

ON GLOBAL ASYMPTOTIC STABILITY OF SOLUTIONS OF SOME IN-ARITHMETIC-MEAN-SENSE MONOTONE STOCHASTIC DIFFERENCE EQUATIONS IN \mathbb{R}^1

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Abstract. Global almost sure asymptotic stability of the trivial solution of some nonlinear stochastic difference equations with in-the-arithmetic-mean-sense monotone drift part and diffusive part driven by independent (but not necessarily identically distributed) random variables is proven under appropriate conditions in \mathbb{R}^1 . This result can be used to verify asymptotic stability of stochastic-numerical methods such as partially drift-implicit trapezoidal methods for nonlinear stochastic differential equations with variable step sizes.

Key Words. Stochastic difference equations, global asymptotic stability, almost sure stability, stochastic differential equations, and partially drift-implicit numerical methods.

1. Introduction

Suppose that a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P})$ with filtrations $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is given. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a one-dimensional real-valued $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ -martingale-difference (for details see [6]) and $\mathcal{B}(S)$ denote the set of all Borel-sets of the set S . Furthermore, let $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers and $k > 0$ be a fixed integer constant.

Throughout this paper we consider discrete time stochastic difference equations (DSDEs) of the type

$$(1) \quad x_{n+1} - x_n = -\alpha_n x_n^{2k} \left(\frac{x_{n+1} + x_n}{2} \right) + \sigma_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}, \quad n \in \mathbb{N}$$

with in-the-arithmetic-mean-sense monotone drift parts

$$a_n(x_n, x_{n+1}) = -\alpha_n x_n^{2k} \left(\frac{x_{n+1} + x_n}{2} \right),$$

driven by square-integrable martingale-differences $\xi = \{\xi_n\}_{n \in \mathbb{N}}$ in \mathbb{R}^1 with $\mathbb{E}[\xi_n] = 0$ and $\mathbb{E}[\xi_n]^2 < +\infty$. We are especially interested in conditions ensuring the asymptotic stability of trivial solutions of these DSDEs (1). The main result should be such that it can be applied to numerical methods for related continuous time

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stochastic differential equations (CSDEs) as its potential limits. For example, consider

$$(2) \quad dX_t = a(t, X_t) dt + b(t, X_t) dW_t$$

driven by standard Wiener process $W = \{W_t\}_{t \geq 0}$ and interpreted in the Itô sense, where $a, b : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are smooth vector fields. Such CSDEs (2) can be discretized in many ways, e.g., see [20] for an overview. However, only a few of those discretization methods are appropriate to tackle the problem of almost sure asymptotic stability of its trivial solutions. One of the successful classes is that of **partially drift-implicit trapezoidal methods** with the scheme

$$(3) \quad x_{n+1} = x_n + \frac{1}{2} A(t_n, x_n) (x_{n+1} + x_n) \Delta_n + b(t_n, x_n) \Delta W_n$$

applied to equation (2), where $a(t, x) = A(t, x)x$, $\Delta_n = t_{n+1} - t_n$ and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$, for a discretization $0 = t_0 \leq t_1 \leq \dots \leq t_N = T$ of the time interval $[0, T]$. These methods provide L^2 -converging approximations to (2) with rate 0.5 in the worst case under appropriate conditions on a, b . For details, see [17]. Obviously, schemes (3) applied to Itô-type CSDEs

$$(4) \quad dX_t = -\gamma^2 [X_t]^{2k+1} dt + b(t, X_t) dW_t$$

possess the form of (1) with $\alpha_n = \gamma^2 \Delta_n$, $\sigma_n((x_l)_{0 \leq l \leq n}) = b(t_n, x_n)$, $A(t, x) = -\gamma^2 x^{2k}$ and $\Delta W_n = \xi_{n+1}$. Thus, assertions on the asymptotic stability of the trivial solution of (1) help us to understand the qualitative-asymptotic behavior of methods (3) and give criteria for choosing possibly variable step sizes Δ_n for long term numerical integration such that asymptotic stability can also be guaranteed for the discretization of the related continuous time system too.

In passing we note that, that several authors have dealt with asymptotic moment-stability of stochastic-numerical methods for CSDEs. Just to name a few of them, Abukhaled and Allen [1] on expectation stability, and Artemiev [3], Artemiev and Averina [4], Mitsui and Saito [8] and Schurz [15], [16], [18], [20] with respect to mean square stability and Schurz [19] on estimates of (nonlinear) moment-stability exponents. Most of them have only treated linear equations. Moreover, very little is known on almost sure asymptotic stability for stochastic numerical methods when applied to (nonlinear) CSDEs (2). In view of equation (1), more precisely speaking, the bilinear case with $k = 0$, moment stability issues have been examined for the corresponding drift-implicit trapezoidal methods with equidistant step sizes Δ in [18]. Here we concentrate us on the nonlinear and nonautonomous subclasses of (1) exclusively, in particular, on the case with variable step sizes Δ_n .

The paper is organized in 5 sections. We suppose that the reader is familiar with basic facts on stochastic calculus, although we provide some in Section 2. This section lists some of the most important auxiliary results known from literature to prove our main result on asymptotic stability of difference equations (1) in Section 3. Section 4 discusses its applicability to the numerical approximation of stochastic differential equations illustrated by partially drift-implicit methods. Eventually, section 5 closes this paper with some final concluding remarks.

2. Auxiliary statements and Definition

The following Lemma 1 is a generalization of Doob decomposition of submartingales (for details, see [6]). Throughout the paper, we abbreviate the expression

" \mathbb{P} -almost surely" by "a.s.". For more details on stochastic concepts and notation, consult [6], [10], [21]. We only list the most needed ones here.

Lemma 1. *Let $\{\xi_n\}_{n \in \mathbb{N}}$ be an $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ -martingale-difference. Then there exists an $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ -martingale-difference $\mu = \{\mu_n\}_{n \in \mathbb{N}}$ and a positive $(\mathcal{F}_{n-1}, \mathcal{B}(\mathbb{R}^1))$ -measurable (i.e. predictable) stochastic sequence $\eta = \{\eta_n\}_{n \in \mathbb{N}}$ such that, for every $n = 1, 2, \dots$ almost surely (a.s.),*

$$(5) \quad \xi_n^2 = \mu_n + \eta_n.$$

The process $\{\eta_n\}_{n \in \mathbb{N}}$, as in Lemma 1, can be represented by $\eta_n = \mathbb{E}(\xi_n^2 | \mathcal{F}_{n-1})$. Moreover, $\eta = (\eta_n)_{n \in \mathbb{N}}$ is a non-random sequence when ξ_n are independent random variables. In this case, we have

$$\eta_n = \mathbb{E}(\xi_n^2) \quad \text{and} \quad \mu_n = \xi_n^2 - E(\xi_n^2).$$

To establish asymptotic stability we shall also make use of a certain application of well-known martingale convergence theorems (cf. [21]) in form of Lemma 2 (see [7]). Lemma 2 below can be considered as a generalization of Theorem 7 (Chapter 2, p. 139) proved in [6] and Lemma A (p. 243) proved in [22].

Lemma 2. *Let $\{A_n^1\}_{n \in \mathbb{N}}$, $\{A_n^2\}_{n \in \mathbb{N}}$, $\{B_n^1\}_{n \in \mathbb{N}}$ and $\{B_n^2\}_{n \in \mathbb{N}}$ with $A_0^1 = A_0^2 = B_0^1 = B_0^2 = 0$ be a.s. non-decreasing $(\mathcal{F}_{n-1}, \mathcal{B}(\mathbb{R}^1))$ -measurable stochastic sequences with $B_n^1 \leq A_n^1$, $B_n^2 \geq A_n^2$ and $A_n = B_n^1 - B_n^2$ for $n \geq 1$. Assume that $Z = \{Z_n\}_{n \in \mathbb{N}}$ is a non-negative $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ -semimartingale with its Doob-Meyer decomposition $Z_n = Z_0 + M_n + A_n$ for $n \in \mathbb{N}$. Then $\{\omega : A_\infty^1 < \infty\} \subseteq \{Z \rightarrow\} \cap \{\omega : A_\infty^2 < \infty\}$ a.s., where $\{Z \rightarrow\}$ denotes the set of all $\omega \in \Omega$ for which $Z_\infty = \lim_{t \rightarrow \infty} Z_t$ exists and is finite.*

The concept of almost sure asymptotic stability under investigation is defined as follows.

Definition 1. The difference equation (1) is said to have a **trivial (equilibrium) solution** $x^* = 0$ if, for all $n \in \mathbb{N}$, we have $\sigma_n((x_l^*)_{0 \leq l \leq n}) = 0$ where all $x_l^* = 0$. A trivial (equilibrium) solution $x^* = 0$ of (1) is said to be **globally a.s. asymptotically stable** if, for all $x_0 \neq 0$ (a.s.) which are independent of the σ -algebra $\sigma(\xi_n : n \in \mathbb{N})$, we have $\mathbb{P}(\lim_{n \rightarrow +\infty} x_n = 0) = 1$.

3. Almost Sure Global Asymptotic Stability of (1)

We suppose that all ξ_n are independent random variables with $\mathbb{E}[\xi_n] = 0$, $\mathbb{E}[\xi_n]^2 < +\infty$, and there exists non-negative numbers $\lambda_n \in \mathbb{R}_+$ for all $n \in \mathbb{N}$ such that

$$(6) \quad |\sigma_n((x_l)_{0 \leq l \leq n})|^2 \leq \lambda_n (1 + x_n^{2k}), \quad \sum_{n=0}^{+\infty} \max\left(1, \frac{1}{\alpha_n}\right) \lambda_n \mathbb{E}[\xi_{n+1}^2] < +\infty$$

and the sequence $\alpha = \{\alpha_i\}_{i \in \mathbb{N}}$ of positive numbers $\alpha_i \in \mathbb{R}_+$ satisfies

$$(7) \quad \forall \varepsilon > 0 \quad \sum_{i=0}^{+\infty} \frac{\alpha_i}{(\varepsilon + \alpha_i)^2} = +\infty.$$

Theorem 1. *Let conditions (6) and (7) be fulfilled. Then the solution $\{x_n\}_{n \in \mathbb{N}}$ of equation (1) has the property that, for every initial condition x_0 which is independent of the σ -algebra $\sigma(\xi_n : n \in \mathbb{N})$, $\lim_{n \rightarrow +\infty} x_n = 0$ almost surely, i.e. if additionally σ has 0 as its trivial equilibrium then 0 is an asymptotically stable equilibrium with probability one.*

Proof. We may suppose that x_0 is non-random since x_0 is assumed to be independent of the σ -algebra $\sigma(\xi_n : n \in \mathbb{N})$. Now, multiply (1) by $x_{n+1} + x_n$ and obtain

$$(8) \quad x_{n+1}^2 - x_n^2 = -\frac{\alpha_n x_n^{2k}}{2} (x_{n+1} + x_n)^2 + \sigma_n((x_l)_{0 \leq l \leq n}) (x_{n+1} + x_n) \xi_{n+1}.$$

Exploiting the equation (1) we explicitly express x_{n+1} in terms of x_n as follows

$$(9) \quad \begin{aligned} x_{n+1} - x_n &= -\frac{\alpha_n x_n^{2k} x_{n+1}}{2} - \frac{\alpha_n x_n^{2k+1}}{2} + \sigma_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}, \\ x_{n+1} \left(1 + \frac{\alpha_n x_n^{2k}}{2}\right) &= x_n - \frac{\alpha_n x_n^{2k+1}}{2} + \sigma_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}, \\ x_{n+1} &= \frac{x_n}{1 + \frac{\alpha_n x_n^{2k}}{2}} - \frac{\frac{\alpha_n x_n^{2k+1}}{2}}{1 + \frac{\alpha_n x_n^{2k}}{2}} + \frac{\sigma_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}}{1 + \frac{\alpha_n x_n^{2k}}{2}} \\ &= F_n(x_n) + G_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}, \end{aligned}$$

where

$$(10) \quad F_n(x_n) = \frac{x_n}{1 + \frac{\alpha_n x_n^{2k}}{2}} - \frac{\frac{\alpha_n x_n^{2k+1}}{2}}{1 + \frac{\alpha_n x_n^{2k}}{2}}, \quad G_n((x_l)_{0 \leq l \leq n}) = \frac{\sigma_n((x_l)_{0 \leq l \leq n})}{1 + \frac{\alpha_n x_n^{2k}}{2}}.$$

Substituting (9) in (8) and applying Lemma 1 yields that

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= -\frac{\alpha_n x_n^{2k}}{2} (x_n + F(x_n) + G_n((x_l)_{0 \leq l \leq n}) \xi_{n+1})^2 + \\ &\quad + \sigma_n((x_l)_{0 \leq l \leq n}) (x_n + F(x_n) + G_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}) \xi_{n+1} \\ &= -\frac{\alpha_n x_n^{2k}}{2} [(x_n + F(x_n))^2 + 2(x_n + F(x_n))G_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}] + \\ &\quad - \frac{\alpha_n x_n^{2k}}{2} [G_n^2((x_l)_{0 \leq l \leq n}) (\eta_{n+1} + \mu_{n+1})] + \sigma_n((x_l)_{0 \leq l \leq n}) G_n((x_l)_{0 \leq l \leq n}) \eta_{n+1} + \\ &\quad + \sigma_n((x_l)_{0 \leq l \leq n}) (x_n + F(x_n)) \xi_{n+1} + \sigma_n((x_l)_{0 \leq l \leq n}) G_n((x_l)_{0 \leq l \leq n}) \mu_{n+1} \\ &= -\frac{\alpha_n x_n^{2k}}{2} ((x_n + F(x_n))^2 + G_n^2((x_l)_{0 \leq l \leq n}) \eta_{n+1}) + \\ &\quad + \sigma_n((x_l)_{0 \leq l \leq n}) G_n((x_l)_{0 \leq l \leq n}) \eta_{n+1} + \Delta M_{n+1}, \end{aligned}$$

where

$$(11) \quad \begin{aligned} \Delta M_{n+1} &= -\alpha_n x_n^{2k} \left[(x_n + F(x_n)) G_n((x_l)_{0 \leq l \leq n}) \xi_{n+1} + \frac{1}{2} G_n^2((x_l)_{0 \leq l \leq n}) \mu_{n+1} \right] + \\ &\quad + \sigma_n((x_l)_{0 \leq l \leq n}) (x_n + F(x_n)) \xi_{n+1} + \sigma_n((x_l)_{0 \leq l \leq n}) G_n((x_l)_{0 \leq l \leq n}) \mu_{n+1} \end{aligned}$$

is an $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ -martingale-difference. We note that

$$\begin{aligned} (x_n + F(x_n))^2 &= \left(x_n + \frac{x_n}{1 + \frac{\alpha_n x_n^{2k}}{2}} - \frac{\frac{\alpha_n x_n^{2k+1}}{2}}{1 + \frac{\alpha_n x_n^{2k}}{2}} \right)^2 \\ &= \left(\frac{2x_n - \frac{\alpha_n x_n^{2k+1}}{2} + \frac{\alpha_n x_n^{2k+1}}{2}}{1 + \frac{\alpha_n x_n^{2k}}{2}} \right)^2 = \frac{4x_n^2}{\left(1 + \frac{\alpha_n x_n^{2k}}{2}\right)^2}. \end{aligned}$$

Then

$$\begin{aligned} x_{n+1}^2 &= x_n^2 - \left[\frac{2\alpha_n x_n^{2k+2}}{\left(1 + \frac{\alpha_n x_n^{2k}}{2}\right)^2} + \frac{\alpha_n x_n^{2k}}{2} G_n^2((x_l)_{0 \leq l \leq n}) \eta_{n+1} \right] + \\ (12) \quad &+ \sigma_n((x_l)_{0 \leq l \leq n}) G_n((x_l)_{0 \leq l \leq n}) \eta_{n+1} + \Delta M_{n+1}. \end{aligned}$$

After summation of (12) we obtain

$$x_n^2 = x_0^2 + B_n^1 - B_n^2 + M_n,$$

where $M_n = \sum_{i=0}^n \Delta M_i$ with $\Delta M_0 = 0$, $M_0 = 0$ and $\Delta M_i = M_i - M_{i-1}$ satisfying (11), and

$$\begin{aligned} B_n^1 &= \sum_{i=0}^{n-1} \sigma_i((x_l)_{0 \leq l \leq i}) G_i((x_l)_{0 \leq l \leq i}) \eta_{i+1}, \quad B_0^1 = B_0^2 = 0, \\ B_n^2 &= \sum_{i=0}^{n-1} \left[\frac{2\alpha_i x_i^{2k+2}}{\left(1 + \frac{\alpha_i x_i^{2k}}{2}\right)^2} + \frac{\alpha_i x_i^{2k}}{2} G_i^2((x_l)_{0 \leq l \leq i}) \eta_{i+1} \right] \text{ for } n \geq 1. \end{aligned}$$

Note that $B^1 = \{B_n^1\}_{n \in \mathbb{N}}$ and $B^2 = \{B_n^2\}_{n \in \mathbb{N}}$ form predictable (i.e. $(\mathcal{F}_{n-1}, \mathcal{B}(\mathbb{R}^1))$ -measurable) non-decreasing processes (Recall that we have $\eta_{i+1} = \mathbb{E} \xi_{i+1}^2 \geq 0$ and the identities $\sigma_i((x_l)_{0 \leq l \leq i}) G_i((x_l)_{0 \leq l \leq i}) = \sigma_i^2((x_l)_{0 \leq l \leq i}) / (1 + \alpha_i x_i^{2k}/2) \geq 0$). From (6) we have a.s.

$$\begin{aligned} \lim_{n \rightarrow +\infty} B_n^1 &= \sum_{i=0}^{\infty} \sigma_i((x_l)_{0 \leq l \leq i}) G_i((x_l)_{0 \leq l \leq i}) \eta_{i+1} \\ &= \sum_{i=0}^{\infty} \sigma_i^2((x_l)_{0 \leq l \leq i}) \left(1 + \frac{\alpha_i x_i^{2k}}{2}\right)^{-1} \eta_{i+1} \\ &\leq 2 \sum_{i=0}^{\infty} \lambda_i \left(1 + \frac{x_i^{2k}}{2}\right) \left(1 + \frac{\alpha_i x_i^{2k}}{2}\right)^{-1} \eta_{i+1} \\ (13) \quad &\leq 2 \sum_{i=0}^{\infty} \max\left(1, \frac{1}{\alpha_i}\right) \lambda_i \eta_{i+1} < +\infty. \end{aligned}$$

Now, we set $A_n^1 = 2 \sum_{i=0}^{n-1} \max\left(1, \frac{1}{\alpha_i}\right) \lambda_i \eta_{i+1}$ and $A_n^2 = B_n^2$ for all integers $n \geq 1$, and $A_0^1 = A_0^2 = 0$. Next, apply Lemma 2 to the sequence $Z = \{Z_n\}_{n \in \mathbb{N}}$ with $Z_n = x_n^2$ and obtain that a.s. $\lim_{n \rightarrow +\infty} x_n^2$ and $\lim_{n \rightarrow +\infty} A_n^2 = \lim_{n \rightarrow +\infty} B_n^2$ exist and are finite. It remains to prove that $\lim_{n \rightarrow +\infty} x_n^2 = 0$. Suppose, indirectly, that the opposite is true. Then there exists a.s. a finite number $a_0^2(\omega) > 0$ on $\Omega_1 = \{\omega : \lim_{n \rightarrow +\infty} x_n^2(\omega) = a_0^2(\omega) > 0\}$ with $\mathbb{P}(\Omega_1) = p_1 > 0$. There also exists

an integer $N(\omega) \geq 0$ such that $\frac{3a_0^2}{2}(\omega) \geq x_n^2(\omega) \geq \frac{a_0^2}{2}(\omega)$ for all $n \geq N(\omega)$ on $\omega \in \Omega_1$. Then, for $\omega \in \Omega_1$, some a.s. finite $a_1^2(\omega) > 0$ and for all $n > N(\omega)$, we have

$$\begin{aligned} B_n^2(\omega) &= A_n^2(\omega) = \sum_{i=0}^{n-1} \left[\frac{2\alpha_i x_i^{2k+2}(\omega)}{\left(1 + \frac{\alpha_i x_i^{2k}(\omega)}{2}\right)^2} + \frac{\alpha_i x_i^{2k}(\omega)}{2} G_i^2((x_l)_{0 \leq l \leq i}(\omega)) \eta_{i+1}(\omega) \right] \\ &\geq \sum_{i=0}^{n-1} \left[\frac{2\alpha_i x_i^{2k+2}(\omega)}{\left(1 + \frac{\alpha_i x_i^{2k}(\omega)}{2}\right)^2} \right] \geq \sum_{i=N(\omega)}^{n-1} \left[\frac{2\alpha_i a_0^{2k+2}(\omega)}{2^{2k+2} \left(1 + \alpha_i \frac{3^k a_0^{2k}(\omega)}{2^{k+1}}\right)^2} \right] \\ &= 2a_0^{2k+2}(\omega) \sum_{i=N(\omega)}^{n-1} \frac{\alpha_i}{(2^{k+1} + \alpha_i 3^k a_0^{2k}(\omega))^2} = \frac{2a_0^{2-2k}(\omega)}{3^{2k}} \sum_{i=N(\omega)}^{n-1} \frac{\alpha_i}{(\varepsilon_k(\omega) + \alpha_i)^2} \\ &\xrightarrow{n \rightarrow +\infty} +\infty \end{aligned}$$

for all $\varepsilon_k(\omega) = \frac{2^{k+1}}{3^k a_0^{2k}(\omega)} > 0$ on Ω_1 due to condition (7). This result contradicts to the fact that $\lim_{n \rightarrow +\infty} B_n^2(= \lim_{n \rightarrow +\infty} A_n^2)$ exist and is a.s. finite. Thus, Theorem 1 is proved. \square

Remark. There are plenty of interesting choices of α_n in view of potential applications, e.g. $\alpha_n = \text{constant} > 0$ or periodic $\alpha_n = 2 + \sin((2n+1)\pi/2)$ which could fulfill the conditions (6) and (7). More problematic is the case when $\alpha_n = \gamma^2/n$. Obviously, condition (7) would be satisfied in this case due to Abel's series divergence test. However, the choice of possible random variables ξ_n together with parameter λ_n would be very restricted in order to ensure the convergence of the series occurring in condition (6).

4. An Application to Numerical Methods for CSDEs

As an example of applicability of our main result, consider the Itô-interpreted CSDEs

$$(14) \quad dX_t = -\gamma^2 [X_t]^{2k+1} dt + \sigma \frac{[X_t]^k}{1+t} dW_t$$

with real constants γ, σ , driven by the standard Wiener process $W = \{W_t\}_{t \geq 0}$ and discretized by the **partially drift-implicit trapezoidal method**

$$(15) \quad x_{n+1} = x_n - \frac{1}{2} \gamma^2 [x_n]^{2k} (x_{n+1} + x_n) \Delta_n + \sigma \frac{[x_n]^k}{1+t_n} \xi_{n+1},$$

where $\xi_{n+1} = \Delta W_n$ are independent random variables with moments $\mathbb{E}[\xi_{n+1}] = 0$ and $\eta_{n+1} = \mathbb{E}[\xi_{n+1}]^2 = \Delta_n$ while using non-random step sizes Δ_n . For general theory of CSDEs (14), see Arnold [2], and for an overview on numerical analysis of (14), see Schurz [20]. Obviously, both equations possess the trivial equilibrium 0 for $k > 0$. Let us focus our attention on its qualitative asymptotic behavior.

Theorem 2. *Assume that the initial values $X_0 = x_0$ are independent of the σ -algebra $\sigma(W_s : s \geq 0)$ and $k > 0$. Then the trivial solution 0 of CSDE (14) is globally asymptotically stable (a.s.).*

Proof. Apply Itô formula to the Lyapunov function $V(x) = x^2$ to the solution of the equation (14). Thus, combined with Young's inequality with $p = (2k+2)/2k$,

we obtain

$$(16) \quad \begin{aligned} dX_t^2 &= \left(-2\gamma^2[X_t]^{2k+2} + \sigma^2 \frac{[X_t]^{2k}}{(1+t)^2} \right) dt + dm_t \\ &\leq \left(-2\gamma^2[X_t]^{2k+2} + \sigma^2 \frac{\frac{2}{2k+2} + \frac{2k[X_t]^{2k+2}}{2k+2}}{(1+t)^2} \right) dt + dm_t \end{aligned}$$

for $t \geq 0$, where $m = \{m_t\}_{t \geq 0}$ with $m_t = 2\sigma \int_0^t \frac{[X_s]^{k+1}}{1+s} dW_s$ is a square-integrable martingale. Obviously, we find a non-random real constant $\mu > 0$ such that

$$(17) \quad dX_t^2 \leq \left(-\mu[X_t]^{2k+2} + \frac{\sigma^2}{(k+1)(1+t)^2} \right) dt + dm_t$$

for all sufficiently large $t \geq t_0$. Hence, the asymptotic behavior of the non-negative semimartingale X_t^2 governed by (17) is controlled by the solution $Z = \{Z_t\}_{t \geq t_0}$ of stochastic differential equation

$$(18) \quad dZ_t = \left(-\mu[Z_t]^{k+1} + \frac{\sigma^2}{(k+1)(1+t)^2} \right) dt + dm_t$$

with sufficiently large $t \geq t_0$. Therefore we may decompose its drift into non-decreasing processes $A^1 = \{A_t^1\}_{t \geq t_0}$ and $A^2 = \{A_t^2\}_{t \geq t_0}$ given by

$$(19) \quad A_t^1 = \frac{\sigma^2}{k+1} \int_{t_0}^t \frac{ds}{(1+s)^2}, \quad A_t^2 = \mu \int_{t_0}^t [Z_s]^{k+1} ds.$$

Notice that

$$\lim_{t \rightarrow +\infty} A_t^1 = \frac{\sigma^2}{(k+1)(1+t_0)} < +\infty.$$

Now we may apply Lemma 2 (as already noted, a similar statement is also found by Theorem 7, Chapter 2 from [6], page 139) in order to know about the existence of the finite limits $Z_{+\infty} = \lim_{t \rightarrow +\infty} Z_t^2$ and $\lim_{t \rightarrow +\infty} A_t^2 < +\infty$. It remains to show that X_t converges to 0 (a.s.). Note that $Z_t \geq 0$ for all $t \geq t_0$, $\mu > 0$ and $\lim_{t \rightarrow +\infty} A_t^2 = \mu \int_{t_0}^{+\infty} [Z_s]^{k+1} ds < +\infty$ holds. It is well-known that the convergence of positive integrand $[Z_s]^{k+1}$ to 0 as s tends to $+\infty$ is necessary for the convergence of the improper integral in $\lim_{t \rightarrow +\infty} A_t^2$. Hence, $\lim_{t \rightarrow +\infty} Z_t^{k+1} = 0$ (a.s.) implies that $\lim_{t \rightarrow +\infty} X_t^2 = 0$ and $X_\infty = \lim_{t \rightarrow \infty} X_t = 0$ (a.s.). Therefore, the proof is complete. \square

Remark. The proof of Theorem 2 extends to the more general Itô equation

$$dX_t = -\gamma^2 X_t^{2k+1} dt + \sigma(t) X_t^k dW_t.$$

To guarantee asymptotic stability for this equation one needs to require

$$\lim_{t \rightarrow +\infty} \sigma(t) = 0 \quad \text{and} \quad \exists t_0 \int_{t_0}^{+\infty} \sigma^2(s) ds < +\infty,$$

and one can work through the same proof-steps as before. Remarkable in view of its asymptotic behavior, it does not really matter what value $k > 0$ has.

However, for the DSDE (15), the dynamic behavior with respect to asymptotic stability might depend on the choice of step sizes Δ_n . This fact can be seen immediately from the following corollaries to Theorem 1.

Corollary 1. *Let $x = \{x_n\}_{n \in \mathbb{N}}$ satisfy the stochastic difference equation (15) under the above mentioned conditions with $\gamma^2 > 0$ and non-random variable step sizes $\Delta_n > 0$ which are uniformly bounded such that*

$$(20) \quad \exists \Delta_a, \Delta_b : \forall n \in \mathbb{N} \quad 0 < \Delta_b \leq \Delta_n \leq \Delta_a < +\infty.$$

Then, the equation (15) for sequences $x = \{x_n\}_{n \in \mathbb{N}}$ possesses an a.s. globally asymptotically stable trivial solution.

Proof. Apply Theorem 1 from section 3. For this purpose, note that equation (15) has the form (1) with $\alpha_n = \gamma^2 \Delta_n > 0$. It remains to check the conditions (6) and (7). Define $\lambda_n = \sigma^2 / (1 + t_n)^2$. One easily estimates

$$\begin{aligned} |\sigma_n((x_l)_{0 \leq l \leq n})| &= |\sigma^2 \frac{x_n^{2k}}{(1 + t_n)^2}| < \frac{\sigma^2}{(1 + t_n)^2} (1 + x_n^{2k}) = \lambda_n (1 + x_n^{2k}), \\ \sum_{n=0}^{+\infty} \max(1, \frac{1}{\alpha_n}) \lambda_n \mathbb{E}[\xi_{n+1}^2] &= \sum_{n=0}^{+\infty} \max\left(1, \frac{1}{\gamma^2 \Delta_n}\right) \frac{\sigma^2}{(1 + t_n)^2} \Delta_n \\ &\leq \sigma^2 \max\left(1, \frac{1}{\gamma^2 \Delta_b}\right) \Delta_a \sum_{n=0}^{+\infty} \frac{1}{(1 + n \Delta_b)^2} \\ &< \sigma^2 \max\left(1, \frac{1}{\gamma^2 \Delta_b}\right) \frac{\Delta_a}{(\Delta_b)^2} \frac{\pi^2}{6} < +\infty \end{aligned}$$

and $\forall \varepsilon > 0$

$$\sum_{i=0}^{+\infty} \frac{\alpha_i}{(\varepsilon + \alpha_i)^2} = \sum_{i=0}^{+\infty} \frac{\gamma^2 \Delta_i}{(\varepsilon + \gamma^2 \Delta_i)^2} \geq \frac{\gamma^2 \Delta_b}{(\varepsilon + \gamma^2 \Delta_a)^2} \sum_{i=0}^{+\infty} (1) = +\infty.$$

These computations confirm the validity of (6) and (7), hence Theorem 1 can be applied and the proof of Corollary 1 is complete. \square

Remark. Consequently, our main result says that 0 is an asymptotically stable equilibrium for the method (15) with probability one. Thus, as in the linear case $k = 0$ (cf. [18]), there is no discrepancy between the qualitative behavior of CSDE (14) and its discretization (15) using the reasonable choice of both constant and uniformly bounded, but variable step sizes Δ_n . Again we have some indications that the choice of step sizes following the restriction (20) gives meaningful qualitative results for stochastic-numerical methods (cf. [16], [19], [20]).

More care is needed when choosing variable step size algorithms with variable step sizes tending to zero as n advances in order to achieve adequate convergence and asymptotic stability results. Obviously, we note that conditions (6) are not always fulfilled in the case of variable step sizes Δ_n . However, when $\Delta_n = K/n^\beta$

(note this can run below any natural machine accuracy!) we arrive at the following mathematical results.

Corollary 2. *Let $x = \{x_n\}_{n \in \mathbb{N}}$ satisfy the stochastic difference equation (15) under the above mentioned conditions with $\gamma^2 > 0$ and non-random variable step sizes $\Delta_n > 0$ which are uniformly bounded such that*

$$(21) \quad \exists K_1, K_2, \beta \in (0, 1/2) \forall n \geq 1 : \frac{K_2}{n^\beta} \leq \Delta_n \leq \frac{K_1}{n^\beta}.$$

Then, the equation (15) for sequences $x = \{x_n\}_{n \in \mathbb{N}}$ possesses an a.s. globally asymptotically stable trivial solution.

Proof. Apply Theorem 1 from section 3. For this purpose, note that equation (15) has the form

$$(22) \quad x_{n+1} - x_n = -\alpha_n x_n^{2k} \left(\frac{x_{n+1} + x_n}{2} \right) + \sigma_n((x_l)_{0 \leq l \leq n}) \xi_{n+1}.$$

$\alpha_n = \gamma^2 \Delta_n > 0$. Suppose that $\gamma^2 \Delta_n < 1$. Define $\lambda_n = \sigma^2 / (1 + t_n)^2$. At first, we note that

$$\begin{aligned} K_2 n^{1-\beta} &= n \frac{K_2}{n^\beta} \leq t_n = \sum_{i=0}^{n-1} \Delta_i \\ &\leq n \frac{K_1}{n^\beta} = K_1 n^{1-\beta} \quad \text{and} \\ \frac{1}{1+t_n} &\leq \frac{1}{1+K_2 n^{1-\beta}} \leq \frac{1}{K_2 n^{1-\beta}} \quad \text{for } n \geq 1. \end{aligned}$$

It remains to check the conditions (6) and (7). We have

$$\begin{aligned} |\sigma_n((x_l)_{0 \leq l \leq n})| &= |\sigma^2 \frac{x_n^{2k}}{(1+t_n)^2}| < \frac{\sigma^2}{(1+t_n)^2} (1+x_n^{2k}) = \lambda_n (1+x_n^{2k}), \\ \sum_{n=1}^{+\infty} \max(1, \frac{1}{\alpha_n}) \lambda_n \mathbb{E}[\xi_{n+1}^2] &\leq \sum_{n=1}^{+\infty} \frac{1}{\gamma^2 \Delta_n} \frac{\sigma^2}{(1+t_n)^2} \Delta_n \\ &\leq \frac{\sigma^2}{\gamma^2} \sum_{n=1}^{+\infty} \frac{1}{(1+t_n)^2} \leq \frac{\sigma^2}{\gamma^2 K_2^2} \sum_{n=1}^{+\infty} \frac{1}{n^{2-2\beta}} < +\infty \end{aligned}$$

if $2 - 2\beta > 1$ or equivalently $\beta < 1/2$ by Abel's convergence test. Furthermore, for all $\varepsilon > 0$, estimate

$$\begin{aligned} \sum_{i=0}^{+\infty} \frac{\alpha_i}{\varepsilon + \alpha_i} &\geq \gamma^2 \sum_{i=1}^{+\infty} \frac{K_2 i^{-\beta}}{(\varepsilon + \gamma^2 K_1 i^{-\beta})^2} = \gamma^2 K_2 \sum_{i=1}^{+\infty} \frac{i^{2\beta}}{i^\beta (\varepsilon i^\beta + \gamma^2 K_1)^2} \\ &= \frac{\gamma^2 K_2}{\varepsilon^2} \sum_{i=1}^{+\infty} \frac{i^{2\beta}}{i^\beta (i^\beta + \gamma^2 K_1 / \varepsilon)^2} \geq \frac{\gamma^2 K_2}{(\varepsilon + \gamma^2 K_1)^2} \sum_{i=1}^{+\infty} \frac{1}{i^\beta} = +\infty \end{aligned}$$

by Abel's divergence test. Now, apply Theorem 1. Similarly, we arrive at the conclusion for the case $\gamma^2 \Delta_n \geq 1$. This completes the proof of Corollary 2. \square

Again we recognize that a bound of variable step sizes from below is essential to establish asymptotic stability of numerical methods. A slightly modified example

is given as follows. Suppose that the process $X = \{X_t\}_{t \geq 0}$ satisfies the Itô SDE

$$(23) \quad dX_t = -\gamma^2 [X_t]^{2k+1} dt + \sigma \frac{[X_t]^k}{(1+t)^\tau} dW_t$$

with real constants γ, σ and $\tau > 1/2$, discretized by the partially drift-implicit trapezoidal method

$$(24) \quad x_{n+1} = x_n - \frac{1}{2} \gamma^2 [x_n]^{2k} (x_{n+1} + x_n) \Delta_n + \sigma \frac{[x_n]^k}{(1+t_n)^\tau} \xi_{n+1},$$

where $\xi_{n+1} = \Delta W_n$ with $\eta_{n+1} = \mathbb{E}[\xi_{n+1}]^2 = \Delta_n$ while using non-random step sizes Δ_n .

Corollary 3. *Let $x = \{x_n\}_{n \in \mathbb{N}}$ satisfy the stochastic difference equation (24) under the above mentioned conditions with $\gamma^2 > 0$ and non-random variable step sizes Δ_n which are uniformly bounded such that*

$$(25) \quad \exists K_1, K_2, \beta \in (0, 1 - \frac{1}{2\tau}) \forall n \geq 1 : \quad \frac{K_2}{n^\beta} \leq \Delta_n \leq \frac{K_1}{n^\beta}.$$

Then, the equation (24) for sequences $x = \{x_n\}_{n \in \mathbb{N}}$ possesses an a.s. globally asymptotically stable trivial solution.

Proof. Apply Theorem 1 from section 3. For this purpose, note that equation (24) has the form (22) with $\alpha_n = \gamma^2 \Delta_n > 0$. As in proof before, suppose that $\gamma^2 \Delta_n < 1$. Define $\lambda_n = \sigma^2 / (1 + t_n)^{2\tau}$. We have

$$K_2 n^{1-\beta} = n \frac{K_2}{n^\beta} \leq t_n = \sum_{i=0}^{n-1} \Delta_i \leq n \frac{K_1}{n^\beta} = K_1 n^{1-\beta},$$

for all $n \geq 1$, hence

$$\frac{1}{1+t_n} \leq \frac{1}{K_2 n^{1-\beta}}.$$

It remains to check the conditions (6) since condition (7) was proved in Corollary 2.

$$\begin{aligned} |\sigma_n((x_l)_{0 \leq l \leq n})| &= |\sigma^2 \frac{x_n^{2k}}{(1+t_n)^{2\tau}}| < \frac{\sigma^2}{(1+t_n)^{2\tau}} (1+x_n^{2k}) = \lambda_n (1+x_n^{2k}), \\ \sum_{n=1}^{+\infty} \max(1, \frac{1}{\alpha_n}) \lambda_n \mathbb{E}[\xi_{n+1}^2] &= \sum_{n=1}^{+\infty} \frac{1}{\gamma^2 \Delta_n} \frac{\sigma^2}{(1+t_n)^{2\tau}} \Delta_n \\ &= \frac{\sigma^2}{\gamma^2} \sum_{n=1}^{+\infty} \frac{1}{(1+t_n)^{2\tau}} \leq \frac{\sigma^2}{\gamma^2 K_2^{2\tau}} \sum_{n=1}^{+\infty} \frac{1}{n^{2\tau(1-\beta)}} < +\infty \end{aligned}$$

if $2\tau(1-\beta) > 1$ or equivalently $\beta < 1 - \frac{1}{2\tau}$ by Abel's convergence test. Thus, the proof of Corollary 3 is complete. \square

Remark. It is easy to see that $\tau > \frac{1}{2(1-\beta)} \rightarrow +\infty$ when $\beta \uparrow 1$ and vice versa. For example, from condition (25) we have

- (1) if $\tau = 2/3$, then $\beta < 1 - 3/4 = 1/4$,
- (2) if $\tau = 1$, then $\beta < 1/2$,
- (3) if $\tau = 2$, then $\beta < 1 - 1/4 = 3/4$,
- (4) if $\tau = 4$, then $\beta < 1 - 1/8 = 7/8$ and so on.

5. Concluding Remarks

All in all, we established a fairly general approach to verify asymptotic stability of stochastic difference equations with some monotone structure in its drift. This approach is heavily based on martingale convergence theorems and allows to treat difference equations with nonautonomous and random coefficients. Some of the remarks of this paper might be interesting for the implementation and convergence proofs referring to the use of variable step sizes in stochastic-numerical algorithms instead of the more trivial case of constant ones. In this direction we have provided some illustrative examples in the previous section.

Our results can be also extended to the case of stochastic Volterra-type difference equations while using the method of Lyapunov-Krasovskii functionals as similarly done in [13]. See author's works (e.g. [14]) in the nearest future. An application to the sequence of gains incurred by an insurance company as indicated by [12] is also conceivable.

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