{R}^{k_i}$, where $k_i \in \mathbb{N}$, which is called the *payoff function* (or called *multicriteria*) and $T_i : X \rightarrow 2^{X_i}$, which is called the *constraint correspondence* of the player i .

If an action $x := (x_1, \dots, x_n) \in X$ is played, each player i is trying to find his/her payoff function $F^i(x) := (f_1^i(x), \dots, f_{k_i}^i(x))$, which consists of noncommensurable outcomes under the possible constraint sets $T_i(x)$. Here it should be remarked that in our constrained multiobjective games, the other players can influence the j -th player

- (1) indirectly, by restricting j 's feasible strategies to $T_j(x)$,
- (2) directly, by affecting j 's payoff function F^j .

Here it is noted that every action domain of the constraint correspondence T_j for each $j \in I$ is not the set X_j but the whole strategy set X . In fact, it is reasonable that everyone can choose a possible action in his strategies which are affected by the other's strategies depending on his actions simultaneously; and this is different from the definitions of Ding [14, 15] using different constraint correspondences.

Each player i has a preference ' \succeq_i ' over the outcome space \mathbb{R}^{k_i} . For each player $i \in I$, its preference ' \succeq_i ' is given by

$$z^1 \succeq_i z^2 \iff z_j^1 \geq z_j^2 \quad \text{for every } j = 1, \dots, k_i,$$

where $z^1 = (z_1^1, \dots, z_{k_i}^1)$ and $z^2 = (z_1^2, \dots, z_{k_i}^2)$ are the elements in \mathbb{R}^{k_i} . The players' preference relations induce the preferences on X is defined as follows:

for each player i and their choices $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in X$,

$$x \succeq_i y \iff F^i(x) \succeq_i F^i(y).$$

Also we assume that the model of a game in this paper is a noncooperative game, i.e., there is no replay communicating between players, and so players act as free agents, and each play is trying to minimize his/her own payoff according to his/her preferences and constraints.

For the games with vector payoff functions (or multicriteria) it is well-known that in general, there does not exist a strategy $\hat{x} \in X$ to minimize (or equivalently to say, maximize) all f_j^i 's for each player i in his/her constraint, e.g., see [11] and the references therein. Hence we shall need some solution concepts for generalized multicriteria games.

Throughout this paper, for each $m \in \mathbb{N}$, we shall denote by \mathbb{R}_+^m the non-negative orthant of \mathbb{R}^m , i.e.,

$$\mathbb{R}_+^m := \{u = (u_1, \dots, u_m) \in \mathbb{R}^m \mid u_j \geq 0 \ \forall j = 1, \dots, m\},$$

so that the non-negative orthant \mathbb{R}_+^m of \mathbb{R}^m has a non-empty interior with the topology induced in terms of convergence of vectors respect to the Euclidean metric. That is, we shall use the notation

$$\text{int } \mathbb{R}_+^m := \{u = (u_1, \dots, u_m) \in \mathbb{R}^m \mid u_j > 0 \ \forall j = 1, \dots, m\}.$$

For each $i \in I$, denote $X_i := \prod_{j \in I \setminus \{i\}} X_j$. If $x = (x_1, \dots, x_n) \in X$, we shall write $x_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_i$. If $x_i \in X_i$ and $x_i \in X_i$, we shall use the notation $(x_i, x_i) := (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = x \in X$. For each $u, v \in \mathbb{R}^m$, $u \cdot v$ denote the standard Euclidean inner product.

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$; then we will introduce the following general equilibrium concept of a generalized multiobjective game:

DEFINITION 1. A strategy $\bar{x}_i \in X_i$ of the player i is said to be a *generalized Pareto efficient strategy* (respectively, *generalized weak Pareto efficient strategy*) of a game $G = (X_i, F^i, T_i)_{i \in I}$ respect to \bar{x} if $\bar{x}_i \in T_i(\bar{x})$ and there is no strategy $x_i \in T_i(\bar{x})$ such that

$$F^i(\bar{x}) - F^i(\bar{x}_i, x_i) \in \mathbb{R}_+^{k_i} \setminus \{0\} \text{ (respectively, } F^i(\bar{x}) - F^i(\bar{x}_i, x_i) \in \text{int } \mathbb{R}_+^{k_i}).$$

Then a strategy $\bar{x} \in X$ is said to be a *generalized Pareto equilibrium* (respectively, *generalized weak Pareto equilibrium*) of a game $G =$

$(X_i, F^i, T_i)_{i \in I}$ if, for each player i , $\bar{x}_i \in X_i$ is a generalized Pareto efficient strategy (respectively, generalized weak Pareto efficient strategy) respect to \bar{x} .

The above definition generalizes the corresponding definitions in [9, 10, 12]. And Definition 1 is different from the definition of Ding [14, 15] using different constraint correspondences.

From the above definition, it is clear that every generalized Pareto equilibria is a Pareto equilibria when the constraint set is fixed with $T_i(x) = X_i$ for each $x \in X$ and $i \in I$. And it is also clear that a generalized Pareto equilibria is a generalized weak Pareto equilibria, and in turn also a weak Pareto equilibria. However the converse is not always true, e.g., see [12].

We also introduce the following definition which generalizes the definition in [9]:

DEFINITION 2. A strategy $\bar{x} \in X$ is said to be a *generalized weight Nash equilibrium* respect to the weight vector $W := (W_1, \dots, W_n)$ of a game $G = (X_i, F^i, T_i)_{i \in I}$, if for each player $i \in I$,

- (1) $\bar{x}_i \in T_i(\bar{x})$;
- (2) $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$;
- (3) $W_i \cdot F^i(\bar{x}) \leq W_i \cdot F^i(\bar{x}_i, x_i)$ for each $x_i \in T_i(\bar{x})$.

In particular, when $W_i \in T_+^{k_i}$ for all $i \in I$, the strategy $\bar{x} \in X$ is said to be a *normalized form of generalized weight Nash equilibrium* respect to the weight W , where $T_+^{k_i}$ is the standard simplex of \mathbb{R}^{k_i} , i.e.,

$$T_+^{k_i} := \{u = (u_1, \dots, u_{k_i}) \in \mathbb{R}_+^{k_i} \mid \sum_{j=1}^{k_i} u_j = 1\}.$$

For each $i \in I$, let $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ be fixed. Then, from the above definitions, it is easy to see that a strategy $\bar{x} \in X$ is a generalized weight Nash equilibrium respect to the weight vector $W := (W_1, \dots, W_n)$ of a game $G = (X_i, F^i, T_i)_{i \in I}$, if and only if for each $i \in I$, \bar{x}_i is an optimal solution of the following vector optimization problem:

$$\min_{x_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_i, x_i)$$

3. Existence of generalized weight Nash equilibria

In order to obtain the existence of generalized weight Nash equilibria and generalized Pareto equilibria, we shall need some fixed point theorems or minimax inequalities as efficient proving tools. First we shall

investigate the existence theorems of generalized weight Nash equilibria under general hypotheses by using the fixed point theorems due to Idzik [5] and Kim-Tan [6]. And in the next section, we shall show that generalized Pareto equilibria problem can be reduced to the study of generalized weight Nash equilibria under suitable conditions in non-compact generalized multiobjective game.

Now we discuss the existence of generalized weight Nash equilibria as application of fixed point theorem as follows:

For each $i = 1, \dots, n$, $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X$. Define two correspondences $S^W : X \times X \rightarrow \mathbb{R}$ and $M^W : X \rightarrow 2^X$ by

$$S^W(x, y) := \sum_{i=1}^n W_i \cdot F^i(x_i, y_i);$$

and

$$M^W(x) := \{y' \in T(x) \mid S^W(x, y') = \min_{y \in T(x)} S^W(x, y)\},$$

where $T(x) = \prod_{i \in I} T_i(x)$, for each $(x, y) \in X \times X$.

Then we prove that the existence of generalized weight Nash equilibria is equivalent to the existence of fixed points for the correspondence M^W as follows:

LEMMA 1. *Let I be a finite set of players, and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game. Suppose that for each $i \in I$, $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$. A strategy $\bar{x} \in X$ is a generalized weight Nash equilibrium respect to the weight $W = (W_1, \dots, W_n)$ of the game G if and only if $\bar{x} \in X$ is a fixed point of the correspondence M^W .*

Proof. \Leftarrow : By the definition of M^W , $\bar{x} \in T(\bar{x})$ and

$$S^W(\bar{x}, \bar{x}) \leq S^W(\bar{x}, y), \quad \text{for all } y \in T(\bar{x}).$$

That is, for each $i \in I$,

$$\begin{aligned} & \sum_{i=1}^n W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \\ & \leq \sum_{i=1}^n W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \end{aligned}$$

for all $y_i \in T_i(\bar{x})$. Since \bar{x} is the fixed point for M^W , we have $\bar{x} \in T(\bar{x})$. If we choose any action y having the form $y = (\bar{x}_1, \dots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \in T(\bar{x})$, then

$$W_i \cdot F^i(\bar{x}_i, \bar{x}_i) \leq \min_{y_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_i, y_i);$$

and hence for all $y_i \in T_i(\bar{x})$, we have

$$\begin{aligned} &W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \\ &\leq W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \dots, \bar{x}_n). \end{aligned}$$

Therefore we have that for each $i \in I$,

$$\begin{aligned} &W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_n) \\ &\leq W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \quad \text{for all } y_i \in T_i(\bar{x}). \end{aligned}$$

Therefore \bar{x} is the generalized weight Nash equilibrium respect to the weight W .

\Rightarrow : Suppose that \bar{x} is the generalized weight Nash equilibrium respect to the weight W . Then $\bar{x} \in T(\bar{x})$ and $W_i \cdot F^i(\bar{x}) \leq W_i \cdot F^i(\bar{x}_i, y_i)$ for each $y_i \in T_i(\bar{x})$; and hence we have

$$\begin{aligned} &\sum_{i=1}^n W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_n) \\ &\leq \sum_{i=1}^n W_i \cdot F^i(\bar{x}_1, \dots, \bar{x}_{i-1}, y_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \quad \text{for all } y \in T(\bar{x}). \end{aligned}$$

By the definition of the correspondence M^W , we conclude that \bar{x} is a fixed point of the correspondence M^W . This completes the proof. \square

REMARKS. (1) Lemma 1 generalizes the corresponding Lemma 2.3 in [9]. And Lemma 1 enables us to investigate the generalized weight Nash equilibria by using appropriate fixed point theorems.

(2) In Lemma 1, the set of players is finite. However, if the convergence is well equipped in strategy sets (e.g., X_i is a subset of an l^2 space with inner product), then the infinite set of players is possible.

Before proving existence theorems of generalized weight Nash equilibria, we shall need the following general fixed point theorems of two different types.

We begin with the following particular form of Idzik's theorem [5, Theorem 4.3]:

LEMMA 2 [5]. Let X be a convex subset of a Hausdorff topological vector space E . Let $T : X \rightarrow 2^X$ be an upper semicontinuous correspondence such that $T(x)$ is non-empty closed convex for each $x \in X$. If $\overline{T(X)}$ is a compact and convexly totally bounded subset of X , then there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

LEMMA 3 [6]. Let X be a non-empty convex subset of a Hausdorff topological vector space E and D be a non-empty compact subset of X . Let $T : X \rightarrow 2^D$ be a correspondence satisfying the following:

- (1) for each $x \in X$, $coT(x) \subset D$;
- (2) for each $y \in X$, $T^{-1}(y)$ is open in X .

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in coT(\hat{x})$.

Now we are ready to prove the existence theorem for a generalized weight Nash equilibria in a general Hausdorff topological vector space.

THEOREM 1. Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for each $i \in I$, X_i is a non-empty convex subset of a Hausdorff topological vector space E_i and D_i be a non-empty compact subset of X_i . Let $T_i : X \rightarrow 2^{D_i}$ is a continuous constraint correspondence such that each $T_i(x)$ is a non-empty closed convex subset of D_i and $D = \prod_{i \in I} D_i$ is c.t.b. in $E = \prod_{i \in I} E_i$. If there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ for each $i \in I$ such that

- (1) the correspondence S^W is jointly continuous on $X \times X$;
- (2) $S^W(x, \cdot)$ is quasi-convex on X for each fixed $x \in X$.

Then there exists a generalized weight Nash equilibrium $\bar{x} \in X$ for the game G respect to the weight vector $W = (W_1, \dots, W_n)$.

Proof. We shall apply Lemma 2 to the correspondence $M^W : X \rightarrow 2^X$, defined by

$$M^W(x) := \{y' \in T(x) \mid S^W(x, y') = \min_{y \in T(x)} S^W(x, y)\},$$

where $T(x) := \prod_{i \in I} T_i(x)$ and $S^W(x, y) := \sum_{i \in I} W_i \cdot F^i(x_i, y_i)$, for each $x, y \in X$. Then it suffices to show that the correspondence $M^W : X \rightarrow 2^X$ is upper semicontinuous such that each $M^W(x)$ is a non-empty closed convex subset of D , where $D = \prod_{i \in I} D_i$. Since S^W is jointly continuous on $X \times X$, and $S^W(x, \cdot)$ is quasi-convex on X , it is easy to see that each $M^W(x)$ is a non-empty closed convex subset of D . Also note that since D is a compact and convexly totally bounded subset

of X and $T(x) \subset D, \overline{T(X)} \subset D \subset X$ is also a compact and convexly totally bounded subset of X . And, by Proposition 2.5.3 in [1], M^W is clearly upper semicontinuous. Therefore, by Lemma 2, M^W has a fixed point $\bar{x} \in X$. Thus, by Lemma 1, \bar{x} is the desired generalized weight Nash equilibrium for the game G respect to the weight vector $W = (W_1, \dots, W_n)$. This completes the proof. \square

Next, we shall prove a generalized weight Nash equilibrium in a Hausdorff topological vector space without assuming the local convexity of the constraint sets. Before proving this, we shall need the following:

LEMMA 4. *Let X, Y be Hausdorff topological vector spaces and X be compact. Let $T : X \rightarrow 2^Y$ be a continuous correspondence such that each $T(x)$ is a non-empty compact subset of Y , and let $f : X \times Y \rightarrow \mathbb{R}$ be a continuous function on $X \times Y$.*

Then the function $\phi : X \rightarrow \mathbb{R}$, defined by

$$\phi(x) := \inf_{u \in T(x)} f(x, u), \quad \text{for each } x \in X,$$

is a continuous function on X .

Proof. By Theorem 2.5.1 in [1], ϕ is upper semicontinuous; and by Theorem 2.5.2 in [1], ϕ is lower semicontinuous. Thus ϕ is a continuous function on X . \square

Now we prove a generalized weight Nash equilibrium in a general Hausdorff topological vector space as follows:

THEOREM 2. *Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for each $i \in I$, X_i is a non-empty compact convex subset of a Hausdorff topological vector space E_i . Let $T_i : X \rightarrow 2^{X_i}$ be an upper semicontinuous constraint correspondence such that each $T_i(x)$ is a non-empty closed convex subset of X_i and $T_i^{-1}(y_i)$ is (possibly empty) open in X for each $y_i \in X_i$. If there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$,*

- (1) $(x, y) \mapsto W_i \cdot F^i(x_i, y_i)$ is jointly continuous on $X \times X$;
- (2) $y \mapsto W_i \cdot F^i(x_i, y_i)$ is quasi-convex on X for each $x_i \in X_i$.

Then there exists a generalized weight Nash equilibrium $\bar{x} \in X$ for the game G respect to the weight vector $W = (W_1, \dots, W_n)$.

Proof. For each $k \in \mathbb{N}$, we define a correspondence $S_k : X \rightarrow 2^X$ by

$$S_k(x) := \Pi_{i \in I} \{y_i \in T_i(x) \mid W_i \cdot F^i(x_{\hat{i}}, y_i) < \min_{u_i \in T_i(x)} W_i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k}\},$$

for each $x \in X$. Then we have that for each $x \in X$,

$$\begin{aligned} S_k(x) &= \Pi_{i \in I} \left(T_i(x) \cap \{y_i \in X_i \mid W_i \cdot F^i(x_{\hat{i}}, y_i) \right. \\ &\quad \left. < \min_{u_i \in T_i(x)} W_i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k} \right), \end{aligned}$$

and each $S_k(x)$ is non-empty convex by the assumption (2). Note that for each $y \in X$, we have

$$\begin{aligned} S_k^{-1}(y) &= \{x \in X \mid y \in S_k(x)\} \\ &= \left\{ x \in X \mid y \in \Pi_{i \in I} \{y_i \in T_i(x) \mid W_i \cdot F^i(x_{\hat{i}}, y_i) \right. \\ &\quad \left. < \min_{u_i \in T_i(x)} W_i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k} \right\} \\ &= \{x \in X \mid y_i \in T_i(x) \text{ and } W_i \cdot F^i(x_{\hat{i}}, y_i) \\ &\quad < \min_{u_i \in T_i(x)} W_i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k} \text{ for each } i \in I\} \\ &= \left(\bigcap_{i \in I} T_i^{-1}(y_i) \right) \cap \left(\bigcap_{i \in I} \{x \in X \mid W_i \cdot F^i(x_{\hat{i}}, y_i) \right. \\ &\quad \left. < \min_{u_i \in T_i(x)} W_i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k} \right). \end{aligned}$$

Using Lemma 4, we can obtain $S_k^{-1}(y)$ is open in X by the continuity assumption (1) and the open lower section assumption on T_i . Therefore the whole assumptions of Lemma 3 are satisfied, so that S_k has a fixed point $x(k) \in X$. From the definition of S_k , it follows that for each $i \in I$,

$$W_i \cdot F^i(x_{\hat{i}}(k), x_i(k)) < \min_{u_i \in T_i(x(k))} W_i \cdot F^i(x_{\hat{i}}(k), u_i) + \frac{1}{k}.$$

Here we note that since each T_i has open lower sections, T_i is lower semicontinuous, and so the correspondence T_i must be continuous. Since X is compact, we can assume that the sequence $\{x(k)\}$ in X converges

to $\bar{x} \in X$ without loss of generality. Since $x_i(k) \in T_i(x(k))$ and T_i is continuous for each $i \in I$, we have $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

By the assumption (1) and Lemma 4 again, we have that

$$\begin{aligned} W_i \cdot F^i(\bar{x}_i, \bar{x}_i) &= \lim_{k \rightarrow \infty} W_i \cdot F^i(x_i(k), x_i(k)) \\ &\leq \lim_{k \rightarrow \infty} \min_{u_i \in T_i(x(k))} W_i \cdot F^i(x_i(k), u_i) \\ &= \min_{u_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_i, u_i). \end{aligned}$$

Therefore we have

$$\sum_{i=1}^n W^i \cdot F^i(\bar{x}_1, \dots, \bar{x}_n) \leq \sum_{i=1}^n \min_{u_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_i, u_i).$$

Thus \bar{x} is a generalized weight Nash equilibrium for the game G respect to the weight vector $W = (W_1, \dots, W_n)$. This completes the proof. \square

It is clear that Theorem 2 is closely related to Theorem 1 as a special case of the continuity and convexity of the mapping S^W in Theorem 1. In fact, the correspondence S^W in Theorem 1 automatically satisfy the continuity assumption by the corresponding hypothesis (1) of Theorem 2; however the converse does not hold in general.

Our Theorems 1 and 2 generalize the corresponding results in [9, 12] in several aspects as follows:

(1) when the constraint correspondence T_i is constant, i.e., $T_i(x) = X_i$ for each $i \in I$ and $x \in X$, Theorem 1 reduces to the corresponding Theorem 1 in [12], and so the corresponding theorems in [9] can be obtained;

(2) the strategy set X_i need not be compact as in the corresponding theorems in [12] nor X_i need not be a subset of a normed linear spaces as in [9].

As we have seen, we have proved two existence results of generalized weight Nash equilibria as applications of fixed point theorems, and those results can be useful in showing the existence of equilibrium actions under the appropriate constraint sets.

The following lemma is an easy consequence of the quasi-variational inequality due to Yuan-Tarafdar [13], and it is the basic tool for proving the existence of generalized weight Nash equilibria:

LEMMA 5. Let X be a non-empty compact convex subset of a Hausdorff topological vector space E which has sufficiently many continuous linear functionals. Let $T : X \rightarrow 2^X$ be an upper semicontinuous correspondence such that each $T(x)$ is a non-empty closed convex subset of X . Let $\phi : X \times X \rightarrow \mathbb{R}$ be a function such that

- (1) for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is lower semicontinuous;
- (2) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is quasi-concave;
- (3) the set $\{x \in X \mid \sup_{y \in T(x)} \phi(x, y) \leq 0\}$ is closed in X .

Then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in T(\hat{x}) \quad \text{and} \quad \sup_{y \in T(\hat{x})} \phi(\hat{x}, y) \leq 0.$$

We also need the following lower semicontinuity property:

LEMMA 6. Let X, Y be Hausdorff topological vector spaces and X be compact. Let $T : X \rightarrow 2^Y$ be a lower semicontinuous correspondence such that each $T(x)$ is a non-empty subset of Y , and let $f : X \times Y \rightarrow \mathbb{R}$ be a lower semicontinuous function on $X \times Y$. Then the function $\phi : X \rightarrow \mathbb{R}$, defined by

$$\phi(x) := \sup_{y \in T(x)} f(x, y), \quad \text{for each } x \in X,$$

is a lower semicontinuous function on X .

Proof. By applying Theorem 2.5.2 in [1] to $-f$, we can obtain the conclusion. \square

Now we will prove an existence theorem of a generalized weight Nash equilibrium as follows:

THEOREM 3. Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for each $i \in I$, X_i is a non-empty compact convex subset of a Hausdorff topological vector space E_i which has sufficiently many continuous linear functionals. Let $T_i : X \rightarrow 2^{X_i}$ be a continuous constraint correspondence such that each $T_i(x)$ is a non-empty closed convex subset of X_i . If there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$,

- (1) for each $y_i \in X_i$, $x \mapsto \sum_{i \in I} W_i \cdot F^i(x_i, y_i)$ is upper semicontinuous on X ;
- (2) for each $x_i \in X_i$, $y \mapsto \sum_{i \in I} W_i \cdot F^i(x_i, y_i)$ is quasi-convex on X ;

(3) $(x, y) \mapsto \sum_{i \in I} W_i \cdot F^i(x_i, y_i)$ is jointly lower semicontinuous on $X \times X$.

Then there exists a generalized weight Nash equilibrium $\bar{x} \in X$ for the game G respect to the weight vector $W = (W_1, \dots, W_n)$.

Proof. In order to apply the quasi-variational inequality, we first define a real-valued function $\phi : X \times X \rightarrow \mathbb{R}$ by

$$\phi(x, y) := \sum_{i \in I} W_i \cdot (F^i(x_i, x_i) - F^i(x_i, y_i)), \quad \text{for each } (x, y) \in X \times X.$$

Then by the assumptions (1) - (3) and the fact that finite sum of lower semicontinuous functions is also lower semicontinuous, we can have

- (a) for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is lower semicontinuous;
- (b) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is quasi-concave.

Since the correspondence $T(x) := \prod_{i \in I} T_i(x)$ is lower semicontinuous and the map ϕ is jointly lower semicontinuous, by Lemma 6, the map $x \mapsto \sup_{y \in T(x)} \phi(x, y)$ is lower semicontinuous and hence the set $\{x \in X \mid \sup_{y \in T(x)} \phi(x, y) \leq 0\}$ is closed in X . Therefore the whole assumptions of Lemma 5 are satisfied, and so there exists a point $\bar{x} \in X$ such that

$$\bar{x} \in T(\bar{x}) \text{ and } \phi(\bar{x}, y) = \sum_{i \in I} W_i \cdot (F^i(\bar{x}_i, \bar{x}_i) - F^i(\bar{x}_i, y_i)) \leq 0$$

for all $y \in T(\bar{x})$. Then for each $i \in I$ and every $(\hat{x}_i, y_i) \in T(\bar{x}_i, \bar{x}_i)$, we have $W_i \cdot F^i(\bar{x}_i, \bar{x}_i) - W_i \cdot F^i(\bar{x}_i, y_i) \leq 0$; which implies that for each $i \in I$, $\bar{x}_i \in T_i(\bar{x})$ and

$$W_i \cdot F^i(\bar{x}_i, \bar{x}_i) = \min_{y_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_i, y_i).$$

Thus \bar{x} is a generalized weight Nash equilibrium point of the game G respect to the weight vector W . □

REMARKS. (1) Theorem 3 generalizes the corresponding results in [9, 10, 12]. In fact, when the constraint correspondence T_i is constant, i.e., $T_i(x) = X_i$ for each $i \in I$ and $x \in X$, our Theorem reduces to the corresponding Theorem 1 in [12], and so the corresponding theorems in [9, 10] can be obtained.

(2) We can obtain the existence of equilibria for generalized multi-objective games by using some coercivity conditions; and in this case,

we can assure that the strategy set X_i need not be compact as in the corresponding theorems in [12] nor X_i need be a subset of a normed linear spaces as in [9, 10].

(3) In recent papers [14, 15], Ding obtained some existence of equilibria for generalized multiobjective games by using some coercivity conditions and quasi-variational inequalities, and those results are comparable to our existence results in this paper.

It is well-known that fixed point technique has wide applications in the study of economics and optimizations, e.g., see [7-12]. On the other hand, in a recent paper [12], Yu and Yuan proved the existence of weight Nash equilibria and Pareto equilibria by using Ky Fan's minimax inequality, which would not be widely used before as an efficient tool for investigating the equilibria in economics and optimizations. Furthermore, in this paper, it is our purpose to present how the quasi-variational inequality can be applied to the existence of generalized weight Nash equilibria, and this method can be considered as an efficient tool for the equilibrium theory.

4. Existence of generalized Pareto equilibria

In this section, as applications of generalized weight Nash-equilibria, we shall derive some existence theorems of generalized Pareto equilibria for generalized multiobjective games.

We now prove the following:

LEMMA 7. *Let I be a finite set of players, and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game. Suppose that for each $i \in I$, $W_i \in T_+^{k_i}$. Then a normalized form of generalized weight Nash equilibrium $\bar{x} \in X$ respect to the weight $W = (W_1, \dots, W_n) \in T_+^{k_1} \times \dots \times T_+^{k_n}$ (resp., $W \in \text{int } T_+^{k_1} \times \dots \times \text{int } T_+^{k_n}$) is a generalized weak Pareto equilibrium (resp., a generalized Pareto equilibrium) of the game G .*

Proof. Suppose the contrary, i.e., \bar{x} is not a generalized weak Pareto equilibrium. Then there exists some $i \in I$ and an $x_i \in T_i(\bar{x})$ such that

$$F^i(\bar{x}) - F^i(\bar{x}_i, x_i) \in \text{int } \mathbb{R}_+^{k_i}.$$

Since $W_i \in T_+^{k_i}$ and $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$, we have

$$W_i \cdot F^i(\bar{x}) - W_i \cdot F^i(\bar{x}_i, x_i) > 0,$$

which contradicts the fact that \bar{x} is a normalized form of generalized weight Nash equilibrium respect to the weight $W = (W_1, \dots, W_n)$. Therefore \bar{x} is a generalized weak Pareto equilibrium.

Next, we assume that $W_i \in \text{int } T_+^{k_i}$ for every $i \in I$. Suppose that \bar{x} is not a generalized Pareto equilibrium; then there exists some $i \in I$ and an $x_i \in T_i(\bar{x})$ such that $F^i(\bar{x}) - F^i(\bar{x}_i, x_i) \in \mathbb{R}_+^{k_i} \setminus \{0\}$. Since $W_i \in \text{int } T_+^{k_i}$ and $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$, we can also have

$$W_i \cdot F^i(\bar{x}) - W_i \cdot F^i(\bar{x}_i, x_i) > 0;$$

which contradicts the definition of the corresponding generalized weight Nash equilibrium respect to the weight $W = (W_1, \dots, W_n)$. Hence \bar{x} is a generalized weak Pareto equilibrium of the game $G = (X_i, F^i, T_i)_{i \in I}$. This completes the proof. \square

REMARKS. It should be noted that the conclusion of Lemma 7 still hold true when \bar{x} is a generalized weight Nash equilibrium respect to the weight $W = (W_1, \dots, W_n)$ satisfying that $W_i \in \mathbb{R}_+^{k_i}$ (resp., $W_i \in \text{int } \mathbb{R}_+^{k_i}$) for each $i \in I$. Also it should be noted that the converse of Lemma 7 is not true in general, i.e., a generalized Pareto equilibrium is not necessarily a generalized weight Nash equilibrium (e.g. see [9, 12]).

By combining Lemma 7 and Theorem 2, we can obtain the following existence of a generalized Pareto equilibria for generalized multiobjective games in general Hausdorff topological vector spaces:

THEOREM 4. *Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for each $i \in I$, X_i is a non-empty compact convex subset of a Hausdorff topological vector space E_i . Let $T_i : X \rightarrow 2^{X_i}$ be an upper semicontinuous constraint correspondence such that each $T_i(x)$ is a non-empty closed convex subset of X_i , and $T_i^{-1}(y_i)$ is (possible empty) open in X for each $y_i \in X_i$. If there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$,*

- (1) $(x, y) \mapsto W_i \cdot F^i(x_i, y_i)$ is jointly continuous on $X \times X$;
- (2) $y \mapsto W_i \cdot F^i(x_i, y_i)$ is quasi-convex on X for each $x_i \in X_i$.

Then there exists a generalized weak Pareto equilibrium $\bar{x} \in X$ for the game G respect to the weight vector $W = (W_1, \dots, W_n)$.

Furthermore, if $W_i \in \text{int } T_+^{k_i}$ for all $i \in I$, then the equilibrium \bar{x} is a generalized Pareto equilibrium respect to the weight vector $W = (W_1, \dots, W_n)$.

As an immediate consequence of Theorem 4, we have the following

THEOREM 5. Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for each $i \in I$, X_i is a non-empty compact convex subset of a Hausdorff topological vector space E_i . Let $T_i : X \rightarrow 2^{X_i}$ be an upper semicontinuous constraint correspondence such that $T_i^{-1}(y_i)$ is (possibly empty) open in X for each $y_i \in X_i$. Assume that for each $i \in I$,

- (1) f_j^i is jointly continuous on $X \times X$;
- (2) for each $x_i \in X_i$, $f_j^i(x_i, \cdot)$ is convex on X_i .

Then there exists a generalized Pareto equilibrium $\bar{x} \in X$ for the game G .

Proof. Take any fixed weight vector $W = (W_1, \dots, W_n)$, where $W_i \in \text{int } T_+^{k_i}$ for all $i \in I$. Since $F^i(x) = (f_1^i(x), \dots, f_{k_i}^i(x))$ for each $x \in X$ and $i \in I$, by the assumptions (1) and (2), it is easy to see that all the hypotheses of Theorem 2 are satisfied. Therefore the game G has a generalized weight Nash equilibrium \bar{x} respect to the weight vector W . Since $W_i \in \text{int } T_+^{k_i}$ for all $i \in I$, by Lemma 7, \bar{x} is a generalized Pareto equilibrium for the game G . This completes the proof. \square

Following the method in [15], we can further unify and generalize the above results in this paper to non-compact generalized multiobjective games in H-spaces without assuming the linear structure.

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