

ON GENERATORS OF C_0 -SEMIGROUPS OF COMPOSITION OPERATORS

EVA A. GALLARDO-GUTIÉRREZ AND DMITRY YAKUBOVICH

ABSTRACT. Avicou, Chalendar and Partington proved in [4] that an (unbounded) operator $Af = G \cdot f'$ on the classical Hardy space generates a C_0 semigroup of composition operators if and only if it generates a quasicontractive semigroup. Here we prove that if such an operator A generates a C_0 semigroup, then it is automatically a semigroup of composition operators, so that the condition of quasicontractivity of the semigroup in the cited result is not necessary. Our result applies to a rather general class of Banach spaces of analytic functions in the unit disc.

1. INTRODUCTION

Let \mathcal{B} denote a Banach space. We recall that a one parameter family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on \mathcal{B} is called a semigroup if $T_0 = I$ and $T_t T_s = T_{t+s}$ for all $t, s \geq 0$. It is called a C_0 -semigroup if it is strongly continuous, that is, $\lim_{t \rightarrow 0^+} T_t f = f$ for any $f \in \mathcal{B}$.

Given a C_0 -semigroup $\{T_t\}_{t \geq 0}$, its generator A is defined by

$$Af = \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t}$$

for $f \in \mathcal{D}(A) = \{f \in \mathcal{B} : \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t} \text{ exists}\}$. It is a closed and densely defined linear operator on \mathcal{B} , and it determines the semigroup uniquely. Observe, as a consequence of the uniform boundedness theorem, that if $\{T_t\}_{t \geq 0}$ is a C_0 -semigroup on \mathcal{B} , then there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$(1.1) \quad \|T_t\| \leq M e^{\omega t} \quad \text{for all } t \geq 0,$$

(see [17, Chapter II] or [1, Chapter 3], for instance). A semigroup satisfying (1.1) with $M = 1$ is called quasicontractive.

In 1978, Berkson and Porta [7] gave a complete description of the generator A of semigroups of composition operators acting on the classical Hardy space $H^2(\mathbb{D})$ induced by a *holomorphic flow* of analytic self-maps of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Recall that a *holomorphic flow* in the open unit disc \mathbb{D} is, by definition (see [27] and [26]), a continuous family $\{\varphi_t\}_{t \geq 0}$ of analytic self-mappings of \mathbb{D} that has a semigroup property with respect to composition. More precisely, a holomorphic flow has to meet the following conditions:

- 1) $\varphi_0(z) = z, \forall z \in \mathbb{D}$;
- 2) $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z), \forall t, s \geq 0, \forall z \in \mathbb{D}$;
- 3) For any $s \geq 0$ and any $z \in \mathbb{D}$, $\lim_{t \rightarrow s} \varphi_t(z) = \varphi_s(z)$.

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The holomorphic flow $\{\varphi_t\}$ gives rise to a semigroup $T_t f = f \circ \varphi_t$ of linear operators on $H^2(\mathbb{D})$, which is called a semigroup of composition operators. Berkson and Porta [7] noticed that this semigroup is always strongly continuous on the Hardy space $H^2(\mathbb{D})$; the same statement was proven by Siskakis regarding other classical spaces of analytic functions, such as the Dirichlet space \mathcal{D} . We refer to the survey [28] in this regard.

A straightforward computation shows that, at least for the case of $H^2(\mathbb{D})$ or of \mathcal{D} , the generator A of a semigroup of composition operators is of the form $Af = Gf'$, where G is an analytic function in \mathbb{D} . Indeed, as Berkson and Porta showed, G is the infinitesimal generator of the holomorphic flow $\{\varphi_t\}$, defined by means of the equation

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)), \quad \text{for } t \in \mathbb{R}_+ \text{ and } z \in \mathbb{D}.$$

Very recently, Avicou, Chalendar and Partington [4] have provided a complete description of quasicontractive C_0 -semigroups of bounded operators acting either on the Hardy space $H^2(\mathbb{D})$ or the Dirichlet space \mathcal{D} , whose generator A is of the form $Af = Gf'$, where G is an analytic function in \mathbb{D} . Indeed, their Theorems 3.9 and 4.1 in [4] include, in particular, the following statement:

Theorem A. *Let \mathcal{B} be either the Hardy space $H^2(\mathbb{D})$ or the Dirichlet space \mathcal{D} and let $Af = G \cdot f'$ for $f \in \mathcal{D}(A) = \{f \in \mathcal{B} : G \cdot f' \in \mathcal{B}\}$. Then A generates a C_0 -semigroup of composition operators on \mathcal{B} if and only if A generates a quasicontractive C_0 -semigroup on \mathcal{B} .*

In fact, this assertion was stated in [4] under an additional assumption $G \in \mathcal{B}$, however, as the authors observe in [5, p. 549], the same proofs work without this assumption. We refer to [4, 5] for more details, in particular, to several characterizations of possible functions G that may appear here.

The question whether there may exist an operator $Af = G \cdot f'$, which is a generator of a C_0 -semigroup (on $H^2(\mathbb{D})$ or on \mathcal{D}), which does not consist of composition operators, remained open.

Our main result in this paper gives an answer to this question not only in the context of the Hardy space or the Dirichlet space, but also for more general function spaces. As a particular instance of our main theorem, we will show that an operator $Af = G \cdot f'$, is a generator of a C_0 -semigroup (on $H^2(\mathbb{D})$ or on \mathcal{D}) if and only if A generates on these spaces a C_0 -semigroup of composition operators.

The paper is organized as follows. In Section 2, after some preliminaries, we prove our main result. Section 3 contains some discussions, and we also raise some questions there.

Remark. After this work was finished, W. Arendt and I. Chalendar informed us about their work in progress, where they address the same question and obtain another versions of our main result, which apply to a class of domains in \mathbb{C} . Their conditions on the functional space are incomparable to ours (we comment on it at the end of the next Section).

2. C_0 -SEMIGROUPS ON SPACES OF ANALYTIC FUNCTIONS

In what follows, \mathcal{B} will be a Banach space of holomorphic functions on the unit disc \mathbb{D} . The space of bounded linear operators on \mathcal{B} will be denoted by $\mathcal{L}(\mathcal{B})$. We denote by $\text{Hol}(\mathbb{D})$ the space of all holomorphic functions on \mathbb{D} and by $\mathcal{O}(\overline{\mathbb{D}})$ the set of all functions, holomorphic on (a neighborhood of) the closed unit disc $\overline{\mathbb{D}}$. Both are given a structure of linear topological vector spaces in a usual way (regarding $\mathcal{O}(\overline{\mathbb{D}})$ see [8, pp. 81], for instance). We impose the following natural assumption on \mathcal{B} :

$$(\star) \quad \mathcal{O}(\overline{\mathbb{D}}) \hookrightarrow \mathcal{B} \hookrightarrow \text{Hol}(\mathbb{D}), \text{ and both embeddings are continuous.}$$

We remark that, if $\{\varphi_t\}$ is a holomorphic flow on \mathbb{D} and the linear operators

$$T_t f = f \circ \varphi_t$$

are bounded on \mathcal{B} for $t \geq 0$, then they form a semigroup of linear operators. In this case, we will say that it is a semigroup of composition operators. In fact, it is a C_0 -semigroup whenever \mathcal{B} is reflexive (see Section 3 below).

Under the hypothesis (\star) , the generator of $\{T_t\}_{t \geq 0}$ has the form

$$Af = G \cdot f',$$

and one has $\mathcal{D}(A) = \{f \in \mathcal{B} : G \cdot f' \in \mathcal{B}\}$, see for instance, [9, Theorem 2].

Our main theorem reads as follows:

Main Theorem. Let \mathcal{B} be a Banach space of analytic functions on \mathbb{D} satisfying hypothesis (\star) .

Let G be an analytic function in \mathbb{D} and let A be given $Af = G \cdot f'$ for $f \in \mathcal{D}(A) = \{f \in \mathcal{B} : G \cdot f' \in \mathcal{B}\}$. Then A generates a C_0 -semigroup on \mathcal{B} if and only if A generates a C_0 -semigroup of composition operators.

In particular, by Theorem A, for the cases of the Hardy space $H^2(\mathbb{D})$ and the Dirichlet space \mathcal{D} , this semigroup will be necessarily quasicontractive. (In Section 3, we will give more general statements.)

Proof of Main Theorem. Assume A generates a C_0 -semigroup $\{T_t\}$ on \mathcal{B} . Our goal is to show the existence of a holomorphic flow $\{\varphi_t\}_{t \geq 0}$ such that $T_t f = f \circ \varphi_t$ for all functions $f \in \mathcal{B}$. Fix a radius $r \in (0, 1)$ and consider the Cauchy problem

$$(CP) \quad \begin{cases} \frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)) \\ \varphi_0(z) = z \end{cases} \quad (z \in D(0, r) = \{z \in \mathbb{D} : |z| < r\}).$$

The standard theory of ordinary differential equations in complex domain implies that there exists $t_0 > 0$ and an analytic solution $\{\varphi_t(z)\}$ of (CP), defined for $z \in D(0, r)$ and all complex t , $|t| < t_0$. See, for instance, [21], Theorems 2.2.1 and 2.8.2 (the Cauchy-Kovalevskaya Theorem is also applicable here). Moreover, this solution is unique in the class of smooth functions.

We will only need real times $t \in (-t_0, t_0)$. Since the differential equation in (CP) is autonomous, we have the semigroup property: $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$ whenever $t, s, t+s \in (-t_0, t_0)$ and $z, \varphi_s(z) \in D(0, r)$. Indeed, by (CP), for fixed s and z , both functions $\varphi_{t+s}(z)$ and $\varphi_t(\varphi_s(z))$ satisfy the same differential equation for $t \in (-t_0 + |s|, t_0 - |s|)$ and have the same initial value at $t = 0$.

We can also assume that there is some $r' \in (0, r)$ such that $\varphi_t(z) \in D(0, r)$ whenever $t \in (-t_0, t_0)$ and $z \in D(0, r')$; this is achieved by substituting t_0 with a smaller number. Therefore

$$(2.1) \quad \varphi_{-t} \circ \varphi_t(z) = z \quad \text{if } t \in (-t_0, t_0) \text{ and } z \in D(0, r').$$

In what follows, we denote $g'(z) = \frac{\partial g(z)}{\partial z}$.

Our first goal is to prove the following:

Claim 1: For any $f \in \mathcal{D}(A)$,

$$(2.2) \quad T_t f(z) = f \circ \varphi_t(z), \quad z \in D(0, r), \quad 0 \leq t < t_0.$$

Proof. Fix $f \in \mathcal{D}(A)$ and denote

$$f_t(z) = T_t f(z), \quad z \in \mathbb{D}, t \geq 0.$$

Then $f_t \in \mathcal{D}(A)$ for all $t \geq 0$, and

$$(2.3) \quad \frac{\partial f_t(z)}{\partial t} = A f_t(z) = G(z) f_t'(z), \quad z \in \mathbb{D}, t \geq 0.$$

By (CP), $\frac{\partial \varphi_{-t}(z)}{\partial t} = -G(\varphi_{-t}(z))$. Calculating the derivative of $f_t \circ \varphi_{-t}(z)$ with respect to t , using (2.3) and the chain rule we get

$$\frac{\partial}{\partial t} (f_t \circ \varphi_{-t}(z)) = G(\varphi_{-t}(z)) f_t'(\varphi_{-t}(z)) - f_t'(\varphi_{-t}(z)) G(\varphi_{-t}(z)) = 0, \quad 0 \leq t < t_0.$$

Therefore for any $z \in D(0, r)$ and any $t \in (0, t_0)$, $f_t \circ \varphi_{-t}(z) = f_0 \circ \varphi_0(z) = f(z)$. By (2.1), $f_t(z) = f \circ \varphi_t(z)$ for $z \in D(0, r')$, which by analyticity of both sides on $D(0, r)$ implies Claim 1. \square

Let us denote by E_z the evaluation functional $E_z f \stackrel{\text{def}}{=} f(z)$, which is continuous on \mathcal{B} for any $z \in \mathbb{D}$ by hypotheses. We can write down (2.2) as

$$E_z(T_t f) = E_{\varphi_t(z)} f, \quad \text{for } z \in D(0, r) \text{ and } 0 \leq t < t_0.$$

Since $\mathcal{D}(A)$ is dense in \mathcal{B} , a density argument gives that (2.2) holds for any $f \in \mathcal{B}$. Now, by applying (2.2) to the identity function id , $\text{id}(z) \equiv z$, we get $(T_t \text{id})(z) = \text{id} \circ \varphi_t(z) = \varphi_t(z)$ for $|z| < r$ and $0 \leq t < t_0$. Now *define*

$$\varphi_t(z) := (T_t \text{id})(z), \quad z \in \mathbb{D}, t \geq 0.$$

This agrees with the previous definition if $0 \leq t < t_0$ and $|z| < r$. Moreover, $\varphi_t \in \mathcal{B}$ and in particular, φ_t is analytic on \mathbb{D} , for any $t \geq 0$.

In the same way, (2.2) also gives $(T_t z^n)(a) = (\varphi_t(a))^n$ for $0 \leq t < t_0$ (we get it first for $|a| < r$ and then for all $a \in \mathbb{D}$, by analytic continuation).

Claim 2: There is a positive $t_1 \leq t_0$ such that $|\varphi_t(z)| < 1$ for all $z \in \mathbb{D}$ and all $0 \leq t < t_1$.

Proof. Notice that (\star) implies that $\overline{\lim}_{n \rightarrow \infty} (\|z^n\|_{\mathcal{B}})^{1/n} \leq 1$. Fix some $\varepsilon > 0$. Then there exist some constants C_ε and N such that for any $t \in [0, t_0)$, $a \in \mathbb{D}$ and any integer $n \geq N$,

$$|\varphi_t(a)|^n = |T_t(z^n)(a)| \leq \|E_a\| \|T_t\| \|z^n\|_{\mathcal{B}} \leq C_\varepsilon \|E_a\| \|T_t\| (1 + \varepsilon)^n.$$

Taking n th roots, letting $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we get that $|\varphi_t(a)| \leq 1$ for any $a \in \mathbb{D}$.

Since $\varphi_t(0)$ depends continuously on t and $\varphi_0(0) = 0$, there is some $t_1 \in (0, t_0]$ such that $|\varphi_t(0)| < 1$ for $0 \leq t < t_1$. It follows that $|\varphi_t(z)| < 1$ for $0 \leq t < t_1$ and all $z \in \mathbb{D}$, which completes the proof of Claim 2. \square

It was shown above that the functions $\varphi_t = T_t \text{id}$, $0 \leq t < t_1$, satisfy $\varphi_s \circ \varphi_t(z) = \varphi_{s+t}(z)$ for $s, t \geq 0$, $s + t < t_1$ and $z \in D(0, r)$. Obviously, this equality extends to all $z \in \mathbb{D}$. The function $\varphi_t(z) = E_z(T_t \text{id})$ is continuous in $t \in [0, t_1]$. By [27], Proposition 3.3.1, the family $\{\varphi_t(z)\}$ can be continued to a holomorphic flow, defined on $[0, +\infty) \times \mathbb{D}$.

Finally, given any $t > 0$, fix some N such that $t/N < t_1$. Then, for any $f \in \mathcal{B}$ it follows

$$T_t f = T_{t/N}^N f = f \circ \underbrace{\varphi_{t/N} \circ \cdots \circ \varphi_{t/N}}_{N \text{ times}} = f \circ \varphi_t,$$

which concludes the proof of the Main Theorem. \square

Let us explain the relationship between our Main Theorem and the results of the paper [2] by Arendt and Chalendar (which were obtained independently). Our results and those in [2] are incomparable: their results do not imply ours, and vice versa. The counterparts of our Main Theorem in [2] are Theorems 3.2 and 4.1. Their statements are similar to the statement of our main result, the only differences are the requirements on \mathcal{B} and also the fact that in [2], holomorphic flows on a general domain Ω in the complex plane are studied.

Let us first make the comparison for the case when $\Omega = \mathbb{D}$. Then Theorem 3.2 in [2] does not apply to any space \mathcal{B} contained in the disc algebra $A(\mathbb{D})$, whereas our main hypothesis (\star) on \mathcal{B} permits us to deal with many natural spaces of this type (diverse Banach algebras of smooth analytic functions on $\overline{\mathbb{D}}$ belong to this class). There are also examples of C_0 semigroups spaces \mathcal{B} such that our Main Theorem applies while Theorem 4.1 in [2] does not. For a holomorphic flow $\{\varphi_t\}$ on \mathbb{D} , denote by $[\varphi_t, H^\infty]$ the maximal closed subspace of H^∞ on which the operator semigroup $\{C_{\varphi_t}\}$ is strongly continuous. We refer to [9, Section 3] for the definition of the subspaces $[\varphi_t, X]$, where $X \subset \text{Hol}(\mathbb{D})$ is a function space. By [6], Corollary 1.4 and Proposition 4.3, there exists a flow $\{\varphi_t\}$ such that $\mathcal{B} := [\varphi_t, H^\infty]$ contains a function f which is not in $A(\mathbb{D})$. If f does not have limit at a point w on the unit circle, then the hypothesis (D) of Theorem 4.1 in [2] is violated in this point (whereas our Main Theorem does apply to \mathcal{B} and to the infinitesimal generator G of the flow $\{\varphi_t\}$).

On the other hand, the results of [2] apply to domains different from \mathbb{D} , whereas our result does not. If Ω is simply connected, one can pass from flows on Ω to flows on \mathbb{D} by a conformal transplantation. However, this does not permit one to reduce the results of [2] in the simply connected case to the special case $\Omega = \mathbb{D}$, because the hypotheses there are not conformally invariant. We refer to [5, Section 6] for a discussion of the possibility of conformal transplantation in this context.

Let us also mention that in a recent paper [13], Chalendar and Partington prove some analogues of the results of [4], [5] for generators given by higher order differential expressions in the disc. The corresponding operator semigroups in general do not have such clear geometric interpretation as above.

3. ON QUASICONTRACTIVE COMPOSITION SEMIGROUPS: REMARKS AND QUESTIONS

As it was mentioned before, if the semigroup $T_t f = f \circ \varphi_t$, $t \geq 0$, is bounded on a *reflexive* Banach space \mathcal{B} satisfying (\star) , that is, $\mathcal{O}(\mathbb{D}) \hookrightarrow \mathcal{B} \hookrightarrow \text{Hol}(\mathbb{D})$ and both embeddings are continuous, then $\{T_t\}$ is automatically a C_0 semigroup. To prove it, one just notices that the functionals

$$E_z f = f(z), \quad z \in \mathbb{D}$$

are complete in \mathcal{B}^* . Hence condition (\star) implies that the family $\{T_t\}$ is weakly continuous. By [17, theorem 5.8], it is strongly continuous.

In general, if \mathcal{B} is not reflexive, not all bounded semigroups of composition operators on \mathcal{B} are C_0 semigroups. Moreover, for some spaces \mathcal{B} , there is no nontrivial C_0 semigroups of composition operators on \mathcal{B} . See [6] for spaces between H^∞ and the Bloch space, [3] for certain mixed norm spaces; in these papers one can find references to earlier results. For non-reflexive spaces, it would be desirable to find an analogue of our Main Theorem, where the words “ C_0 semigroup” are substituted by a weaker property, valid for all bounded composition semigroups.

On the other hand, as Avicou, Chalendar and Partington prove in [4], any semigroup of composition operators on $H^2(\mathbb{D})$ is quasicontractive. Their argument extends to a wide range of Banach spaces \mathcal{B} . Consider the following condition

($\star\star$) For any univalent function η , which maps \mathbb{D} to \mathbb{D} and satisfies $\eta(0) = 0$, one has $\|f \circ \eta\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}}$.

Note that, in particular, ($\star\star$) implies that \mathcal{B} is rotation invariant. Moreover, if we denote

$$\alpha_r(z) = \frac{z+r}{1+rz}, \quad r \in (0, 1),$$

it is clear that α_r is a hyperbolic disc automorphism and the arguments of [4] yield the following statement.

Proposition 3.1. Suppose \mathcal{B} has the properties (\star) and ($\star\star$). Then the following holds.

- (i) Every holomorphic flow $\{\varphi_t\}$ generates a bounded semigroup of composition operators on \mathcal{B} (not necessarily a C_0 -semigroup), if and only if the composition operators $C_{\alpha_r} f = f \circ \alpha_r$ are bounded on \mathcal{B} for any $r \in (0, 1)$;
- (ii) Every holomorphic flow $\{\varphi_t\}$ generates a quasicontractive semigroup of composition operators on \mathcal{B} if and only if the composition operators C_{α_r} are bounded on \mathcal{B} for $r \in (0, 1)$ and satisfy an estimate

$$(3.1) \quad \|C_{\alpha_r}\|_{\mathcal{L}(\mathcal{B})} \leq \left(\frac{1+r}{1-r}\right)^a, \quad r \in (0, 1),$$

for some nonnegative constant a . In this case, for any univalent function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ one has

$$(3.2) \quad \|C_{\varphi}\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^a.$$

A few words are in order. First, observe that $\{C_{\alpha_r}\}_{0 \leq r < 1}$ is a semigroup of operators, if one makes the change of variables $t = \frac{1}{2} \log \frac{1+r}{1-r}$, or, equivalently, $r = \tanh t$. Indeed, we have that $T_t = C_{\alpha_{\tanh t}}$, satisfies $T_t T_s = T_{t+s}$ since $\alpha_r \alpha_s = \alpha_{(r+s)/(1+rs)}$. Inequality (3.1) rewrites as $\|T_t\|_{\mathcal{L}(\mathcal{B})} \leq e^{2at}$, $t \geq 0$, therefore it is equivalent to the fact that $\{T_t\}$ is a quasicontractive semigroup. By passing to the parameter t , it follows also that if (3.1) holds for $r \in (0, r_0)$, where $0 < r_0 < 1$, then it holds for all $r \in (0, 1)$, and (3.2) is true for any value of $\varphi(0)$.

Let $\beta = \{\beta_n\}$ be a sequence of positive numbers. Consider the weighted Hardy space $\mathcal{H}^2(\beta)$ consisting of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on a neighborhood of 0 for which the norm

$$\|f\|_{\beta} = \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2\right)^{1/2}$$

is finite. We assume that $\limsup \beta_n^{1/n} \geq 1$, then every $f \in \mathcal{H}^2(\beta)$ is analytic on \mathbb{D} .

Consider the quantity

$$(3.3) \quad \Lambda := \sup \left\{ \operatorname{Re} \left(\beta_0^2 \bar{a}_0 a_1 + \sum_{n=1}^{\infty} \beta_n^2 \bar{a}_n [(n+1)a_{n+1} - (n-1)a_{n-1}] \right) : \|f\|_{\beta} = 1, \|(1-z^2)f'(z)\| < \infty \right\}$$

(where $f(z) = \sum_n a_n z^n$). Gallardo-Gutiérrez and Partington proved in [19, Proposition 2.4] that if $\mathcal{B} = \mathcal{H}^2(\beta)$ is a weighted Hardy space which contains $\mathcal{H}^2(\mathbb{D})$, then $\{C_{\alpha_r}\}$ satisfy (3.1) if and only if $\Lambda < \infty$. Moreover, as they show, the best constant a in the estimate (3.1) equals to $\Lambda/2$. In [19], this was only stated for the case when $\mathcal{H}^2(\beta) \supset H^2$, but its proof is valid without this assumption. Our next observation makes their criterion for quasicontractivity of $\{C_{\alpha_{\tanh t}}\}$ more explicit.

Proposition 3.2. Let $\mathcal{H}^2(\beta)$ be a weighted Hardy space. Then $\{C_{\alpha_{\tanh t}}\}_{t \geq 0}$ is a quasicontractive C_0 -semigroup if and only if

$$(3.4) \quad \sup_n n \left| 1 - \frac{\beta_{n+1}}{\beta_n} \right| < \infty.$$

Proof. Observe that (3.4) is equivalent to

$$(3.5) \quad \sup_n \left| (n+1) \frac{\beta_n}{\beta_{n+1}} - n \frac{\beta_{n+1}}{\beta_n} \right| < \infty.$$

Indeed, from (3.4) one deduces that $\lim_n \frac{\beta_{n+1}}{\beta_n} = 1$ and hence, (3.5) follows. The converse is analogous.

Assume that $\{C_{\alpha_{\tanh t}}\}_{0 \leq t < \infty}$ is quasicontractive. Hence $\Lambda < \infty$, or equivalently

$$(3.6) \quad \sup \left\{ \operatorname{Re} \left(\frac{\beta_0}{\beta_1} \bar{x}_0 x_1 + \sum_{n=1}^{\infty} \bar{x}_n \left[(n+1) \frac{\beta_n}{\beta_{n+1}} x_{n+1} - (n-1) \frac{\beta_n}{\beta_{n-1}} x_{n-1} \right] \right) : x \in \mathcal{E} \right\} < \infty,$$

where $\mathcal{E} \subset \ell^2$ is given by

$$\mathcal{E} = \left\{ x = \{x_n\}_{n \geq 0} : \|x\|_{\ell^2}^2 = 1, \left\{ (n+1) \frac{\beta_n}{\beta_{n+1}} x_{n+1} - (n-1) \frac{\beta_n}{\beta_{n-1}} x_{n-1} \right\}_{n \geq 0} \in \ell^2 \right\}.$$

By considering the vectors $x = (e_k \pm e_{k+1})/\sqrt{2}$, $k \geq 0$, where $\{e_k\}_{k \geq 0}$ is the standard basis of ℓ^2 , we get (3.5).

Conversely, let us assume that (3.4) holds. Then $1/2 \leq \frac{\beta_{n+1}}{\beta_n} \leq 2$, say, for $n \geq n_0$. This allows us to rearrange the summation in (3.6) and rewrite it as follows

$$\sup \left\{ \operatorname{Re} \sum_{n=1}^{\infty} \bar{x}_n \left[(n+1) \frac{\beta_n}{\beta_{n+1}} \bar{x}_n x_{n+1} - n \frac{\beta_{n+1}}{\beta_n} x_n \bar{x}_{n+1} \right] : x \in \mathcal{E} \right\} < \infty.$$

This relation is valid, because (3.5) holds and

$$\operatorname{Re} \left((n+1) \frac{\beta_n}{\beta_{n+1}} \bar{x}_n x_{n+1} - n \frac{\beta_{n+1}}{\beta_n} x_n \bar{x}_{n+1} \right) = \left((n+1) \frac{\beta_n}{\beta_{n+1}} - n \frac{\beta_{n+1}}{\beta_n} \right) \operatorname{Re}(x_n \bar{x}_{n+1}).$$

So, Λ is finite and therefore,

$$\|C_{\alpha_{\tanh t}}\|_{\mathcal{L}(\mathcal{H}^2(\beta))} \leq e^{\Lambda t},$$

which shows that $\{C_{\alpha_{\tanh t}}\}_{0 \leq t < \infty}$ is quasicontractive as we wish. \square

It is easy to see that whenever Λ is finite, (3.3) is equivalent to the expression (2) for Λ , given in [19] (where, to be precise, one should add the condition $(1-z^2)f'(z) \in H^2(\beta)$, where $f = \sum_n a_n z^n$).

The classical Dirichlet space \mathcal{D} corresponds to the weights, given by $\beta_n = \sqrt{n}$ for $n \geq 1$ and $\beta_0 = 1$. This space satisfies $(\star\star)$ and, in fact, satisfies the estimate (3.2). (See [29], [18] and [23] for estimates for the norms of composition operators C_φ on \mathcal{D} .) We get the following statement, which is close to [19], Corollary 2.5.

Proposition 3.3. Let $\mathcal{B} = \mathcal{H}^2(\beta)$. Suppose that the sequence $\{\beta_n/\sqrt{n} : n \geq 1\}$ is monotone decreasing, $\beta_0 \geq \beta_1$, and (3.4) holds. Then any holomorphic flow $\{\varphi_t\}$ generates a quasicontractive semigroup of composition operators on \mathcal{B} and (3.2) holds for any univalent function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$.

Proof. Assuming the hypotheses, we get from [29] that $(\star\star)$ holds for the Dirichlet space. Next, we apply a result by Cowen [14, Theorem 7], and get $\|C_\eta\|_{\mathcal{L}(\mathcal{B})} \leq \|C_\eta\|_{\mathcal{L}(\mathcal{D})} = 1$ for any univalent function $\eta : \mathbb{D} \rightarrow \mathbb{D}$ with $\eta(0) = 0$. Hence $(\star\star)$ holds for \mathcal{B} . Now all our statements follow from Proposition 3.1. \square

We remark that a lemma which implies the cited result by Cowen had been proved in 1972 by Katznel'son [22] (the proofs are different). We refer to [12] for more information and for other applications of this kind of results.

Notice that, whenever \mathcal{B} is contained in the disc algebra $A(\mathbb{D})$, not all holomorphic flows $\{\varphi_t\}$ induce a bounded semigroup of composition operators on \mathcal{B} . This applies, in particular, to Dirichlet spaces, smaller than \mathcal{D} . This follows from the observation that $\varphi_t \in \mathcal{B}$ for all $t > 0$ whenever operators C_{φ_t} are bounded, but there are flows such that $\varphi_t(z)$ does not extend continuously to the closed unit disc. See also Theorem 4.8 in [15].

There are spaces $\mathcal{H}^2(\beta)$ where C_φ is bounded for any univalent $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ but is unbounded for some non-univalent functions, like the Dirichlet space (see [16], for instance).

On the other side, composition operators induced by the Möbius maps α_r are not always bounded even in spaces $\mathcal{H}^2(\beta)$ with fast decreasing weights. See, for instance, Chapter 5 in [15], in particular, Theorem 5.2.

A key observation regarding Proposition 3.2 is that the quasicontractivity property of composition semigroups is very sensitive to changing the norm by an equivalent one. As it follows from Proposition 3.2, if for some sequence of weights $\{\beta_n\}$, the semigroup $\{C_{\alpha_{\tanh t}}\}_{0 \leq t < \infty}$ is quasicontractive, then it will fail to be quasicontractive for weights

$$\tilde{\beta}_n = (2 + (-1)^n)\beta_n,$$

These weights define an equivalent norm, so that the property of boundedness of our semigroup (as well as that of any other composition semigroup) will not be affected by this change.

This phenomenon is related to much more general facts proved by Matolcsi in [24]: given any C_0 -semigroup on a Banach space, whose generator is unbounded, it can be converted to a non-quasicontractive one by passing to an equivalent norm on this space. By [25], the same is true in the context of Hilbert spaces.

Observe that, if \mathcal{B} is a Hilbert space, the generator of any quasicontractive semigroup on \mathcal{B} admits an H^∞ calculus on a half-plane $|\arg(z_0 - z)| < \pi/2$, see the book [20]. The existence of an H^∞ calculus is not affected if one passes to an equivalent norm on \mathcal{B} . This motivates the following question.

Question. Do there exist weights $\{\beta_n\}$ such that the generator of the semigroup $\{C_{\alpha_{\tanh t}}\}_{0 \leq t < \infty}$ on $\mathcal{H}^2(\beta)$ is bounded, but does not admit an H^∞ calculus in a sector $|\arg(z_0 - z)| < \theta$, where $\theta \in [\pi/2, \pi)$?

Finally, we notice that the property $(\star\star)$ and estimates like (3.2) are known for many classical Banach spaces. We refer to [15, Chapter 3] for H^p spaces and to [11] for VMOA. Property $(\star\star)$ also holds true for the case of mixed norm spaces $H(p, q, \alpha)$, see [3].

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E. A. GALLARDO-GUTIÉRREZ
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
FACULTAD DE MATEMÁTICAS,
UNIVERSIDAD COMPLUTENSE DE MADRID,
PLAZA DE CIENCIAS Nº 3, 28040 MADRID, SPAIN
AND INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM),
MADRID, SPAIN
E-mail address: `eva.gallardo@mat.ucm.es`

D. V. YAKUBOVICH
DEPARTAMENTO DE MATEMÁTICAS,
UNIVERSIDAD AUTÓNOMA DE MADRID,
CANTOBLANCO, 28049 MADRID, SPAIN
AND INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM),
MADRID, SPAIN.
E-mail address: `dmitry.yakubovich@uam.es`