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ON GENERIC SUBMANIFOLDS OF MANIFOLDS EQUIPPED WITH A HYPERCOSYMPLECTIC 3-STRUCTURE

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Abstract. Generic submanifolds of a Riemannian manifold endowed with a hypercosymplectic 3-structure are studied. Integrability conditions for certain distributions on the generic submanifold are discussed. Geometry of leaves of certain distributions are also studied.

1. Introduction

Three local (global) almost complex structures which satisfy the quaternionic relations of the imaginary quaternions, constitute the quaternionic analog of almost complex structures, namely the almost quaternion (hypercomplex) structure ([5]). Quaternion Kähler manifolds and hyper-Kähler manifolds are special and interesting cases of Riemannian manifolds with almost quaternion and almost hypercomplex structure, respectively. Quaternion Kähler manifolds are Einstein, hyper-Kähler manifolds are Ricci flat and their respective holonomy groups are included in the Berger list ([2]). An almost contact 3-structure was defined by Kuo ([8]) and it is closely related to both almost quaternion and almost hypercomplex structures. Hypersurfaces of manifolds with almost hypercomplex structure inherit naturally three almost contact structures which constitute an almost contact 3-structure. An almost contact metric 3-structure manifold is always $(4m+3)$-dimensional. The structural group of the tangent bundle of a $(4m+3)$-dimensional manifold equipped with an almost contact 3-structure is reducible to $Sp\,(m) \times I_3$.

In particular, if each almost contact metric structure of an almost contact metric 3-structure is Sasakian, then this structure is called a

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Sasakian 3-structure. Riemannian manifolds with Sasakian 3-structure are called 3-Sasakian manifolds. They are Einstein ([6]) and have many links with quaternion Kähler and hyper-Kähler manifolds. In fact, a 3-Sasakian manifold (with some regularity conditions) fibers over a quaternion Kähler manifold ([5]) and can be imbedded into a hyper-Kähler manifold ([4]). A $(4m+3)$-dimensional sphere is a 3-Sasakian manifold.

Recently, F. Martín Cabrera introduced a hypercosymplectic 3-structure ([9]), where each almost contact metric structure of an almost contact metric 3-structure is cosymplectic ([3]). A $(4m+3)$-dimensional torus is a typical example of a hypercosymplectic 3-structure manifold.

In [7], the authors studied hypersurfaces of a manifold equipped with a hypercosymplectic 3-structure. A. Bejancu studied generic submanifolds of 3-Sasakian manifolds ([1]). In the present paper we study generic submanifolds of a manifold equipped with a hypercosymplectic 3-structure. The paper is organized as follows. Section 2 contains preliminaries. In section 3, some basic results are given. Integrability conditions for certain distributions on the generic submanifold are investigated in section 4. In the last section, geometry of leaves of certain distributions are studied.

2. Hypercosymplectic 3-structures

Let $\widetilde{M}$ be a $(4m+3)$-dimensional manifold endowed with three almost contact structures $(\varphi_a, \xi_a, \eta_a)$, $a = 1, 2, 3$, that is,

$$\varphi_a^2 = -I + \eta_a \otimes \xi_a, \quad \eta_a(\xi_a) = 1, \quad \varphi_a(\xi_a) = 0, \quad \eta_a \circ \varphi_a = 0. \tag{2.1}$$

Let these three almost contact structures satisfy

$$\varphi_a \circ \varphi_b - \eta_b \otimes \xi_a = -\varphi_b \circ \varphi_a + \eta_a \otimes \xi_b = \varphi_c, \tag{2.2}$$

$$\varphi_a \xi_b = -\varphi_b \xi_a = \xi_c, \tag{2.3}$$

$$\eta_a \circ \varphi_b = -\eta_b \circ \varphi_a = \eta_c, \tag{2.4}$$

$$\eta_a(\xi_b) = \eta_b(\xi_a) = 0, \quad a \neq b \tag{2.5}$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$. Then we say that $\widetilde{M}$ is endowed with an *almost contact 3-structure* (Kuo, [8]). If $\widetilde{M}$ is a Riemannian manifold, then there is always a Riemannian metric g on $\widetilde{M}$ such that

$$g(\varphi_a X, \varphi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y), \qquad a = 1, 2, 3 \tag{2.6}$$

(2.7) $$g(X,\xi_a) = \eta_a(X), \qquad a = 1,2,3$$

for all $X, Y \in T\widetilde{M}$. Then we say that $\widetilde{M}$ is endowed with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$ (Kuo, [8]). From (2.7) it follows that ξ_1, ξ_2, ξ_3 are mutually orthogonal. We also have

(2.8) $$\Omega_a(X,Y) \equiv g(X, \varphi_a Y) = -g(\varphi_a X, Y), \qquad a = 1,2,3.$$

We know that an almost contact metric structure (φ, ξ, η, g) is called a cosymplectic structure if (Blair, [3])

(2.9) $$\widetilde{\nabla}\varphi = 0,$$

where $\widetilde{\nabla}$ is the Riemannian connection. From (2.9) it follows that

(2.10) $$\widetilde{\nabla}\xi = 0, \qquad \widetilde{\nabla}\eta = 0.$$

If all the three almost contact metric structures $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, are cosymplectic structures, that is,

(2.11) $$\widetilde{\nabla}\varphi_a = 0,$$

(2.12) $$\widetilde{\nabla}\xi_a = 0 \qquad \widetilde{\nabla}\eta_a = 0,$$

then the manifold $\widetilde{M}$ is said to have a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$ (cf. F. Martin Cabrera, [9]).

EXAMPLE 2.1. We construct a simple example of a hypercosymplectic 3-structure in the 3-dimensional Euclidean space $\mathbb{R}^3$. We define $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$ in $\mathbb{R}^3$ by their matrices as follows:

$$\varphi_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\eta_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and

$$g = I_3.$$

A straightforward calculation shows that the above set provides a hypercosymplectic 3-structure on $\mathbb{R}^3$.

EXAMPLE 2.2. We consider a $(4m+3)$-dimensional torus $\mathbb{T}^{4m+3}$ $(m \geq 1)$ and let $\{\alpha_1, \alpha_2, \ldots, \alpha_{4m+3}\}$ a basis for 1-forms such that each α_i is integral and closed. That is, each α_i defines an element of the first integral cohomology group so that if one does the integral of α_i along any 1-cycle, then the result is an integral number. On $\mathbb{T}^{4m+3}$ we consider the metric tensor field given by

$$g(X,Y) = \sum_{i=1}^{4m+3} \alpha_i(X)\alpha_i(Y)$$

and the almost contact metric 3-structure consisting of the three $(1,1)$ tensor fields

$$\varphi_a = \sum_{i=1}^{4m+3} (e_{am+i} \otimes \alpha_i - e_i \otimes \alpha_{a+i} + e_{cm+i} \otimes \alpha_{bm+i} \\ -e_{bm+i} \otimes \alpha_{cm+i} + e_{4m+c} \otimes \alpha_{4m+b} - e_{4m+b} \otimes \alpha_{4m+c}),$$

where $\{e_1, e_2, \ldots, e_{4m+3}\}$ is the orthonormal frame field dual of $\{\alpha_1, \alpha_2, \ldots, \alpha_{4m+3}\}$ and (a,b,c) is a cyclic permutation of $(1,2,3)$; the three 1 -forms

$$\eta_1 = \alpha_{4m+1}, \quad \eta_2 = \alpha_{4m+2}, \quad \eta_3 = \alpha_{4m+3};$$

and the three vector fields

$$\xi_1 = e_{4m+1}, \quad \xi_2 = e_{4m+2}, \quad \xi_3 = e_{4m+3}.$$

Then the torus $\mathbb{T}^{4m+3}$ contains a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$ ([9]).

3. Generic submanifolds

Let M be an $(n+3)$-dimensional submanifold of a $(4m+3)$-dimensional manifold $\widetilde{M}$ endowed with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, such that the structure vector fields ξ_a, $a = 1,2,3$ are tangential to M. The submanifold M is said to be a *generic submanifold* (Bejancu, [1]) of $\widetilde{M}$ if $\varphi_a(T_x^\perp M)$ is orthogonal to $T_x^\perp M$, that is, $\varphi_a(T_x^\perp M) \subset T_x M, a = 1,2,3$. Thus we can define three distributions $\mathcal{D}_a$, $a = 1,2,3$ on M by

(3.1) $$\mathcal{D}_a : x \longrightarrow \mathcal{D}_{ax} \equiv \varphi_a(T_x^\perp M) \subset T_x M, \qquad a = 1,2,3, \quad x \in M.$$

The 1-dimensional distributions $\{\xi_1\}, \{\xi_2\}, \{\xi_3\}$ are mutually orthogonal and we denote by

(3.2) $$\mathcal{E} \equiv \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}.$$

Each hypersurface M, of a manifold $\widetilde{M}$ endowed with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M, is a generic submanifold of $\widetilde{M}$.

THEOREM 3.1. *Let M be a generic submanifold of $\widetilde{M}$ equipped with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then*
(a) *$\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are mutually orthogonal,*
(b) *$\mathcal{D}^{\perp} \equiv \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ is orthogonal to $\mathcal{E}$.*
Consequently, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \{\xi_1\}, \{\xi_2\}, \{\xi_3\}$ are mutually orthogonal.

PROOF. **(a)** Let $Y_1 \in \mathcal{D}_1, Y_2 \in \mathcal{D}_2$. By definition of $\mathcal{D}_1$ and $\mathcal{D}_2$ there exist normal vector fields V_1, V_2 such that $Y_1 = \varphi_1 V_1$ and $Y_2 = \varphi_2 V_2$. Then by using (2.6), (2.7), (2.8) and (2.2) we get

$$\begin{aligned} g(Y_1, Y_2) &= g(\varphi_1 V_1, \varphi_2 V_2) = -g(V_1, (\varphi_1 \circ \varphi_2) V_2) \\ &= g(V_1, \varphi_3 V_2 + \eta_2(V_2)\xi_1) = 0, \end{aligned}$$

since ξ_1 and $\varphi_3(T_x^{\perp} M)$ are tangential to M. Thus $\mathcal{D}_1 \perp \mathcal{D}_2$. Similarly we can prove that $\mathcal{D}_1 \perp \mathcal{D}_3$ and $\mathcal{D}_2 \perp \mathcal{D}_3$.

(b) Let $Y_a \in \mathcal{D}_a$, $a = 1, 2, 3$. Then by (3.1) there exists a normal vector field V_a such that $Y_a = \varphi_a V_a$ and we have

$$g(Y_a, \xi_b) = g(\varphi_a V_a, \xi_b) = -g(V_a, \varphi_a \xi_b) = 0,$$

since we have $\varphi_a \xi_a = 0$ and $\varphi_a \xi_b = \xi_c$, where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. This completes the proof. □

Now, we denote by $\mathcal{D}$ the orthogonal complementary distribution to $\mathcal{D}^{\perp} \oplus \mathcal{E}$ in M.

THEOREM 3.2. *Let M be a generic submanifold of $\widetilde{M}$ endowed with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then the distribution $\mathcal{D}$ is invariant by each φ_a, that is*

$$\varphi_a(\mathcal{D}) = \mathcal{D}, \qquad a = 1, 2, 3. \tag{3.3}$$

PROOF. Let $X \in \mathcal{D}, Y \in \mathcal{D}^{\perp}$. Then by using (2.8), we get

$$g(\varphi_a X, Y) = -g(X, \varphi_a Y) = 0.$$

Thus $\varphi_a \mathcal{D} \perp \mathcal{D}^{\perp}$. Next, we get

$$g(\varphi_a X, N) = -g(X, \varphi_a N) = 0,$$

which implies that $\varphi_a \mathcal{D} \perp T^{\perp}M$. Finally, we have

$$g(\varphi_a X, \xi_b) = -g(X, \varphi_a \xi_b) = 0,$$

showing that $\varphi_a \mathcal{D} \perp \mathcal{E}$. Hence $\varphi_a \mathcal{D} = D$, that is, $\mathcal{D}$ is invariant by each φ_a. □

With respect to the behavior of distributions $\mathcal{D}_a$ to the action of φ_a we have

$$\varphi_a(\mathcal{D}_a) = T^{\perp}M; \qquad \varphi_a(\mathcal{D}_b) = \mathcal{D}_c, \tag{3.4}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$.

4. Integrability of certain distributions

4.1. Some basic results

Let M be an $(n+3)$-dimensional generic submanifold of a $(4m+3)$-dimensional manifold $\widetilde{M}$ endowed with an almost contact metric 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. We choose a local field of orthonormal frames $\{N_1, \ldots, N_s\}$ on the normal bundle $T^{\perp}M$, where $s = \text{codim}M$. Then on the distribution $\mathcal{D}^{\perp}$ we can take the local field of orthonormal frames

$$\{X_{11}, \ldots, X_{1s}, X_{21}, \ldots, X_{2s}, X_{31}, \ldots, X_{3s}\}, \tag{4.1}$$

where

$$X_{ai} = \varphi_a N_i, \quad a = 1, 2, 3, \quad i = 1, 2, \ldots, s. \tag{4.2}$$

We denote by U the projection operator of TM on to the invariant distribution $\mathcal{D}$. Then any arbitrary vector field X on M can be written locally as follows

$$Y = UY + \sum_{a=1}^{3} \sum_{i=1}^{s} (\omega_{ai}(Y) X_{ai}) + \sum_{a=1}^{3} \eta_a(Y) \xi_a, \tag{4.3}$$

where ω_{ai}, $a = 1, 2, 3$ are 1-forms locally defined on M by

$$\omega_{ai}(Y) = g(Y, X_{ai}). \tag{4.4}$$

We recall the Gauss and Weingarten formulae:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{4.6}$$

for all $X, Y \in TM$, where ∇ is the induced Riemannian connection on M, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal vector N. It is well known that

$$g\left(h(X,Y),N\right) = g(A_N X, Y). \tag{4.7}$$

Applying φ_a to equation (4.3), we obtain ([1])

$$\begin{aligned} \varphi_a Y &= \varphi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b \\ &\quad + \sum_{i=1}^{s} \left(\omega_{bi}(Y)X_{ci} - \omega_{ci}(Y)X_{bi} - \omega_{ai}(Y)N_i\right). \end{aligned} \tag{4.8}$$

If $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$ is a hypercosymplectic 3-structure, then we have

$$\begin{aligned} 0 &= \widetilde{\nabla}_X \varphi_a Y - \varphi_a \widetilde{\nabla}_X Y \\ &= \widetilde{\nabla}_X \Bigg(\varphi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b \\ &\quad + \sum_{i=1}^{s} \left(\omega_{bi}(Y)X_{ci} - \omega_{ci}(Y)X_{bi} - \omega_{ai}(Y)N_i\right) \Bigg) \\ &\quad - \varphi_a \left(\nabla_X Y + h(X,Y)\right) \\ &= \nabla_X \left(\varphi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b + \sum_{i=1}^{s} \left(\omega_{bi}(Y)X_{ci} - \omega_{ci}(Y)X_{bi}\right) \right) \\ &\quad + \sum_{i=1}^{s} \left(\omega_{ai}(Y)A_{N_i}X\right) - \varphi_a U \nabla_X Y - \eta_b(\nabla_X Y)\xi_c + \eta_c(\nabla_X Y)\xi_b \\ &\quad - \sum_{i=1}^{s} \left(\omega_{bi}(\nabla_X Y)X_{ci} - \omega_{ci}(\nabla_X Y)X_{bi}\right) - \varphi_a h(X,Y) \\ &\quad + h\Bigg(X, \varphi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b \\ &\quad + \sum_{i=1}^{s} \left(\omega_{bi}(Y)X_{ci} - \omega_{ci}(Y)X_{bi}\right) \Bigg) \\ &\quad - \sum_{i=1}^{s}((\nabla_X \omega_{ai})Y)N_i - \sum_{i=1}^{s} \left(\omega_{ai}(Y)\nabla_X^{\perp} N_i\right). \end{aligned}$$

Equating normal parts we get

$$
\begin{aligned}
0 &= h(X,\varphi_a UY) + \eta_b(Y)h(X,\xi_c) - \eta_c(Y)h(X,\xi_b) \\
&\quad + \sum_{i=1}^{s} \Big(\omega_{bi}(Y)h(X,X_{ci}) - \omega_{ci}(Y)h(X,X_{bi}) \\
&\quad - ((\nabla_X \omega_{ai})Y)N_i - \omega_{ai}(Y)\nabla^{\perp}_X N_i\Big). \qquad (4.9)
\end{aligned}
$$

LEMMA 4.1. *Let M be a submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a,\xi_a,\eta_a,g)$, $a=1,2,3$, such that $\xi_a \in TM$, $a=1,2,3$. Then*

$$\nabla_X \xi_a = 0, \qquad X \in TM, \tag{4.10}$$

$$h(X,\xi_a) = 0, \qquad X \in TM, \tag{4.11}$$

$$A_N X \in \mathcal{E}^{\perp}, \qquad N \in T^{\perp}M, X \in TM, \tag{4.12}$$

$$A_N \xi_a = 0, \qquad N \in T^{\perp}M. \tag{4.13}$$

PROOF. By using (2.12) and (4.5), for all $X \in TM$ we get

$$\nabla_X \xi_a + h(X,\xi_a) = \widetilde{\nabla}_X \xi_a = 0. \tag{4.14}$$

Equating tangential and normal parts in (4.14) we get (4.10) and (4.11) respectively. In view of (4.11) and (4.7), we get

$$0 = g(h(X,\xi_a),N) = g(A_N X,\xi_a) = g(A_N\xi_a, X),$$

which gives (4.12) and (4.13). □

LEMMA 4.2. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a,\xi_a,\eta_a,g)$, $a=1,2,3$. Then we have*

$$g([X,\xi_a],Y) = g([Y,\xi_a],X) = 0, \qquad X \in \mathcal{D}, Y \in \mathcal{D}^{\perp}\oplus\mathcal{E}, \tag{4.15}$$

$$g([X,\xi_a],Y) = 0, \qquad X \in \mathcal{D}_a,\ Y \in \mathcal{D}_b \oplus \mathcal{D}_c \oplus \{\xi_b\} \oplus \{\xi_c\}, \tag{4.16}$$

$$g([X,\xi_a],\xi_b) = g([Y,\xi_a],\xi_b) = 0, \qquad X \in \mathcal{D},\ Y \in \mathcal{D}^{\perp},\ a \neq b. \tag{4.17}$$

PROOF. By using (4.10) for all $X \in \mathcal{D}$ and $Y \perp X$, we get

$$
\begin{aligned}
g([X,\xi_a],Y) &= g\left(\widetilde{\nabla}_X \xi_a - \widetilde{\nabla}_{\xi_a} X, Y\right) \\
&= g(X,Y) - g\left(\widetilde{\nabla}_{\xi_a} X, Y\right) = g\left(X, \widetilde{\nabla}_{\xi_a} Y\right).
\end{aligned}
$$

Since $\mathcal{D}$ is invariant by each φ_a and φ_a is an isomorphism on $\mathcal{D}$, therefore $X = \varphi_a Z_a$, for some $Z_a \in \mathcal{D}$, $a = 1,2,3$, in the above equation we get

$$\begin{aligned} g([X,\xi_a],Y) &= g\left(X,\widetilde{\nabla}_{\xi_a}Y\right) = g\left(\varphi_a Z,\widetilde{\nabla}_{\xi_a}Y\right) \\ &= g\left(Z,-\varphi_a\widetilde{\nabla}_{\xi_a}Y\right) = g\left(Z,-\widetilde{\nabla}_{\xi_a}\varphi_a Y\right). \end{aligned}$$

In particular, if $Y \in \mathcal{D}_a \oplus \{\xi_a\}$, then from the above equation, we get

$$g([X,\xi_a],Y) = g(Z, A_{\varphi_a Y}\xi_a) = g(h(Z,\xi_a),\varphi_a Y) = 0,$$

which implies the first equality of (4.15). Similarly we can prove other equalities. □

LEMMA 4.3. *Let M be a generic submanifold of $\widetilde{M}$ endowed with a hypercosymplectic 3-structure $(\varphi_a,\xi_a,\eta_a,g)$, $a = 1,2,3$. Then we have*

$$A_{\varphi_a X}Y = A_{\varphi_a Y}X \qquad X,Y \in \mathcal{D}_a, \tag{4.18}$$

$$g([X,Y],\xi_a) = 0, \qquad X,Y \perp \xi_a. \tag{4.19}$$

PROOF. For $X,Y \in \mathcal{D}_a$, $Z \in TM$, we have

$$\begin{aligned} g(A_{\varphi_a X}Y,Z) &= g(h(Y,Z),\varphi_a X) = g\left(\widetilde{\nabla}_Z Y,\varphi_a X\right) \\ &= g\left(-\varphi_a\left(\widetilde{\nabla}_Z Y\right),X\right) = g\left(-\widetilde{\nabla}_Z\varphi_a Y,X\right) \\ &= g(A_{\varphi_a Y}Z,X) = g(A_{\varphi_a Y}X,Z,), \end{aligned}$$

where (4.7), (4.5), (2.8), (2.11) have been used. This proves (4.18). Next, for $X,Y \perp \xi_a$, we have

$$g\left(\widetilde{\nabla}_X Y,\xi_a\right) = -g\left(\widetilde{\nabla}_X\xi_a,Y\right) = 0,$$

which proves (4.19). □

4.2. The distribution $\mathcal{E}$

LEMMA 4.4. *For a manifold with a hypercosymplectic 3-structure $(\varphi_a,\xi_a,\eta_a,g)$, we have*

$$[\xi_a,\xi_b] = 0, \quad a \neq b. \tag{4.20}$$

Consequently, the distributions $\{\xi_{ab}\}$ spanned by ξ_a and ξ_b, $a \neq b$, are integrable.

PROOF. We have

$$[\xi_a,\xi_b] = \widetilde{\nabla}_{\xi_a}\xi_b - \widetilde{\nabla}_{\xi_b}\xi_a = 0 - 0 = 0.$$

□

THEOREM 4.5. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distribution $\mathcal{E}$ is integrable.*

PROOF. In view of Lemma 4.4, the proof follows immediately. □

4.3. The distributions $\mathcal{D}$, $\mathcal{D}^{\perp}$, and $\mathcal{D} \oplus D^{\perp}$

THEOREM 4.6. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distribution $\mathcal{D} \oplus D^{\perp}$ is integrable.*

PROOF. Let $X, Y \in \mathcal{D} \oplus D^{\perp}$. Then we have

$$g([X, Y], \xi_a) = g(\nabla_X Y - \nabla_Y X, \xi_a) = -g(Y, \nabla_X \xi_a) + g(X, \nabla_Y \xi_a) = 0,$$

where Lemma 4.1 has been used. So $[X, Y] \in \mathcal{D} \oplus D^{\perp}$. □

Unlike in the case of Sasakian 3-structure, where $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are not integrable, in view of Theorem 4.6, we can state the following two theorems.

THEOREM 4.7. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$g([X, Y], Z) = 0, \qquad X, Y \in \mathcal{D},\ Z \in \mathcal{D}^{\perp}.$$

THEOREM 4.8. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distribution $\mathcal{D}^{\perp}$ is integrable if and only if*

$$g([X, Y], Z) = 0, \qquad X, Y \in \mathcal{D}^{\perp},\ Z \in \mathcal{D}.$$

4.4. The distribution $\mathcal{D} \oplus \mathcal{E}$

THEOREM 4.9. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the following statements are equivalent:*
(a) *the generic submanifold M is $\mathcal{D}$-geodesic.*
(b) *the distribution $\mathcal{D} \oplus \mathcal{E}$ is integrable.*
(c) *the second fundamental form h of the immersion of M satisfies*

$$h(X, \varphi_a Y) = h(\varphi_a X, Y), \qquad a = 1, 2, 3, \quad X, Y \in \mathcal{D}. \tag{4.21}$$

PROOF. Let the generic submanifold M be $\mathcal{D}$-geodesic, that is,

$$h(X, Y) = 0, \qquad X, Y \in \mathcal{D}. \tag{4.22}$$

In view of (4.15) and Theorem 4.5, to prove the integrability of $\mathcal{D} \oplus \mathcal{E}$ it is sufficient to prove that $[X, Y] \in \mathcal{D} \oplus \mathcal{E}$ for all $X, Y \in \mathcal{D}$. Since the differential forms ω_{ai} and η_a, $a = 1, 2, 3$; $i = 1, 2, \dots s$, vanish on $\mathcal{D}$, using (4.22) in (4.9), we have for all $X, Y \in \mathcal{D}$:

$$0 = -(\nabla_X \omega_{ai})Y \qquad \text{or} \qquad \omega_{ai}(\nabla_X Y) = 0.$$

Thus we get

$$g([X, Y], X_{ai}) = \omega_{ai}([X, Y]) = \omega_{ai}(\nabla_X Y) - \omega_{ai}(\nabla_Y X) = 0.$$

Hence $[X, Y] \in \mathcal{D} \oplus \mathcal{E}$ for all $X, Y \in \mathcal{D}$. This proves **(*a*) ⇒ (*b*)**.

Next, we assume that the distribution $\mathcal{D} \oplus \mathcal{E}$ is integrable. Then for all $X, Y \in \mathcal{D}$ we obtain

$$0 = g([X, Y], X_{ai}) = \omega_{ai}(\nabla_X Y) - \omega_{ai}(\nabla_Y X).$$

Hence using (4.9) we get for each $a = 1, 2, 3$

$$h(X, \varphi_a Y) = -\sum_{i=1}^{s} \omega_{ai}(\nabla_X Y) N_i = -\sum_{i=1}^{s} \omega_{ai}(\nabla_Y X) N_i = h(Y, \varphi_a X),$$

which proves (**b**) ⇒ (**c**). In last, we show that (**c**) ⇒ (**a**). We assume (**c**). Then, for all $X, Y \in \mathcal{D}$, we get

$$\begin{aligned} h(\varphi_3 X, Y) &= h(X, \varphi_3 Y) = h(X, (\varphi_1 \circ \varphi_2)Y) \\ &= h((\varphi_2 \circ \varphi_1)X, Y) = -h(\varphi_3 X, Y). \end{aligned}$$

Thus $h(\varphi_3 X, Y) = 0$. This implies that the generic submanifold M is $\mathcal{D}$-geodesic because φ_3 is an automorphism on $\mathcal{D}$. □

4.5. The distributions $\mathcal{D}_a$ and $\mathcal{D}_a \oplus \{\xi_a\}$

THEOREM 4.10. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distributions $\mathcal{D}_a$ are integrable.*

PROOF. For $X, Y \in \mathcal{D}_a$, we get

$$0 = \widetilde{\nabla}_X \varphi_a Y - \varphi_a \widetilde{\nabla}_X Y = -A_{\varphi_a Y} X + \nabla^{\perp}_X \varphi_a Y - \varphi_a (\nabla_X Y) - \varphi_a h(X, Y),$$

which in view of the equation (4.18) gives

$$\varphi_a [X, Y] = \nabla^{\perp}_X \varphi_a Y - \nabla^{\perp}_Y \varphi_a X.$$

In view of (4.19) we get

$$-[X, Y] = \varphi_a^2 [X, Y] = \varphi_a \left(\nabla^{\perp}_X \varphi_a Y - \nabla^{\perp}_Y \varphi_a X\right) \in \mathcal{D}_a,$$

which makes $\mathcal{D}_a$ integrable. □

In view of the above theorem and the equation (4.16), we get the following theorem.

THEOREM 4.11. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the distributions $\mathcal{D}_a \oplus \{\xi_a\}$ are integrable.*

4.6. The distribution $\mathcal{D}^\perp \oplus \mathcal{E}$

THEOREM 4.12. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Then the following statements are equivalent:*
(a) *the distribution $\mathcal{D}^\perp \oplus \mathcal{E}$ is integrable.*
(b) *the generic submanifold M is $(\mathcal{D}, \mathcal{D}^\perp)$-geodesic, that is,*

$$h(X, Y) = 0, \qquad X \in \mathcal{D}, Y \in \mathcal{D}^\perp. \tag{4.23}$$

PROOF. Let (a, b, c) be a cyclic permutation of $(1, 2, 3)$. Then, for all $U, V \in T^\perp M$ and $X \in \mathcal{D}$, we obtain

$$\begin{aligned} & g\left([\varphi_a U, \varphi_b V], \varphi_c X\right) \\ = \ & g\left(\widetilde{\nabla}_{\varphi_a U}\varphi_b V - \widetilde{\nabla}_{\varphi_b V}\varphi_a U, \varphi_c X\right) \\ = \ & g\left(\varphi_c\left(\widetilde{\nabla}_{\varphi_b V}\varphi_a U\right) - \varphi_c\left(\widetilde{\nabla}_{\varphi_a U}\varphi_b V\right), X\right) \\ = \ & g\left(\widetilde{\nabla}_{\varphi_b V}\varphi_c\varphi_a U, X\right) - g\left(\widetilde{\nabla}_{\varphi_a U}\varphi_c\varphi_b V, X\right) \\ = \ & g\left(\nabla_{\varphi_b V}\left(\varphi_b U + \eta_a(U)\xi_c\right), X\right) - g\left(\nabla_{\varphi_a U}\left(-\varphi_a V + \eta_b(V)\xi_c\right), X\right) \\ = \ & g\left(\nabla_{\varphi_a U}\varphi_a V + \nabla_{\varphi_b V}\varphi_b U, X\right). \end{aligned}$$

Taking in to account the equation (4.15) and Theorem 4.10, we conclude that the distribution $\mathcal{D}^\perp \oplus E$ is integrable if and only if for all $U, V \in T^\perp M$ and $X \in \mathcal{D}$

$$g\left(\nabla_{\varphi_a U}\varphi_a V, X\right) = 0, \qquad a = 1, 2, 3. \tag{4.24}$$

On the other hand, we have for all $U, V \in T^\perp M$ and $X \in \mathcal{D}$

$$\begin{aligned} g\left(\nabla_{\varphi_a U}\varphi_a V, X\right) &= g\left(\widetilde{\nabla}_{\varphi_a U}\varphi_a V, X\right) = g\left(\varphi_a\widetilde{\nabla}_{\varphi_a U}\varphi_a V, \varphi_a X\right) \\ &= g\left(\widetilde{\nabla}_{\varphi_a U}\left(-V + \eta_a(V)\xi_a\right), \varphi_a X\right) \\ &= g\left(-\widetilde{\nabla}_{\varphi_a U}V, \varphi_a X\right) = g\left(A_V\varphi_a U, \varphi_a X\right), \end{aligned}$$

or

$$g\left(\nabla_{\varphi_a U}\varphi_a V, X\right) = g\left(h\left(\varphi_a U, \varphi_a X\right), V\right). \tag{4.25}$$

Thus from (4.24) and (4.25) the two statements are equivalent. □

5. Geometry of leaves

In this section we obtain sufficient condition for leaves of distribution $\mathcal{D}\oplus\mathcal{E}$ (resp. $\mathcal{D}^{\perp}\oplus\mathcal{E}$) to be totally geodesic immersed in $\widetilde{M}$ (resp. M).

THEOREM 5.1. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. If the distribution $\mathcal{D}\oplus\mathcal{E}$ is integrable then each leaf of $\mathcal{D}\oplus\mathcal{E}$ is totally geodesic immersed in $\widetilde{M}$.*

PROOF. Let M' be a leaf of $\mathcal{D}\oplus\mathcal{E}$. We denote by h' the second fundamental form of the immersion of M' in $\widetilde{M}$ and by ∇' the Riemannian connection induced by $\widetilde{\nabla}$ on M'. Then we get

$$\widetilde{\nabla}_X Y = \nabla'_X Y + h'(X,Y), \qquad X, Y \in TM'. \tag{5.1}$$

Since $\mathcal{D}\oplus\mathcal{E}$ is invariant by each φ_a, $a = 1, 2, 3$, therefore from (5.1) taking account of (2.11), we obtain

$$\begin{aligned} h'(X,\varphi_a Y) &= \widetilde{\nabla}_X \varphi_a Y - \nabla'_X \varphi_a Y = \varphi_a \widetilde{\nabla}_X Y - \nabla'_X \varphi_a Y \\ &= \varphi_a \nabla'_X Y + \varphi_a h'(X,Y) - \nabla'_X \varphi_a Y. \end{aligned} \tag{5.2}$$

Taking normal parts to TM' in (5.2), we get

$$\begin{aligned} h'(X,\varphi_3 Y) &= \varphi_3 h'(X,Y) = (\varphi_1 \circ \varphi_2) h'(X,Y) \\ &= \varphi_1 h'(X,\varphi_2 Y) = h'(\varphi_1 X, \varphi_2 Y) \\ &= h'((\varphi_2 \circ \varphi_1) X, Y) = -h'(\varphi_3 X, Y) \\ &= -h'(X,\varphi_3 Y). \end{aligned}$$

Hence we get

$$h'(X,\varphi_3 Y) = 0, \qquad X, Y \in \mathcal{D}\oplus\mathcal{E}. \tag{5.3}$$

Since φ_3 is an automorphism of $\mathcal{D}\oplus\{\xi_1\}\oplus\{\xi_2\}$ from (5.3) it follows that

$$h'(X,Z) = 0, \qquad X \in \mathcal{D}\oplus\mathcal{E},\ Z \in \mathcal{D}\oplus\{\xi_1\}\oplus\{\xi_2\}. \tag{5.4}$$

Next, (5.3) is valid if we replace φ_3 by φ_1, φ_2. Thus we get

$$h'(X,\xi_3) = h'(X,\varphi_1\xi_2) = 0, \qquad X \in \mathcal{D}\oplus\mathcal{E}. \tag{5.5}$$

By (5.4) and (5.5) it follows that M' is totally geodesic immersed in $\widetilde{M}$. □

THEOREM 5.2. *Let M be a generic submanifold of $\widetilde{M}$ equipped with a hypercosymplectic 3-structure* $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. *If the distribution $\mathcal{D}^\perp \oplus \mathcal{E}$ is integrable then each leaf of $\mathcal{D}^\perp \oplus \mathcal{E}$ is totally geodesic immersed in M.*

PROOF. Let M^* be a leaf of $\mathcal{D}^\perp \oplus \mathcal{E}$. We denote by h^* the second fundamental form of the immersion of M^* in M. By using (2.8), (2.11) and Gauss and Weingarten formulae we get for $X \in \mathcal{D}^\perp \oplus \mathcal{E}$, $Z \in \mathcal{D}$, $V \in T^\perp M$ we have

$$\begin{aligned} g(\nabla_X \varphi_a V, Z) &= g\left(\widetilde{\nabla}_X \varphi_a V, Z\right) = g\left(\varphi_a \widetilde{\nabla}_X V, Z\right) = -g\left(\widetilde{\nabla}_X V, \varphi_a Z\right) \\ &= g\left(A_V X, \varphi_a Z\right) = g\left(h\left(X, \varphi_a Z\right), V\right). \end{aligned}$$

Therefore, in view of Theorem 4.12 and Lemma 4.1, we get

$$g(\nabla_X \varphi_a V, Z) = 0, \qquad a = 1, 2, 3 \tag{5.6}$$

for all $X \in \mathcal{D}^\perp \oplus \mathcal{E}$, $Z \in \mathcal{D}$, $V \in T^\perp M$. On the other hand for $X \in \mathcal{D}^\perp \oplus \mathcal{E}$, $Z \in \mathcal{D}$ we get

$$g(\nabla_X \xi_a, Z) = g\left(\widetilde{\nabla}_X \xi_a, Z\right) = 0. \tag{5.7}$$

Hence from (5.6) and (5.7) by using the Gauss formula for the immersion of M^* in M we obtain

$$g(h^*(X, Y), Z) = 0, \qquad X, Y \in \mathcal{D}^\perp \oplus \mathcal{E},\ Z \in \mathcal{D}.$$

Thus each leaf M^* of $\mathcal{D}^\perp \oplus \mathcal{E}$ is totally geodesic immersed in M. □

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